Viscoacoustic Born Series Continued: Toward Scattering-based Q Compensation/Estimation

Kristopher A. Innanen† and Arthur B. Weglein‡
†University of British Columbia, ‡University of Houston

Abstract

The forward scattering series in 1D has available to it a mapping (Matson, 1996) which allows one to follow terms, or groups of terms, from their individual computation through to their overall effect on the closed form wave field expression. This provides a tool to gain insight into the likely behaviour of the inverse scattering series, and the subseries' which have elsewhere been identified and separated as means to eliminate/attenuate multiples, and image and invert primaries. Here this tool is used to (1) gain basic understanding of the types of scattering interaction which give rise to macroscopic properties of the viscoacoustic wave field, and (2) gain focused understanding of where in the inverse scattering series the machinery for Q compensation and estimation must reside. In casting the problem with an acoustic reference medium a conceptually compelling result is obtained: a viscoacoustic wave field, attenuated and dispersed, is correctly computed through the scaled, nonlinear, combination of propagations which all occur in a lossless (reference) medium. Beyond this, the key conclusion is that the "imaging subseries", which in the acoustic case is seen as an engine for moving reflectors to their correct depths, must be generalized to include the removal of all propagation effects. This includes the phase and amplitude distortions associated with attenuation. In other words, the imaging subseries is also the basic Q-compensation machine; it is to this subseries that we must direct our attention in devising scattering-based processing strategies.

1 Introduction: Viscoacoustic Scattering Potentials

This paper involves refining our understanding of how the forward scattering series, or Born series, functions in media which attenuate the wave field. The first section develops some ideas of how various types of scattering diagram conspire to construct aspects of the viscoacoustic wave field – for instance, the absorptive propagation effects, and the negative of the direct wave. The second section is more focused on predicting the nature of an inverse scattering-based scheme for Q compensation and estimation. In it we consider the role of the so-called "separated" and "self-interacting" scatter-type comparatively, contrasting a purely acoustic case with a purely absorptive one. The results suggest where in the inverse series we must look for tools to accomplish these tasks of estimation and compensation.

To begin, we review the Born series for viscoacoustic media, and construct some appropriate scattering potentials.

The Born series representation of a wave field arises from a perturbation of the coefficients of the wave equation around a reference value. For instance, in a 1D constant density acoustic medium, the equation

$$\left[\frac{d^2}{dz^2} + \frac{\omega^2}{c(z)^2}\right] \psi(z|z_s;\omega) = \delta(z - z_s), \tag{1}$$

which describes the behaviour of the wave field $\psi(z|z_s;\omega)$, measured at z and due to an impulsive source at z_s, in a medium characterized by the wavespeed profile $c(z)$, is re-written

$$\left[\frac{d^2}{dz^2} + k_0^2(1 - \alpha(z))\right] \psi(z|z_s;k_0) = \delta(z - z_s), \tag{2}$$

where $k_0^2 = \omega^2/c_0^2$. Usually the reference model, here represented by the constant wavespeed c_0, is assumed to be known, so the perturbation $\alpha(z) = 1 - \frac{c_0^2}{c^2(z)}$ is the *de facto* model. The scattering potential $V(k_0, z)$ is the difference between the "true" and reference wave operators:

$$V(k_0, z) = \left[\frac{d^2}{dz^2} + \frac{\omega^2}{c(z)^2}\right] - \left[\frac{d^2}{dz^2} + \frac{\omega^2}{c_0^2}\right] = k_0^2\alpha(z). \tag{3}$$

The acoustic Born series is a representation of the solution of equation (1) in orders of $V(k_0, z)$. A straightforward derivation involves placing the term $V(k_0, z)\psi(z|z_s;k_0)$, onto the right-hand side of equation (2), multiplying these "sources" by the Green's function $G_0(z|z_s;k_0)$, which satisfies

$$\left[\frac{d^2}{dz^2} + k_0^2\right] G_0(z|z_s;\omega) = \delta(z - z_s), \tag{4}$$

and integrating:

$$\psi(z|z_s;k_0) = G_0(z|z_s;k_0) + \int_{-\infty}^{\infty} G_0(z|z';k_0)V(k_0, z')\psi(z'|z_s;k_0)dz'. \tag{5}$$

Finally, equation (5), which is the 1D version of the Lippmann-Schwinger equation, is expanded to produce the Born series:

$$\psi(z|z_s;k_0) = \psi_0(z|z_s;k_0) + \psi_1(z|z_s;k_0) + \psi_2(z|z_s;k_0) + \psi_3(z|z_s;k_0) + ..., \tag{6}$$

where

$$\begin{aligned}
\psi_0(z|z_s;k_0) &= G_0(z|z_s;k_0), \\
\psi_1(z|z_s;k_0) &= \int_{-\infty}^{\infty} G_0(z|z';k_0)V(k_0,z')G_0(z'|z_s;k_0)dz', \\
\psi_2(z|z_s;k_0) &= \int_{-\infty}^{\infty} G_0(z|z';k_0)V(k_0,z')\int_{-\infty}^{\infty} G_0(z'|z'';k_0)V(k_0,z'')G_0(z''|z_s;k_0)dz''dz',
\end{aligned} \tag{7}$$

etc. Clearly, ψ_1 is first order in V, whereas ψ_2 is second order in V, and so forth. Because the Green's function $G_0(z|z';k_0)$ describes propagation in the reference medium from point z' to point z, the term ψ_N may be interpreted as a wave field which has propagated in the reference medium $N+1$ times, and has N times interacted with the perturbation $\alpha(z)$ via the scattering potential. Since the reference medium is characterized by constant wavespeed c_0, the Green's function is (e.g. DeSanto, 1993):

$$G_0(z|z_s;k_0) = \frac{e^{ik_0|z-z_s|}}{2ik_0}. \tag{8}$$

Therefore, having defined $V(k_0,z)$ via some desired Earth model, and knowing G_0, one may compute as many terms as desired in equation (6) to approximate the solution.

One may define a wide variety of scattering potentials, differing in what the "true" medium properties are with respect to the reference medium. Here we consider two variants on the acoustic case, each utilizing wavenumbers which permit attenuation to be modeled in addition to acoustic behaviour. This requires moving away from the acoustic $k_0 = \omega/c_0$, and adopting for the true medium:

$$k(z) = \frac{\omega}{c(z)}\left[1+\beta(\omega,z)\right], \tag{9}$$

where $\beta(\omega,z)$, a complex number, is the spatial distribution of an attenuation parameter which instills absorption and dispersion character into the wave field. From equation (9), and guided by equation (3), two related scattering potentials are defined. The first corresponds to media in which both wavespeed contrasts and attenuation contrasts are permitted:

$$\alpha_{cq}(z) = 1 - \frac{k^2(z)}{k_0^2} = 1 - \frac{c_0^2}{c^2(z)}\left[1+2\beta(\omega,z)\right], \tag{10}$$

neglecting terms quadratic in β. The second corresponds to a medium in which the wavespeed is constant throughout, and contrasts are only permitted in β. Since in such a case $c(z) = c_0$, from equation (10) the form is

$$\alpha_q(z) = 2\beta(\omega,z). \tag{11}$$

The remarkable simplicity of this perturbation and its association with the actual value of β arises partly because there is no attenuation in the reference medium, and also because of the attenuating wavenumber of equation (9) already resembles a perturbation away from the acoustic case. In this paper we consider only cases in which the reference medium is acoustic, and non-attenuative.

If we choose the constant Q model of Kjartansson (1979), for instance, we have

$$\beta(\omega, z) = \frac{i}{2Q(z)} - \frac{1}{\pi Q(z)} \ln\left(\frac{\omega}{\omega_r}\right), \tag{12}$$

where ω_r is a chosen reference frequency. This arises due to the form of a 1D constant Q wavenumber:

$$k_1 = \frac{\omega}{c_0}\left[1 + \frac{i}{2Q} - \frac{1}{\pi Q}\ln\left(\frac{\omega}{\omega_r}\right)\right]. \tag{13}$$

2 The Terms and Diagrams of the Born Series

The Born series, equation (6), is a decomposition of the full wave field into those components which have interacted with the non-reference portion of the medium a certain number of times; each interaction is separated by a propagation in the reference medium. Scattering diagrams, which, as used here, are the wave-theoretic analogues of the Feynman diagrams of quantum field theory (e.g. Weglein et al., 2002), arise from further decomposition of the terms in this series. In this section, the scattering diagrams associated with a simple 1D transmission case are developed from the form of the Born series integrals, and linked to the explicitly computed terms in the series.

The integrals which give rise to the terms ψ_n in equation (6) subdivide because of the geometric constraints on the propagation imposed by the Green's functions (equation 8). To see this, consider equations (7) with the condition that the source z_s is less than all z for which the perturbation is non-zero. Then:

$$\psi_0(z|z_s; k_0) = \frac{e^{ik_0(z-z_s)}}{2ik_0}. \tag{14}$$

Also,

$$\begin{aligned}\psi_1(z|z_s;k_0) &= \int_{-\infty}^{\infty} \frac{e^{ik_0|z-z'|}}{2ik_0} k_0^2 \alpha(z') \frac{e^{ik_0(z'-z_s)}}{2ik_0} dz' \\ &= \int_{-\infty}^{z} \frac{e^{ik_0(z-z')}}{2ik_0} k_0^2 \alpha(z') \frac{e^{ik_0(z'-z_s)}}{2ik_0} dz' + \int_{z}^{\infty} \frac{e^{ik_0(z'-z)}}{2ik_0} k_0^2 \alpha(z') \frac{e^{ik_0(z'-z_s)}}{2ik_0} dz' \\ &= -\frac{1}{4} e^{ik_0(z-z_s)} \int_{-\infty}^{z} \alpha(z') dz' - \frac{1}{4} e^{-ik_0(z+z_s)} \int_{z}^{\infty} e^{i2k_0 z'} \alpha(z') dz' \\ &= \psi_{11} + \psi_{12}.\end{aligned} \tag{15}$$

The integrals are likewise broken up in the computation of ψ_2:

$$\begin{aligned}\psi_2(z|z_s;k_0) &= \int_{-\infty}^{\infty} \frac{e^{ik_0|z-z'|}}{2ik_0} k_0^2 \alpha(z') \int_{-\infty}^{\infty} \frac{e^{ik_0|z'-z''|}}{2ik_0} k_0^2 \alpha(z'') \frac{e^{ik_0(z''-z_s)}}{2ik_0} dz'' dz' \\ &= \psi_{21} + \psi_{22} + \psi_{23} + \psi_{24},\end{aligned} \tag{16}$$

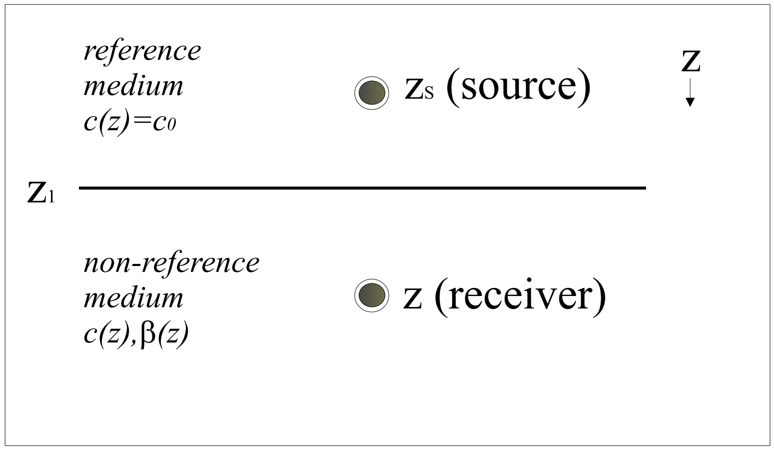
where

$$\begin{aligned}\psi_{21}(z|z_s;k_0) &= \frac{1}{8} ik_0 e^{ik_0(z-z_s)} \int_{-\infty}^{z} \alpha(z') \int_{-\infty}^{z'} \alpha(z'') dz'' dz', \\ \psi_{22}(z|z_s;k_0) &= \frac{1}{8} ik_0 e^{ik_0(z-z_s)} \int_{-\infty}^{z} e^{-i2k_0 z'} \alpha(z') \int_{z'}^{\infty} e^{i2k_0 z''} \alpha(z'') dz'' dz', \\ \psi_{23}(z|z_s;k_0) &= \frac{1}{8} ik_0 e^{-ik_0(z+z_s)} \int_{z}^{\infty} e^{i2k_0 z'} \alpha(z') \int_{-\infty}^{z'} \alpha(z'') dz'' dz', \\ \psi_{24}(z|z_s;k_0) &= \frac{1}{8} ik_0 e^{-ik_0(z+z_s)} \int_{z}^{\infty} \alpha(z') \int_{z'}^{\infty} e^{i2k_0 z''} \alpha(z'') dz'' dz'.\end{aligned} \tag{17}$$

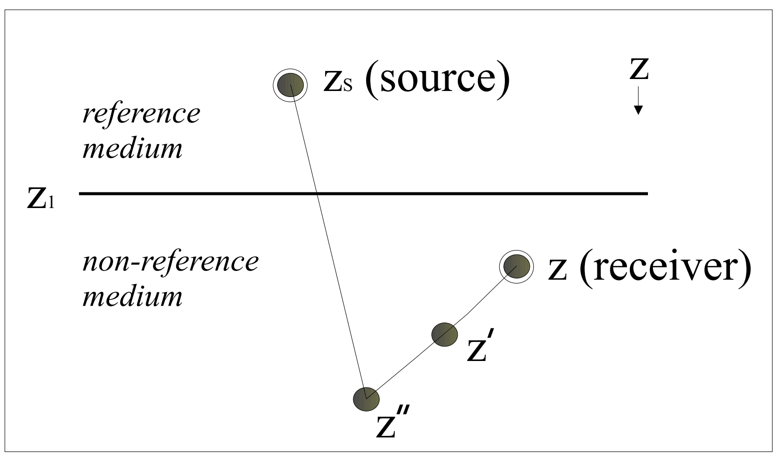
In the "new" series,

$$\psi(z|z_s;k_0) = \psi_0 + \psi_{11} + \psi_{12} + \psi_{21} + \psi_{22} + \psi_{23} + \psi_{24} + ..., \tag{18}$$

the terms have been divided into, first, the number of interactions, and second, the relative location of the interactions. For instance, the term ψ_{23} represents the totality of second order interactions in which $z' > z$ and $z' > z''$, whence arise the scattering diagrams, which symbolize this particular scattering geometry. Since the addition of a further order of interaction, e.g. going from second to third order, involves the inclusion of one further Green's function, which must be subdivided into two cases, in general the n'th order term produces 2^n sub-terms. In this way, the eight terms of ψ_3, followed by the 16 terms of ψ_4, and so forth may be produced. Scattering diagrams may now be drawn based on the terms in equation (18). This is preceded by a description of the chosen 1D model.



(a)



(b)

Figure 1: 1D transmission model and framework for scattering diagrams: (a) a homogeneous acoustic wholespace is chosen as the reference medium, in which the source is located; a homogeneous (visco-)acoustic half-space is chosen as the non-reference medium, in which the receiver is located; the step-like interface is located at z_1; (b) an example (ψ_{23}) of the form and construction of the scattering diagrams is superimposed on the chosen transmission model; the arrows and labels included in this example are assumed but omitted in subsequent diagrams.

The model is a 1D homogeneous acoustic whole-space, characterized by constant density and the wavespeed c_0. Overlaying this reference whole-space is the perturbation, a homogeneous half-space in which the medium parameters, i.e. $c(z)$ and/or $\beta(z)$, are constant and may or may not differ from that of the reference medium. A source is located at $z_s = 0$, in the reference medium, and a receiver is located at z, in the non-reference medium, thus mimicking a transmission experiment; the interface between the reference medium and the non-reference medium is at z_1. The model is illustrated in Figure 1a. This configuration is geometrically identical to one used by Matson (1996), such that the mapping developed therein, from Born series to closed-form, may be utilized. Figure 1b illustrates, as an example, the scattering diagram associated with ψ_{23}, within the context of the chosen model. In later diagrams the arrows are omitted, nevertheless, all propagations go from z_s to z. Further, it is worth mentioning that in this 1D model the lateral separation of scattering points has no meaning other than as an aid to visualization.

Figure 2 contains the scattering diagrams associated with ψ_0 (one diagram), ψ_1 (two diagrams), ψ_2 (four diagrams), and ψ_3 (eight diagrams), pictured without the context of the 1D model discussed above. As in Figure 1b, the the "top-left" endpoint is z_s, and the "bottom-right" is z.

The next step is to evaluate these integrals, given the chosen transmission model of Figure 1, and to use the Matson approach to produce the closed form expression for the transmitted wave field, all the while tracking the diagrams through the organization and collapsing of series. To remain general in the sense of acoustic/viscoacoustic perturbation, the perturbation $\alpha(z)$ is assigned the spatial dependence

$$\alpha(z) = \alpha_1 H(z - z_1), \tag{19}$$

such that the details of the contrast (i.e. acoustic or viscoacoustic) are hidden in the amplitude α_1, which could have a form like either of equations (10) or (11). The step-like behaviour is explicitly present in the Heaviside function $H(z - z_1)$. Substituting equation

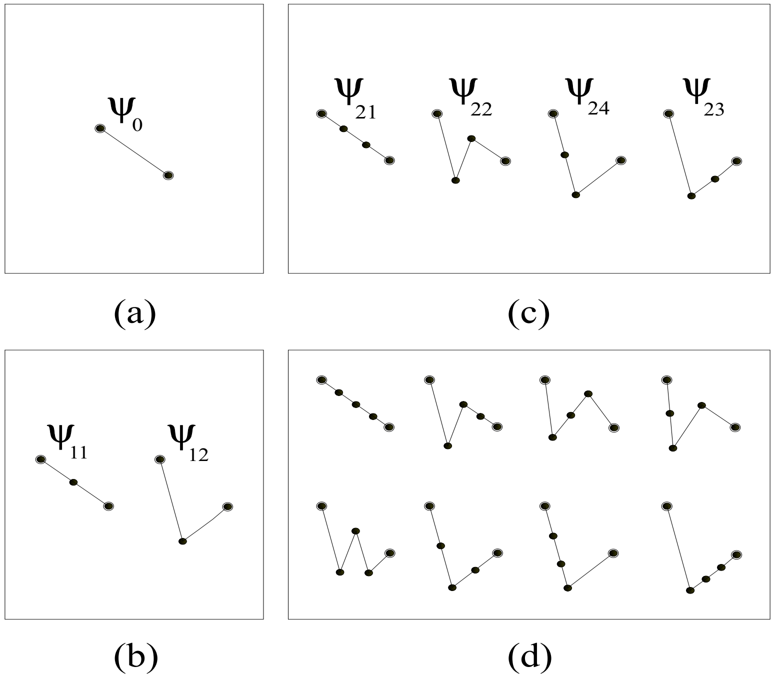


Figure 2: Scattering diagrams, for the 1D transmission example of Figure 1a, are illustrated up to third order in the perturbation α. (a) 0'th order, ψ_0; (b) 1st order, $\psi_{11} - \psi_{12}$; (c) 2nd order, $\psi_{21} - \psi_{24}$; (d) the eight diagrams of the 3rd order. Those diagrams associated with terms that have been explicitly computed in this paper are labelled as in the calculations.

(19) and $z_s = 0$ into the terms of equation (18) results in:

$$
\begin{aligned}
\psi_0(z|0;k_0) &= \frac{e^{ik_0 z}}{2ik_0}, \\
\psi_{11}(z|0;k_0) &= -\frac{\alpha_1}{4}\frac{e^{ik_0 z}}{2ik_0}2ik_0(z-z_1), \\
\psi_{12}(z|0;k_0) &= \frac{\alpha_1}{4}\frac{e^{ik_0 z}}{2ik_0}, \\
\psi_{21}(z|0;k_0) &= -\frac{\alpha_1^2}{8}\frac{e^{ik_0 z}}{2ik_0}[ik_0(z-z_1)]^2, \\
\psi_{22}(z|0;k_0) &= -\frac{\alpha_1^2}{16}\frac{e^{ik_0 z}}{2ik_0}2ik_0(z-z_1), \\
\psi_{23}(z|0;k_0) &= -\frac{\alpha_1^2}{16}\frac{e^{ik_0 z}}{2ik_0}2ik_0(z-z_1), \\
\psi_{24}(z|0;k_0) &= \frac{\alpha_1^2}{16}\frac{e^{ik_0 z}}{2ik_0}.
\end{aligned}
\tag{20}
$$

The straightforward computation of the terms in equation (20) raises an issue that is of at least conceptual importance. The wave field associated with a single interface, whether reflected or transmitted, must ultimately consist of a local event due to that interface. Scattering series terms involve nested integrals over all (or much of) space. One might well ask why integrals over all space are necessary for the determination of the character of a local event. What, for instance, has the Earth at $100km$ depth got to say about a reflection coefficient at $50m$ depth? The answer is, not much, of course. The computation of equation (20) makes clear a more appropriate way of interpreting the "job" of these integrals which range over all space. The form of the perturbation places the interface at z_1 with a Heaviside function, which alters the integration limits of the integral to include z_1. In such a (definite) integral, then, the antiderivative is "picked out" at the limit z_1. This would happen anywhere an interface was placed in the form of $\alpha(z)$. As such the results are local terms, which depend on measurement location z and interface location z_1 only. So these integrals should be thought of as the process of *scanning* the model for discontinuities which give rise to local events in the series terms. This is the conceptual link – albeit simplified to include only stepwise constant media – between the integrals of the scattering series and the local nature of reflected and transmitted events.

Figure 3 contains an organization of the terms comprising the Born series representation of the wave field $\psi(z|0;k_0)$, including those of equations (20); above the low order terms, the associated scattering diagram is included. These terms are, together,

$$\begin{aligned}\psi(z|0;k_0) = \frac{e^{ik_0z}}{2ik_0} + \frac{e^{ik_0z}}{ik_0}&\left\{\frac{1}{8}\alpha_1 + \frac{1}{16}\alpha_1^2 + \frac{5}{128}\alpha_1^3 + ...\right.\\ &- 2ik_0(z-z_1)\left[\frac{1}{8}\alpha_1 + \frac{1}{16}\alpha_1^2 + \frac{5}{128}\alpha_1^3 + ...\right]\\ &\left.-k_0^2(z-z_1)^2\left[\frac{1}{16}\alpha_1^2 + \frac{3}{64}\alpha_1^3 + \frac{9}{256}\alpha_1^4 + ...\right] + ...\right\}.\end{aligned} \tag{21}$$

Equation (21) makes use of two definitions:

$$R = 2\left\{\frac{1-\alpha_1/2-(1-\alpha_1)^{1/2}}{\alpha_1}\right\}, \tag{22}$$

and

$$\gamma = (1-\alpha_1)^{1/2} = \frac{k_1}{k_0}, \tag{23}$$

where k_1 is the wavenumber in the non-reference medium (which may describe acoustic or viscoacoustic propagation). Noting that γ may be expanded in Taylor series:

$$\gamma = (1-\alpha_1)^{1/2} = 1 - \frac{1}{2}\alpha_1 - \frac{1}{8}\alpha_1^2 - \frac{1}{16}\alpha_1^3 - \frac{5}{128}\alpha_1^4 - ..., \tag{24}$$

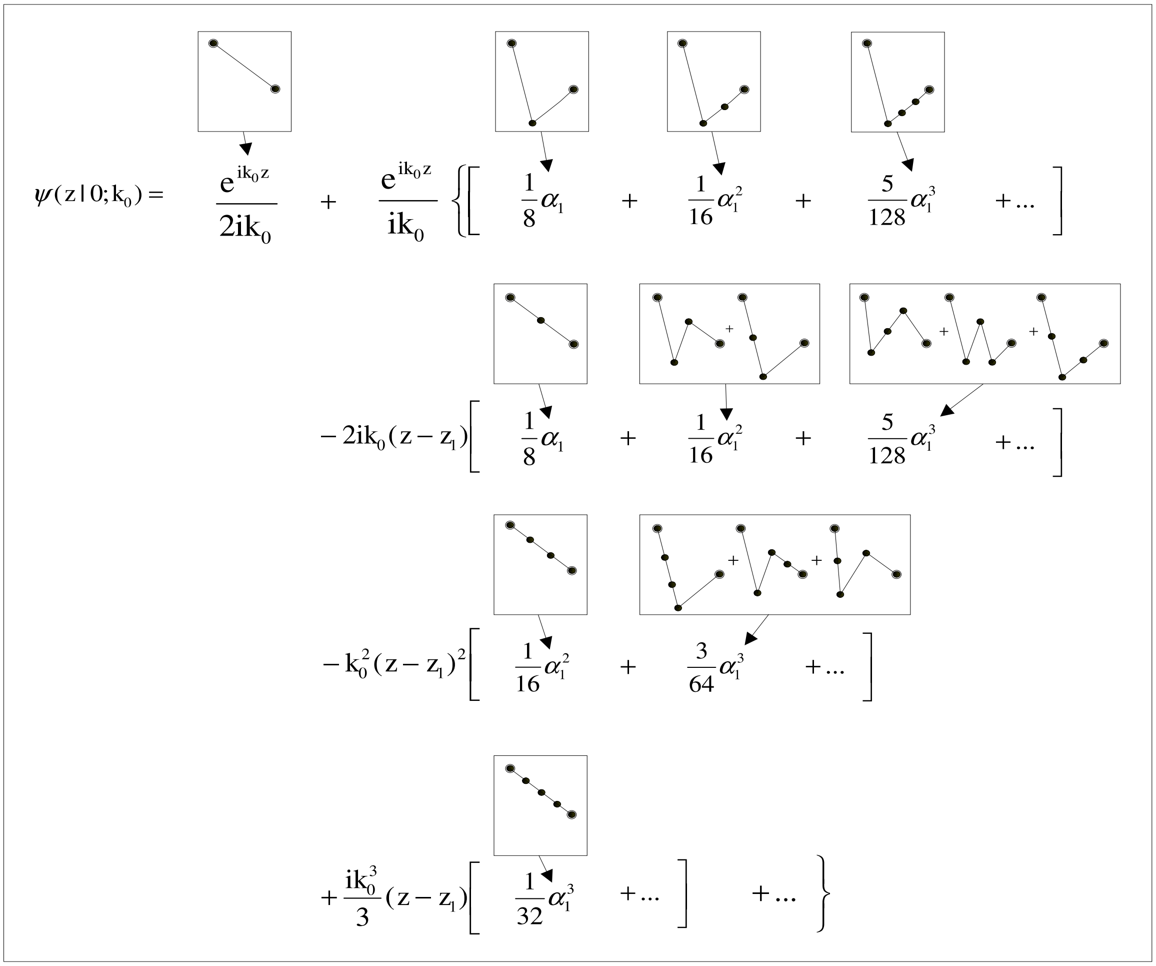


Figure 3: The Born series terms for the 1D transmission case of equation (20) are illustrated with their associated scattering diagrams.

and keeping in mind the terms found in Figure 3, one may write

$$\frac{1}{8}\alpha_1 + \frac{1}{16}\alpha_1^2 + \frac{5}{128}\alpha_1^3 + ... = \frac{R}{2}, \tag{25}$$

$$\frac{1}{16}\alpha_1^2 + \frac{3}{64}\alpha_1^3 + \frac{9}{256}\alpha_1^4 + ... = (1-\gamma)\frac{R}{2}, \tag{26}$$

and so on. The series' in powers of α_1, i.e. the rows of Figure 3, therefore collapse into single terms in R and $1-\gamma$. In fact:

$$\psi(z|0;k_0) = \frac{e^{ik_0z}}{2ik_0} + \frac{e^{ik_0z}}{ik_0}\frac{R}{1-\gamma}\left\{\frac{(1-\gamma)}{2} - ik_0(z-z_1)(1-\gamma) - \frac{k_0^2(z-z_1)^2}{2}(1-\gamma)^2 + ...\right\}. \tag{27}$$

These collapsed series' in α_1 are therefore coefficients of a further series in orders of $-ik_0(z-z_1)$. Notice that one may expand as a Taylor's series:

$$\begin{aligned} & e^{-ik_0(z-z_1)(1-\gamma)} - \frac{(1+\gamma)}{2} \\ &= \left[1 - \frac{(1+\gamma)}{2}\right] - ik_0(z-z_1)(1-\gamma) - \frac{k_0^2}{2}(z-z_1)^2(1-\gamma)^2 + ... \\ &= \frac{(1-\gamma)}{2} - ik_0(z-z_1)(1-\gamma) - \frac{k_0^2}{2}(z-z_1)^2(1-\gamma)^2 + ... \end{aligned} \tag{28}$$

This is the form of the series in equation (27). Substituting equation (28) into equation (27) produces the closed-form expression:

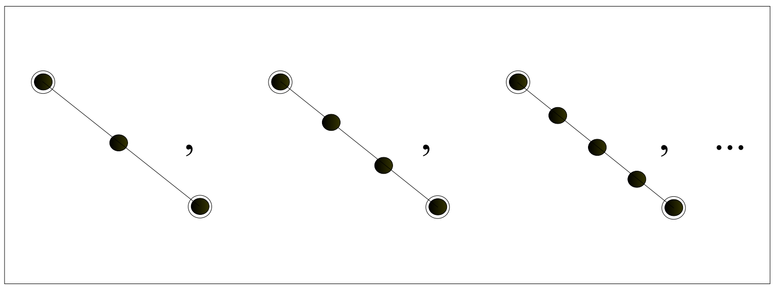
$$\begin{aligned} \psi(z|0;k_0) &= \frac{e^{ik_0 z}}{2ik_0} + \frac{e^{ik_0 z}}{ik_0}\frac{R}{1-\gamma}\left\{e^{-ik_0(z-z_1)(1-\gamma)} - \frac{(1+\gamma)}{2}\right\} \\ &= \frac{e^{ik_0 z}}{2ik_0} + \frac{k_0}{k_0+k_1}\frac{e^{ik_0 z_1}}{ik_0}e^{ik_1(z-z_1)} - \frac{e^{ik_0 z}}{2ik_0} \\ &= \frac{k_0}{k_0+k_1}\frac{e^{ik_0 z_1}}{ik_0}e^{ik_1(z-z_1)}, \end{aligned} \tag{29}$$

using the definitions of equations (22) and (23). This expression for the transmitted wave field, generalized to accomodate viscoacoustic or acoustic propagation, propagates according to k_1, which is the non-reference wavenumber.

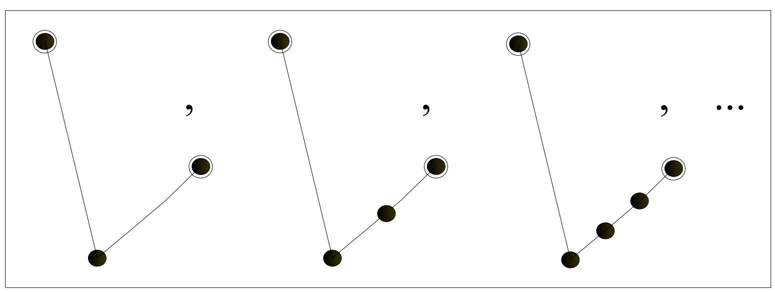
Observing the process of collapsing the series, from the form of Figure 3 to that of equation (27), the roles of the types of scattering interaction in the construction of the eventual wave field become clear.

The wave field, propagating correctly in the non-reference medium via the term $e^{ik_0 z}e^{-ik_0(z-z_1)(1-\gamma)} = e^{ik_0 z}e^{ik_0(k_1/k_0)(z-z_1)} = e^{ik_1(z-z_1)}$, is generated in the Born series by a weighted series in orders of $-ik_0(z-z_1)$, i.e. each row of equation (21) and Figure 3. Notice from Figure 3 that the leading terms in each order $n > 0$ of $[-ik_0(z-z_1)]^n$, arise due to the same scattering interaction-type, that is, those with no direction-change from source to receiver (see Figure 4a). It is therefore justifiable to attribute much of the burden of alteration of propagation (wavespeed and/or attenuation) to this type of scattering interaction in the Born series. Of course, to correctly alter k_0 to k_1 requires $\gamma = k_1/k_0$, which in turn requires terms at all orders of α_1; nevertheless, the leading terms are the most signficant, especially for small α_1.

Next consider the amplitude of the transmitted wave field, $k_0/(k_0+k_1) = R/(1-\gamma)$, which is correctly produced by the series of 0'th order in $-ik_0(z-z_1)$, i.e. the first "row" of equation (21) and Figure 3. While it is true that every order of $-ik_0(z-z_1)$ has such a series embedded



(a)



(b)

Figure 4: Certain scattering-types are seen to be associated with the generation of macroscopic properties of the transmitted wave field. (a) These interaction types contain all information necessary to properly alter the amplitude of the transmitted wave field; also, these interaction-types are solely responsible for producing the negative of the direct wave; (b) These interaction types are responsible for the leading order terms in the alteration of the propagation wavenumber, from k_0 (reference medium) to k_1 (true medium).

in it, so in truth the whole series produces the amplitude coefficient, the information required for its correct computation is laid down by this first row of equation (21). Inspection of Figure 3 reveals that this series is also characterized by scattering interactions of common type (see Figure 4b). This amplitude produces the expected transmission coefficient, as was noted in Matson (1996) for the acoustic case:

$$T = \frac{|\psi(z|0;k_0)|}{|G_0(z|0;k_0)|} = \frac{2k_0}{k_0+k_1}; \tag{30}$$

and so one is justified in looking to these types of interaction as being central to amplitude adjustment.

In "mixed" terms, of order higher than $n = 1$ in $[\alpha_1]^n$ and $n = 0$ in $[-ik_0(z - z_1)]^n$, more than one type of scattering diagram is associated with each term. In other words, certain scattering interaction-types are not distinct in the solution. Inspection of Figure 3 suggests that these indistinct components of the solution are related in that they share the same number of "up-" and "down-"scattering directions. For instance, the term $-k_0^2(z - z_1)^2(3/64)\alpha_1^3$ has three contributing diagrams: from left to right, "down-down-down-up", "down-up-down-down", and "down-down-up-down". Since the source is fixed to be above all interactions, the first direction must be "down"; therefore, these three diagrams represent all permutations of "two downs + one up-" interaction type.

Finally, consider the third key task of the Born series: the elimination of the direct wave $\psi_0 = e^{ik_0z}/2ik_0$. The series accomplishes this, in equation (28), concurrently to the wave field construction, by creating the negative of the direct wave such that they destructively interfere. This is also a conclusion of Matson (1996). This "task" is accomplished by the terms which are 0'th order in $-ik_0(z-z_1)$; the unit first term is split into two parts, $(1+\gamma)/2$ and $1 - (1 + \gamma)/2$, the former of which ultimately becomes the negative of the direct wave (equation 28). This direct wave eliminator, unlike the amplitude term, owes its existence *solely* to scattering interactions of the type seen in Figure 4b.

Lastly, we might underscore a remarkable aspect of the propagations and interactions which constitute the terms in the Born series. The reference Green's function propagates from interaction point to interaction point in every term – no other type of propagation ever occurs in this formalism. The reference Green's function is, here, the solution to the acoustic wave equation, yet the final wave field is viscoacoustic: this means that an attenuated and dispersed wave field is being correctly generated by a sophisticated interplay of non-attenuating wave propagations.

In this section the scattering potentials, generalized to permit viscoacoustic wave propagation, have been confirmed as producing the expected wave field expression for a simple 1D transmission case. Concurrently, scattering diagrams, which are a byproduct of the subdivision of the Born series terms into computable units, are carried through the calculations. Thus, when the mapping of Matson (1996) is used to produce the closed form expression for the wave field, the "scattering interaction-types" that produced each term may be categorized as to their contribution to overall wave field properties, such as amplitude, phase (and attenuation), and the destruction of unwanted wave field components.

3 Toward Scattering-based Q Compensation/Estimation

In this section we utilize a different categorization of scattering diagram, that of "separated" vs. "self-interaction" (Weglein et al., 2002). The insight gained from such a categorization directly guides the search in the inverse series for task-oriented subseries.

The promise of the inverse scattering series, convergence issues notwithstanding, is to compute the model $\alpha(z)$ via certain nonlinear operations carried out upon the data. If the measured wave field is distorted and smoothed because it propagated through an attenuating medium that has sharp transitions, then the reconstruction of these sharp transitions *must* include some process of Q-compensation. Furthermore, since Q can be cast as part of the perturbation, the computation of the model *must* include Q-estimation. To determine how and where such processes take place in the inversion, it is, as ever, useful to turn to the forward scattering series.

It is particularly compelling to consider a 1D case in which only Q contrasts, and no wavespeed contrasts, exist. These contrasts produce reflections, but the wave never travels at a speed different from that of the reference medium. So, in the inverse, no beyond-Born imaging *per se* will be required. All events will be correctly located by imaging with the reference wavespeed. The only processing step necessary will be to remove the smoothing and distortion effects. Weglein et al. (2002) have identified a separable subseries responsible for imaging in the presence of wavespeed contrasts – will this subseries shut down in a Q-only case? How then will Q compensation occur? Let us next address these questions by looking at the forward analogue of the imaging subseries.

Consider the two models shown in Figures 5a and b, both of which represent 1D media. Both are geometrically identical, but the first (5a) represents a purely acoustic variation: the wave field propagates at wavespeed c_0 for $z < z_1$ and $z > z_2$, but at c_1 in the layer between (z_1, z_2). Meanwhile, the second corresponds to a situation akin to that described above: the wavespeed never changes, $c(z) = c_0$ everywhere. The absorption parameter Q, which we assume obeys the dispersion relation associated with Kjartansson's constant Q model (1979), undergoes a contrast in this case (Figure 5b), from ∞ outside (z_1, z_2) to Q_1 within.

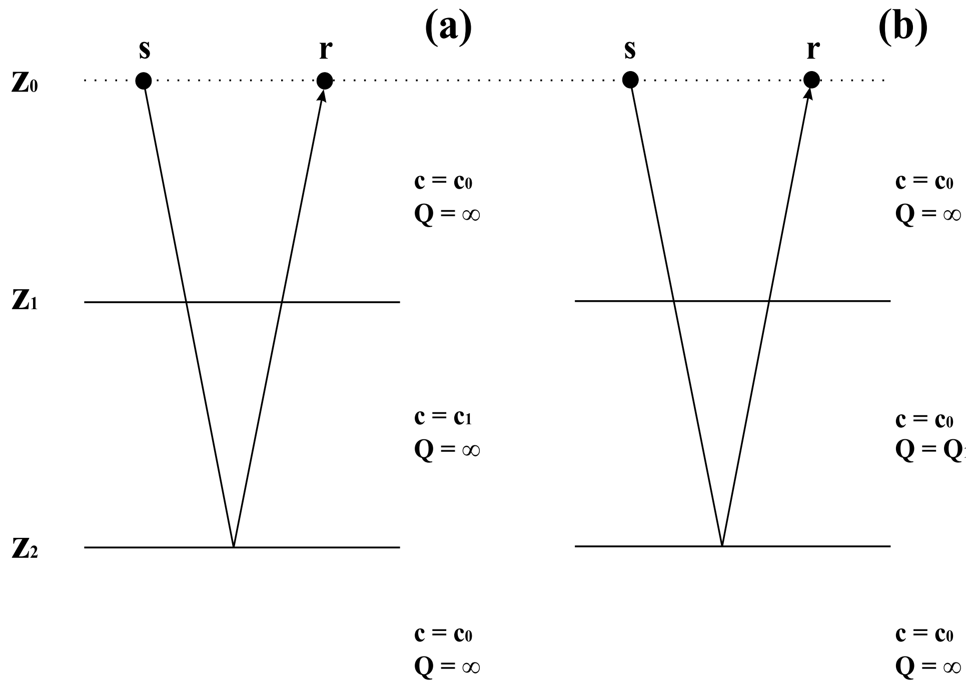


Figure 5: Schematic of two models for a 1D reflected wavefield. (a) The acoustic case: we consider the "second primary", i.e. that which has reflected at the z_2 interface, which in this case corresponds to a wavespeed contrast. (b) The viscoacoustic case: an identical geometry as that of (a), here we consider a contrast in Q only. This latter wave field travels everywhere with the wavespeed c_0, and hence no alteration of the Born approximate arrival time is required; we wish to ascertain what if any effect the "mover", or timing-related terms in the series have on the construction of this field.

Matson (1996) finds an expression for the primary reflection associated with the lower interface (z_2) in a model of this sort (whose ray-path is illustrated in Figures 5a and b). Previous work of Innanen (M-OSRP Report, 2001) has shown that the general form of Matson's expressions are the same for both the acoustic and viscoacoustic examples. After the amplitude series' have been collapsed, we have

$$\psi_{pr2}(z < z_1|0;k_0) = \frac{e^{-ik_0 z}e^{i2k_0 z_2}}{2ik_0}R_2T_{10}T_{01}\left[1 + ik_0 2(z_2 - z_1)(\gamma - 1) - \frac{k_0^2}{2!}4(z_2 - z_1)^2(\gamma - 1)^2 - \frac{ik_0^3}{3!}8(z_2 - z_1)^3(\gamma - 1)^3 + ...\right], \tag{31}$$

which further collapses to

$$\psi_{pr2}(z < z_1|0;k_0) = \frac{e^{-ik_0 z}e^{i2k_0 z_2}}{2ik_0}R_2T_{10}T_{01}e^{i2k_0(z_2-z_1)(\gamma-1)}, \tag{32}$$

recalling that $\gamma = k_1/k_0$, i.e. the ratio of the reference and non-reference wavenumbers. The reflection and transmission coefficients (R_2, and T_{10} and T_{01}, for the reflection, transmission from medium 1 to 0 and transmission from medium 0 to 1 respectively) have been produced similarly to the amplitude of the transmitted wave field in the previous section.

The terms in the Born series which have conspired to produce the bulk of the $e^{i2k_0(z_2-z_1)(\gamma-1)}$ component in equation (32) correspond to "separated" diagrams, by virtue of the presence of powers of $(z_2 - z_1)$; hence this component is due to the forward analogue of the *imaging* subseries. We expect, therefore, that in the acoustic case, this term will do much of the work required to take the incorrectly-timed arrival of the Born approximation, and alter it such that it arrives having travelled everywhere at the correct wavespeed. Let us first see that this is the case.

In the acoustic case

$$\gamma = \frac{k_1}{k_0} = \frac{c_0}{c_1}, \tag{33}$$

so, expanding equation (32),

$$\psi_{pr2}(z < z_1|0;k_0) = \frac{e^{-ik_0 z}}{2ik_0}R_2T_{10}T_{01}\left[e^{i2k_0 z_2}e^{-i2k_0 z_2}\right]e^{i2k_0 z_1}e^{i2k_0(z_2-z_1)\frac{c_0}{c_1}}. \tag{34}$$

Consider first equation (32). The incorrect arrival time of the Born approximation appears in the uncorrected leftmost term $\frac{e^{-ik_0 z}e^{i2k_0 z_2}}{2ik_0}$, specifically in the part which produces a phase delay over the distance z_2 with wavenumber k_0: $e^{i2k_0 z_2}$. Next consider equation (34). Notice that the -1 portion of $(\gamma - 1)$ in the correction produces a term opposing this arrival (both are in square brackets $[\cdot]$) when the expression is expanded. So in the acoustic case, the first task of the series' "corrector" term, $e^{i2k_0(z_2-z_1)(\gamma-1)}$, is to delete the incorrectly timed event. Next,

$$\begin{aligned}\psi_{pr2}(z < z_1|0;k_0) &= \frac{e^{-ik_0 z}}{2ik_0}R_2T_{10}T_{01}e^{i2k_0 z_1}e^{i2[\frac{\omega}{c_0}\frac{c_0}{c_1}](z_2-z_1)} \\ &= \frac{e^{-ik_0 z}}{2ik_0}R_2T_{10}T_{01}e^{i2k_0 z_1}e^{i2k_1(z_2-z_1)}.\end{aligned} \tag{35}$$

Here, the second task of the series corrector is engaged: the remainder of $(\gamma - 1)$, namely c_0/c_1, multiplies the reference wavenumber k_0 in the square brackets, deleting the incorrect wavespeed and replacing it with the correct wavespeed and thus the correct wavenumber k_1.

In their totality, the "separated diagram" type terms therefore have the effect of (1) deleting the Born approximate arrival, and (2) replacing it with the correctly-timed true arrival. Thus arises the interpretation, in the inverse analogue, of these separated diagram terms as being "movers". One might indeed expect that, in the event of a true medium with no wavespeed variation, these terms would shut down. There are a number of problems with this expectation, however, not least of which is: they don't.

Let us proceed by examining the purely-Q contrast case. We have

$$\gamma = \frac{k_1}{k_0} = \frac{\frac{\omega}{c_0}\left[1 + \frac{i}{2Q_1} - \frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}{\frac{\omega}{c_0}} = 1 + \frac{i}{2Q_1} - \frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right), \tag{36}$$

following the constant Q law of Kjartansson (1979). Once again expanding equation (32), but this time with the viscoacoustic γ, we have

$$\begin{aligned}\psi_{pr2}(z < z_1|0;k_0) &= \frac{e^{-ik_0z}e^{i2k_0z_2}}{2ik_0}R_2T_{10}T_{01}e^{i2k_0(z_2-z_1)\left[1+\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)-1\right]}\\ &= \frac{e^{-ik_0z}e^{i2k_0z_2}}{2ik_0}R_2T_{10}T_{01}e^{i2k_0(z_2-z_1)\left[\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}.\end{aligned} \tag{37}$$

So the correcting term doesn't become unity, or by any means vanish, even though there are no timing changes to be made – the imaging subseries analogue stays alive. What happens instead is that the form of the viscoacoustic γ extinguishes the -1 from $(\gamma - 1)$ – see equation (37). This was the mechanism in the acoustic case that deleted the incorrect arrival. The Born arrival time of this primary is kept! What is left in place of a "mover" term, that deletes and replaces primaries, is an operator, $e^{i2k_0(z_2-z_1)\left[\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}$, that distorts the amplitude and phase of the Born primary according to the Q model. We finally arrive at

$$\psi_{pr2}(z < z_1|0;k_0) = \frac{e^{-ik_0z}}{2ik_0}R_2T_{10}T_{01}e^{i2k_0z_2\left[1+\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}e^{-i2k_0z_1\left[\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}, \tag{38}$$

Here, via the term $e^{i2k_0z_2\left[1+\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}$, the wave field propagates the entire distance $2z_2$, attenuating with Q_1. This is the correct arrival time, but the incorrect amount of attenuation – too much. The rightmost term, $e^{-i2k_0z_1\left[\frac{i}{2Q_1}-\frac{1}{\pi Q_1}\ln\left(\frac{\omega}{\omega_r}\right)\right]}$, corrects this by deconvolving the attenuation effects (not the propagation effects) associated with the distance $2z_1$. The final result is that the second primary has experienced the expected amount of attenuation, that is, through the distance $2(z_2 - z_1)$.

The important point here is that the "separated diagram"-type terms have played an enormously important role in the Q contrast only case, in spite of the fact that no "moving" was required. It seems clear that a redefinition of the forward analogue of the imaging subseries is in order. We surmise that these terms are responsible for generating *propagation effects* rather than simply *timing changes* – with the former reducing to the latter in the acoustic case. The "movers", which correspond to the imaging subseries in the inverse case, must be generalized to "depropagators".

4 Conclusions

The work described in this paper reflects the course of the initial investigation into the use of scattering theory as a means to process data with non-negligible attenuation effects. It contains, first, the results of computing terms in the forward (or Born) series, and following the scattering diagram associated with each of these terms. Secondly, it concerns itself with the effect on a reflected primary of terms that involve "separated" interactions. These are the forward series analogues of the imaging terms of the inverse scattering series. We reach the conclusion that the meaning of the "moving" terms, those whose diagrammatical representation involves separated scattering interactions, must be generalized to accomodate the inclusion of all propagation effects into the true wave field.

The consequences with regards to how the inverse scattering series must treat attenuation are clear – we must look to the imaging subseries to remove the effects of Q. In an example pathologically created with no wavespeed changes, we may indeed find that the imaging series does only this. In such an example the inversion subseries (Zhang's work in this report) must therefore be involved with Q-estimation only.

Beyond ascertaining that this is the case, the next challenge is to cast the inverse scattering series problem with at least two parameters (perhaps c and Q), and use a pre-stack experimental milieu to investigate both the imaging (depropagation) and inversion (c and Q estimation) subseries.

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