Effective Scalar P-Wave Equation for Multi-Dimensional Inhomogeneous Elastic Media

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We consider a single channel pressure wave equation that allows for determining the behavior, amplitude, and phase fidelity of pressure wave propagation in an inhomogeneous elastic medium. We will illustrate this for the 2D inhomogeneous case, where there are only SV and P waves.

I. 2D ELASTIC MEDIA

We choose (here for convenience, although not necessary) to describe the medium as a homogeneous background plus a perturbation in background material properties that results in actual medium properties. For this specific case, the coupled equations for P and S waves, ϕ_P and ϕ_S respectively, can be written as:

$$\begin{aligned} \left[\nabla^2 + \frac{\omega^2}{{\alpha_o}^2}\right]\phi_P &= V_{PP}\phi_P + V_{PS}\phi_S + f_P \\ \left[\nabla^2 + \frac{\omega^2}{{\beta_o}^2}\right]\phi_S &= V_{SS}\phi_S + V_{SP}\phi_P + f_S \end{aligned} \tag{I.1}$$

where α_o is the reference P-wave velocity, β_o is the reference S-wave velocity, $(V_{PP}, V_{PS}, V_{SP}, V_{SS})$ are the matrix elements of the perturbation operator in the P-S representation, and (f_P, f_S) are the components of the source function in the P-S representation.

We rewrite the second equation presented above as

$$\left[\nabla^2 + \frac{\omega^2}{{\beta_o}^2} - V_{SS}\right]\phi_S = V_{SP}\phi_P + f_S \tag{I.2}$$

and define the Green's function G_S to satisfy

$$\left[\nabla^2 + \frac{\omega^2}{{\beta_o}^2} - V_{SS}\right]G_S = \delta. \tag{I.3}$$

The causal solution to (I.2) is

$$\phi_S = \int G_S\left(V_{SP}\phi_P + f_S\right). \tag{I.4}$$

Substituting this result back into the first equation of (I.1), we obtain

$$\left[\nabla^2 + \frac{\omega^2}{{\alpha_o}^2}\right]\phi_P = V_{PP}\phi_P + V_{PS}\left\{\int G_S\left(V_{SP}\phi_P + f_S\right)\right\} + f_P \tag{I.5}$$

For the case where

$$\vec{f} = \begin{pmatrix} f_P \\ 0 \end{pmatrix} \tag{I.6}$$

and the source generates only P waves, $f_S = 0$ and (I.5) can be written

$$\phi_P = G_P^o V_{PP}\phi_P + G_P^o V_{PS}\int G_S V_{SP}\phi_P + G_P^o f_P, \tag{I.7}$$

where G_P^o is the causal solution to

$$\left[\nabla^2 + \frac{\omega^2}{{\alpha_o}^2}\right]G_P^o = \delta. \tag{I.8}$$

Equation (I.7) can thus be rewritten

$$\left[\nabla^2 + \frac{\omega^2}{{\alpha_o}^2} - \left(V_{PP} + V_{PS}G_S V_{SP}\right)\right]\phi_P = f_P. \tag{I.9}$$

This is a single channel equation for ϕ_P that builds in all of the S channel interactions that influence the P channel, without solving for ϕ_S. The equation for ϕ_P can be written as an integral equation, as in (I.7), or a differential equation, as in (I.9). G_S can be found by solving

$$\left[\nabla^2 + \frac{\omega^2}{{\beta_o}^2} - V_{SS}\right]G_S = \delta \tag{I.10}$$

directly by, e.g., finite difference methods or by numerically evaluating the Fredholm II integral equation

$$G_S = G_S^o + G_S^o V_{SS} G_S = \sum_{k=0}^{\infty} G_S^o (V_{SS} G_S^o)^k, \tag{I.11}$$

where G_S^o satisfies the relation

$$\left[\nabla^2 + \frac{\omega^2}{{\beta_o}^2}\right]G_S^o = \delta. \tag{I.12}$$

In order to facilitate the calculation of G_S for computational purposes, we can use the Born approximation

$$G_S = G_S^o + G_S^o V_{SS} G_S^o \ . \tag{I.13}$$

For simple, preliminary model systems, we choose a layer sandwiched between two semi-infinite half-spaces where in each region the P and S wave speed is constant. For simplicity, we let the density be constant throughout the entire medium. This single channel P-wave only calculation is being performed by Mason Biamonte (a junior undergraduate who already published papers in Phys. Rev. and a chapter in a Quantum Mechanics textbook).

If we develop a variable background theory, then a small perturbation could be realized and a Born form for G_S would be reasonable. As a modeling tool, we know the medium properties and using a smooth but proximal background could allow a WKBJ solution with P and S remaining uncoupled in the background.

We could also develop a discontinuous background version where P and S waves are coupled in the background as well as in the actual medium.

This single effective and complicated channel for a multi-channel problem was pioneered for electron-atom, projectile-nucleus problems by Herman Feshbach where there are infinite coupled channels (for each excited state of the target) and a single effective elastic channel is sought. The 2-channel or 3-channel (3D) solution we propose are also derivable by the Feshbach projection operator approach that he developed.