# Inverse scattering series for laterally-varying media

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**Abstract**

We consider the extension of the current casting of the inverse scattering series for imaging and inverting primary seismic energy (Weglein et al., 2002) to cases involving lateral variation in the medium parameters. The extension is developed theoretically to the second term in the series, and a form is presented which is interpretable in the same way as previously-derived terms in the absence of lateral variation. We present a preliminary numerical result that shows lateral and vertical correction of location and amplitude to occur, in accordance with current understanding of second order imaging and inversion terms/numerics. These results are very encouraging, and represent the first concrete numerical indication of multidimensional imaging in the absence of a correct velocity model.

## 1 Introduction

The inverse scattering series is the only known theory permitting the multidimensional direct inversion of seismic wave field measurements. To date (Weglein et al., 2003; Shaw et al., 2003; Zhang and Weglein, 2003) focus has been on the detailed understanding of the operations associated with subseries' for imaging (reflector location) and inversion (target identification) in 1D, for both normal incidence and 1D-with-offset cases (and various acoustic and elastic models). We have also considered the analogous development of terms given a variable reference medium (Liu and Weglein, 2002). Encouraging numerical results in the case of data with missing zero-frequency in 1D (Shaw and Weglein, 2004) and/or methods for extrapolation to low- and zero-frequency (Innanen et al., 2004) have been found and/or developed. But the true power of these imaging and inversion methods is in their ability to handle multi-dimensional media. In this paper we approach this issue, by developing and testing these subseries' for media which vary in lateral as well as vertical coordinates.

We make conscious choices in this derivation to produce terms which mirror those of the 1D cases previously investigated. At an intermediate point in the development, we separate into two alternate derivations, one that is more efficient for a numerical implementation, the other being better suited to interpretation in a task-specific framework. We further note:

(1) The development is expressed in the midpoint-conjugate/depth domain, i.e. $(k_m, z)$.

(2) We consider a medium which varies in $P$-wave velocity only, i.e. we assume constant density everywhere.

(3) The extra degree of freedom occurring in this single-parameter development is fixed by choosing offset-conjugate $k_h = 0$ in the numerical development.

# 2 Theoretical Development

We begin by reviewing the first and second order portions of the inverse scattering series, closely following the development of, e.g., Weglein et al. (2003). The desired scattering potential $V$, which for our purposes describes perturbations of wavespeed away from an at most slowly-varying background wavespeed $c_0(x,z)$:

$$V(x,z,\omega) = \frac{\omega^2}{c_0^2(x,y)}\alpha(x,z), \tag{1}$$

(in which $\alpha(x,z) = 1 - c_0^2(x,z)/c^2(x,z)$ for a true wavespeed distribution $c(x,z)$), has the linear component

$$V_1(x,z,\omega) = \frac{\omega^2}{c_0^2(x,z)}\alpha_1(x,z), \tag{2}$$

and second order component

$$V_2(x,z,\omega) = \frac{\omega^2}{c_0^2(x,z)}\alpha_2(x,z). \tag{3}$$

The linear portion of the inverse scattering series is an exact relationship between $V_1$ and the scattered field evaluated on a measurement surface (i.e. the data $D$). In operator form, this relationship is

$$\mathbf{G_0 V_1 G_0} = \mathbf{D} = \mathbf{G} - \mathbf{G_0}. \tag{4}$$

Causal Green's functions for homogeneous media are given by (following Clayton and Stolt, 1981):

$$G_0(k_g, z_g, x', z', \omega) = \frac{\rho_r}{2i}\frac{e^{-ik_g x'}e^{iq_g|z_g - z'|}}{q_g}, \tag{5}$$

and

$$G_0(x', z', k_s, z_s, \omega) = \frac{\rho_r}{2i}\frac{e^{ik_s x'}e^{iq_s|z_s - z'|}}{q_s}, \tag{6}$$

where

$$q_g = \text{sgn}(\omega)\sqrt{\left(\frac{\omega}{c_0}\right)^2 - k_g^2}, \tag{7}$$

and

$$q_s = \mathrm{sgn}(\omega)\sqrt{\left(\frac{\omega}{c_0}\right)^2 - k_s^2}. \tag{8}$$

In the above expressions, $k_g$ is the Fourier conjugate to $x_g$, and likewise $k_s$ is conjugate to $x_s$. Note that the "sign convention" of the Fourier transform is different for the source and geophone coordinates. This is a convenient choice in the present formalism.

Equation (4) may be solved by applying a Fourier transform over the lateral source and receiver coordinates to obtain $\alpha_1$. Here we consider the solution of the second-order inverse scattering series equations:

$$\mathbf{G_0 V_2 G_0} = -\mathbf{G_0 V_1 G_0 V_1 G_0}. \tag{9}$$

Expressing equation (9) explicitly given our choice of $V_2$, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' G_0(x_g, z_g, x', z', \omega) V_2(x', z', \omega) G_0(x', z', x_s, z_s, \omega) \\
&= -\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dz' G_0(x_g, z_g, x', z', \omega) V_1(x', z', \omega) \\
&\times \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dz'' G_0(x', z', x'', z'', \omega) V_1(x'', z'', \omega) G_0(x'', z'', k_s, z_s, \omega).
\end{aligned} \tag{10}$$

We next Fourier transform over lateral geophone and shot coordinates: $\int_{-\infty}^{\infty} dx_g \int_{-\infty}^{\infty} dx_s e^{ik_s x_s - ik_g x_g}$, and express the internal Green's function as

$$G_0(x', z', x'', z'', \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_\lambda e^{ik_\lambda x'} G_0(k_\lambda, z', x'', z'', \omega) \tag{11}$$

(in which $k_\lambda$ is conjugate to $x'$). This results in

$$\begin{aligned}
-\frac{\rho_r}{4c_0^2}\frac{\omega^2}{q_g q_s}\widetilde{\widetilde{\alpha}}_2(k_g - k_s, k_z) = &-\frac{i\rho_r}{16\pi c_0^4}\int_{-\infty}^{\infty} dk_\lambda \frac{\omega^4}{q_g q_\lambda q_s}\int_{-\infty}^{\infty} dz' \widetilde{\alpha}_1(k_g - k_\lambda, z') \\
&\times \int_{-\infty}^{\infty} dz'' \widetilde{\alpha}_1(k_\lambda - k_s, z'') e^{i[q_g(z'-z_g) + q_\lambda|z''-z'| + q_s(z''-z_s)]},
\end{aligned} \tag{12}$$

where $k_z = \sqrt{\omega^2/c_0^2 - k_g^2} + \sqrt{\omega^2/c_0^2 - k_s^2}$ is the vertical wavenumber. The quantity $\widetilde{\alpha}_1(k_m, z)$ is the Fourier transform of $\alpha_1(x_m, z)$, and $\widetilde{\widetilde{\alpha}}_1(k_m, k_z)$ is the Fourier transform over both $x_m$ and $z$. For computational reasons and to allow an easy comparison with 1D acoustic earth, we will compute $\alpha_2$ in the $(k_m, z)$ domain.

With this in mind, we apply the inverse Fourier transform $(1/2\pi)\int_{-\infty}^{\infty} e^{ik_z z}dk_z$ to equation (12), resulting in:

$$\begin{aligned}\widetilde{\alpha}_2(k_g - k_s, z) =& \frac{1}{8\pi^2 c_0^2}\int_{-\infty}^{\infty} dk_\lambda \int_{-\infty}^{\infty} dz' \int_{-\infty}^{\infty} dz'' \int_{-\infty}^{\infty} dk_z \\ & \frac{i\omega^2}{q_\lambda}\widetilde{\alpha}_1(k_g - k_\lambda, z')\widetilde{\alpha}_1(k_\lambda - k_s, z'')e^{i[q_g(z'-z)+q_\lambda|z''-z'|+q_s(z''-z)]}.\end{aligned} \tag{13}$$

This expression can be shown to reduce to the 1D form of $\alpha_2(z)$ discussed and analyzed elsewhere. See Appendix C for a detailed account of this reduction.

From this point (equation 13) we may proceed with development in two ways; one way, shown in the next section, produces quantities that are more convenient for numerical implementations. The section after that summarizes a development that more closely mirrors the task-separation strategy adopted in previous simpler cases (i.e. $\alpha_1(z)$).

# 3 Derivation I: Numerically Implemented Form

The innermost integral of equation (13) contains $\widetilde{\alpha}_1$, which depends on the measurement of the wave field; it can be taken out of this integral (with respect to $k_z$) if it can be shown that $k_g - k_\lambda$ and $k_\lambda - k_s$ are not a function of $k_z$. This can be accomplished in many ways, for instance by fixing the Fourier conjugate $x_h$ of the lateral offset coordinate to be constant. For convenience, we choose that constant to be $k_h = 0$. (See Clayton and Stolt, 1981 for a more detailed discussion.) Making this choice, we have:

$$\begin{aligned} k_h &= k_g + k_s = 0, \\ \frac{\omega}{c_0} &= \frac{1}{2}\mathrm{sgn}(k_z)\sqrt{k_z^2 + k_m^2}, \\ k_g &= -k_s = \frac{1}{2}k_m, \end{aligned} \tag{14}$$

which results in a simplified expression for $\widetilde{\alpha}_2(k_m, z)$:

$$\begin{aligned}\widetilde{\alpha}_2(k_m, z) =& \frac{1}{16\pi^2}\int_{-\infty}^{\infty} dk_\lambda \int_{0}^{\infty} dz' \widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \\ & \times \left\{\int_0^{\infty} dz'' \widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'')\right\} \gamma(z, z', z'', k_\lambda),\end{aligned} \tag{15}$$

where

$$\gamma(z,z',z'',k_\lambda) = i\int_{-\infty}^{\infty} dk_z \frac{k_z^2+k_m^2}{\text{sgn}(k_z)\sqrt{k_z^2+k_m^2-4k_\lambda^2}} e^{i\left[z_1k_z+z_2\text{sgn}(k_z)\sqrt{k_z^2+k_m^2-4k_\lambda^2}\right]}, \tag{16}$$

and

$$\begin{aligned} z_1 &= 0.5(z'+z'')-z, \\ z_2 &= 0.5|z'-z''|. \end{aligned} \tag{17}$$

Notice that the expression $\gamma(z,z',z'',k_\lambda)$ does not depend on the measured data; we may compute it once and use it repeatedly, saving on computation. The computation is further simplified by taking advantage of the symmetries:

$$\begin{aligned} &\gamma(z,z',z'',k_\lambda) \\ =&\gamma(z,z',z'',-k_\lambda) \\ =&\gamma(z,z'',z',k_\lambda) \\ =&\gamma(z,z'',z',-k_\lambda), \end{aligned} \tag{18}$$

which reduces the integration interval by half:

$$\begin{aligned} \widetilde{\alpha}_2(k_m,z) =&\frac{1}{8\pi^2}\int_{-\infty}^{\infty} dk_\lambda \int_0^\infty dz'\widetilde{\alpha}_1(0.5k_m-k_\lambda,z') \\ &\times\left\{\int_0^{z'} dz''\widetilde{\alpha}_1(k_\lambda+0.5k_m,z'')\right\}\gamma(z,z',z'',k_\lambda). \end{aligned} \tag{19}$$

## 3.1 An Evaluation of the Integral $\gamma$

We consider the evaluation of the integral $\gamma(z,z',z'',k_\lambda)$ as seen in the foregoing derivation of the second order perturbation. We define: $|k_m^2-4k_\lambda^2| = a^2$. The quantity $\widetilde{\alpha}_2(k_m,z)$ may be written

$$\widetilde{\alpha}_2(k_m,z) = \frac{1}{8\pi^2}\left(I_1+I_{12}+I_{22}+I_{\text{pre}}+I_{\text{post}}+I_{\text{sc}}\right), \tag{20}$$

where

$$\begin{aligned} I_1 =& -2\pi\int_{-\infty}^{\infty} dk_\lambda\widetilde{\alpha}_1(0.5k_m-k_\lambda,z)\widetilde{\alpha}_1(k_\lambda+0.5k_m,z) \\ &-2\pi\int_{-\infty}^{\infty} dk_\lambda\frac{\partial\widetilde{\alpha}_1(0.5k_m-k_\lambda,z)}{\partial z}\int_0^z dz'\widetilde{\alpha}_1(k_\lambda+0.5k_m,z'), \end{aligned} \tag{21}$$

$$\begin{aligned} I_{21} =& -2\int_{-0.5k_m}^{0.5k_m} dk_\lambda \int_{-\infty}^{\infty} dz'\widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \int_0^z dz''\widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'') \\ &\times \frac{0.5a(z'-z'')\cos(0.5a(z'-z'')) - \sin(0.5a(z'-z''))}{(z-z')^2}, \end{aligned} \tag{22}$$

$$\begin{aligned} I_{22} =& -2\int_{|k_\lambda|>0.5k_m} dk_\lambda \int_{-\infty}^{\infty} dz'\ \widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \int_0^z dz''\widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'') \\ &\times \frac{0.5a(z'+z''-2z)\cos(0.5a(z'+z''-2z)) - \sin(0.5a(z'+z''-2z))}{(z-z')^2}, \end{aligned} \tag{23}$$

$$\begin{aligned} I_{\text{pre}} =& -2\pi\int_{-0.5k_m}^{0.5k_m} dk_\lambda \int_{-\infty}^{\infty} dz'\text{sgn}(z'-z)\widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \int_{-\infty}^{z'} dz''\widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'') \\ &\times \left\{\frac{a^2 z_f}{|z_e|} f_{31}(z,z',z'',k_\lambda) - \frac{4k_\lambda^2 z_2^2 - k_m^2 z_1^2}{z_e} f_{41}(z,z',z'',k_\lambda)\right\}, \end{aligned} \tag{24}$$

$$\begin{aligned} I_{\text{post}} =& -2\pi\int_{|k_\lambda|>0.5k_m} dk_\lambda \int_{-\infty}^{\infty} dz'\text{sgn}(z'-z)\widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \int_{-\infty}^{z'} dz''\widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'') \\ &\times \left\{\frac{a^2 z_f}{|z_e|} f_{32}(z,z',z'',k_\lambda) - \frac{4k_\lambda^2 z_2^2 - k_m^2 z_1^2}{z_e} f_{42}(z,z',z'',k_\lambda)\right\}, \end{aligned} \tag{25}$$

and

$$\begin{aligned} I_{\text{sc}} =& 2\pi i \times \int_{|k_\lambda|>0.5k_m} dk_\lambda \int_{-\infty}^{\infty} dz'\widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \int_{-\infty}^{z'} dz''\widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'') \\ &\times \left\{4k_\lambda^2 H_{J2}(az_1, \text{fluc} = az_2) + k_m^2 H_{J3}(az_1, \text{fluc} = az_2)\right\}. \end{aligned} \tag{26}$$

The functions $f$ in the above expressions are given by

$$
\begin{aligned}
f_{31}(z,z',z'',k_\lambda) &= H(z_e)H_{k2}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_2}{\sqrt{|z_e|}}\right) \\
&\quad + H(-z_e)H_{N2}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_2}{\sqrt{|z_e|}}\right), \\
f_{32}(z,z',z'',k_\lambda) &= H(z_e)H_{N2}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_1}{\sqrt{|z_e|}}\right) \\
&\quad + H(-z_e)H_{k2}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_1}{\sqrt{|z_e|}}\right), \\
f_{41}(z,z',z'',k_\lambda) &= H(z_e)H_{k}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_2}{\sqrt{|z_e|}}\right) \\
&\quad + H(-z_e)H_{N}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_2}{\sqrt{|z_e|}}\right), \\
f_{42}(z,z',z'',k_\lambda) &= H(z_e)H_{N}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_1}{\sqrt{|z_e|}}\right) \\
&\quad + H(-z_e)H_{k}\left(a\sqrt{|z_e|},\mathrm{trunc}=\frac{\mathrm{sgn}(z'-z)z_1}{\sqrt{|z_e|}}\right),
\end{aligned}
\tag{27}
$$

also

$$
\begin{aligned}
z_e &= z_1^2 - z_2^2, \\
z_f &= z_1^2 + z_2^2.
\end{aligned}
\tag{28}
$$

The functions $H$ are given in Appendix B. In Appendix A we provide the remaining definitions of variables in the expressions for $I_1$, $I_{21}$, $I_{22}$, $I_{\mathrm{pre}}$, $I_{\mathrm{post}}$, and $I_{\mathrm{sc}}$ above, and develop the strategy for their derivation.

Consider $I_1$ first. Notice that if the $\gamma(z,z',z'',k_\lambda)$ integral is approximated by its leading order Taylor's series term of $\gamma(q_\lambda)$ calculated at $q_\lambda = q$, then the expression for $\widetilde{\alpha}_2(k_m,z)$ reduces to this term only. The rest of the terms can hence be interpreted as compensating for this approximation.

# 4 Derivation II: Task-Separated Form

The development thus far has been for an amalgamated second order perturbation, in the sense that we have not identified separable terms that appear to concern themselves only with either imaging or inversion tasks. In this section we demonstrate the separation of the second-order relationship given by equation (19) to reflect the 1D separated form, i.e. into terms that are related exclusively to the tasks of imaging and inversion (c.f. Weglein et al., 2003).

Consider the formula in the $(k_m, k_z)$ domain; we follow an "integration by parts logic" similar to that done for the 1D case and as reported in previously referenced literature. We may write:

$$
\begin{aligned}
\tilde{\tilde{\alpha}}_2(k_m, k_z) = & \frac{k_0^2}{4\pi i} \int_{-\infty}^{\infty} dk_\lambda \frac{1}{q_\lambda} \\
& \times \left\{ \int_{-\infty}^{\infty} dz' e^{iz'(q_s+q_g)} \left( \frac{1}{i(q_g+q_\lambda)} + \frac{1}{i(q_s+q_\lambda)} \right) \tilde{\alpha}_1(k_g - k_\lambda, z') \tilde{\alpha}_1(k_\lambda - k_s, z') \right. \\
& + \int_{-\infty}^{\infty} dz' \frac{e^{iz'(q_g+q_\lambda)}}{i(q_g+q_\lambda)} \tilde{\alpha}_1'(k_g - k_\lambda, z') \int_{-\infty}^{z'} dz'' e^{iz''(q_g-q_\lambda)} \tilde{\alpha}_1(k_\lambda - k_s, z'') \\
& \left. + \int_{-\infty}^{\infty} dz' \frac{e^{iz'(q_s+q_\lambda)}}{i(q_s+q_\lambda)} \tilde{\alpha}_1'(k_\lambda - k_s, z') \int_{-\infty}^{z'} dz'' e^{iz''(q_g-q_\lambda)} \tilde{\alpha}_1(k_g - k_\lambda, z'') \right\},
\end{aligned} \tag{29}
$$

where $k_m = k_g - k_s$ and $k_z = -q_g - q_s$. The topmost term has a form that resembles that of the self-interaction type term in the 1D case, i.e. in which the multiplicatively/nonlinearly contributing portions of the linear perturbation $\alpha_1$ are constrained to be collocated. The remaining two terms likewise greatly resemble the imaging portion of the 1D case, i.e. they involve the first derivative in depth of the linear perturbation whose amplitude is modified by the first integral of the same linear perturbation.

For $k_h = 0$, the choice that was made in the derivation for the numerical computations, we obtain

$$
\begin{aligned}
\tilde{\tilde{\alpha}}_2(k_m, k_z) = & -\frac{k_0^2}{2\pi} \int_{-\infty}^{\infty} dk_\lambda \frac{1}{q_\lambda(q+q_\lambda)} \left[ \int_{-\infty}^{\infty} dz' e^{iz'2q_g} \tilde{\alpha}_1(k_m/2 - k_\lambda z') \tilde{\alpha}_1(k_m/2 + k_\lambda z') \right. \\
& \left. + \int_{-\infty}^{\infty} dz' e^{iz'(q+q_\lambda)} \tilde{\alpha}_1'(k_m/2 - k_\lambda z') \int_{-\infty}^{z'} dz'' e^{iz''(q-q_\lambda)} \tilde{\alpha}_1(k_\lambda - k_m/2, z'') \right],
\end{aligned} \tag{30}
$$

where $q = q_g = q_s$. If we consider the leading order Taylor's series term of the coefficient and the phase only as functions of $q_\lambda$ calculated at $q_\lambda = q$ (as mentioned in our above discussion on the evaluation of $\gamma$), and then inverse Fourier transform on $k_z$ we find the approximate expression

$$
\tilde{\alpha}_2(k_m, z) \approx I_1. \tag{31}
$$

In other words, the two derivations are consistent with one another.

# 5 Numerical Example

In this section we present the result of a preliminary numerical implementation of the second order inverse scattering series terms for the recovery of laterally-varying medium parameters. The geological model used is pictured in Figure 1. The model is laterally-invariant at its edges, but at its center we place an incline in the top interface. Below this is a horizontal reflector. The first interface is at $z = 500$m depth on the left side, and at $z = 300$m depth on the right; the lower interface is fixed at $z = 700$m. At the surface ($z = 0$m) sources and receivers are placed on a regular grid. (Pictured many receivers and a single shot at the center.)

Figure 2 illustrates a typical shot gather from synthetic data. The data is generated using the aforementioned model and a finite difference scheme, and utilizing the first derivative of a Gaussian function as the source wavelet.

Figure 3 illustrates the linear inverse result, i.e. $\alpha_1(x, z)$. The key is in the deflection of the lower interface away from horizontal, due to the laterally-varying overburden. This reflector, correctly imaged, would naturally lie closer to fully horizontal.

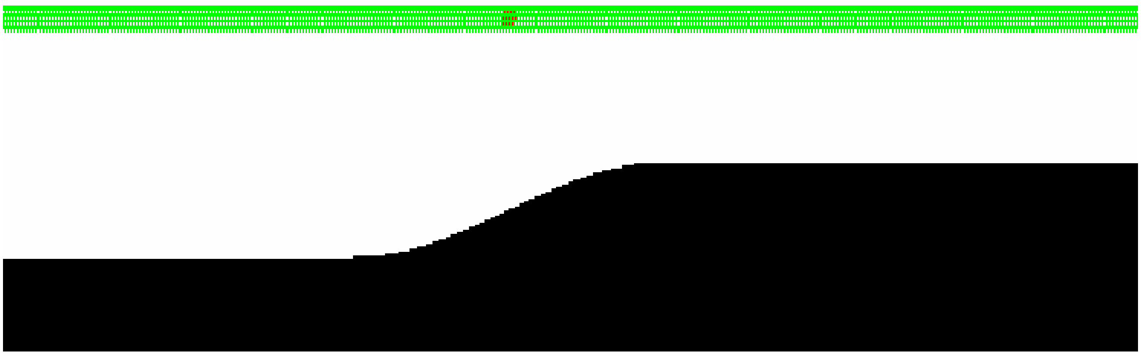

Figure 1: *Model used in numerical test of second order inverse scattering series terms for laterally-varying media. Top interface varies in the lateral coordinate direction near the center of the model.*

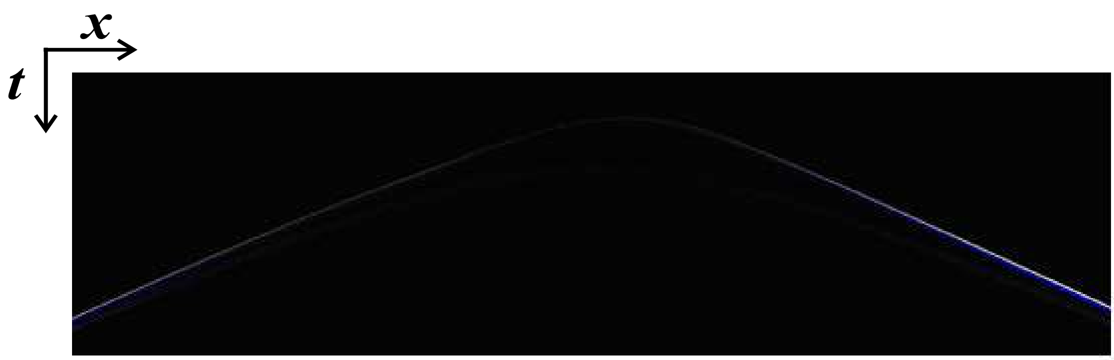


Figure 2: *Typical shot gather computed using a finite difference numerical scheme. The source wavelet is the first derivative of a Gaussian.*

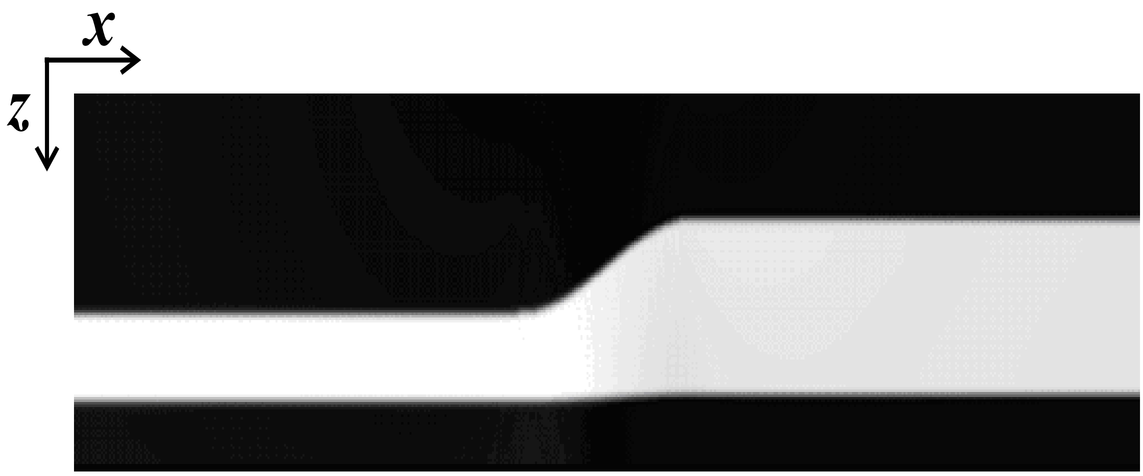


Figure 3: *The linear inverse results, i.e.* $\alpha_1(x, z)$. *Notice the deflection of the lower reflector from horizontal.*

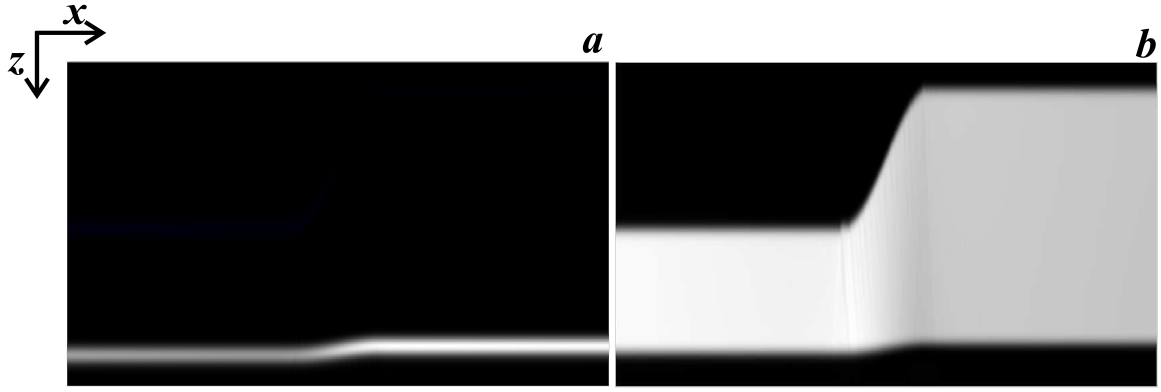


Figure 4: *(a) The portion of* $\alpha_2$ *which is responsible for imaging, calculated using the second part of Eq.(21). The single strong event in this figure will move the second reflector towards its correct location. Notice the right half is stronger than the left; this half has experienced a thicker high velocity zone, so we need a stronger correction. There is no need to move the first event because its location is already correct. (b) Illustrated is the inversion, or amplitude altering, portion of* $\alpha_2$.

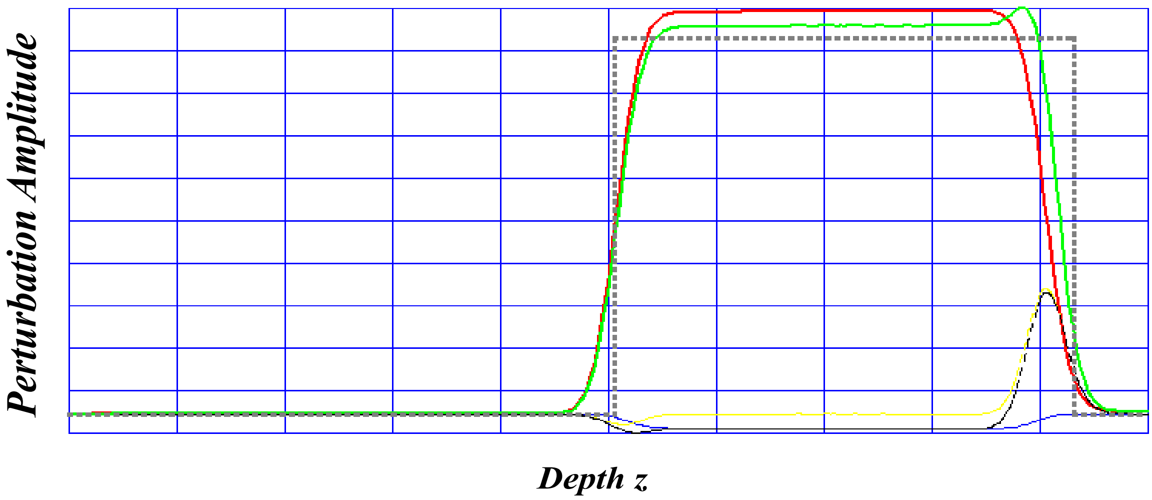


Figure 5: *Comparison of various terms near the edge of the left half. In red is the original* $\alpha_1$*, in green is* $\alpha_1+\alpha_2$*, in yellow is the imaging part of* $\alpha_2$*, in blue is the parameter inversion part of* $\alpha_2$*, and in black is the sum of the imaging and inversion part of* $\alpha_2$*. The dashed gray line shows the actual location of the first and second reflector. Both* $\alpha_1$ *and* $\alpha_1+\alpha_2$ *show the correct location of the first reflector, but* $\alpha_1+\alpha_2$ *successfully moves the second reflector towards its correct location.*

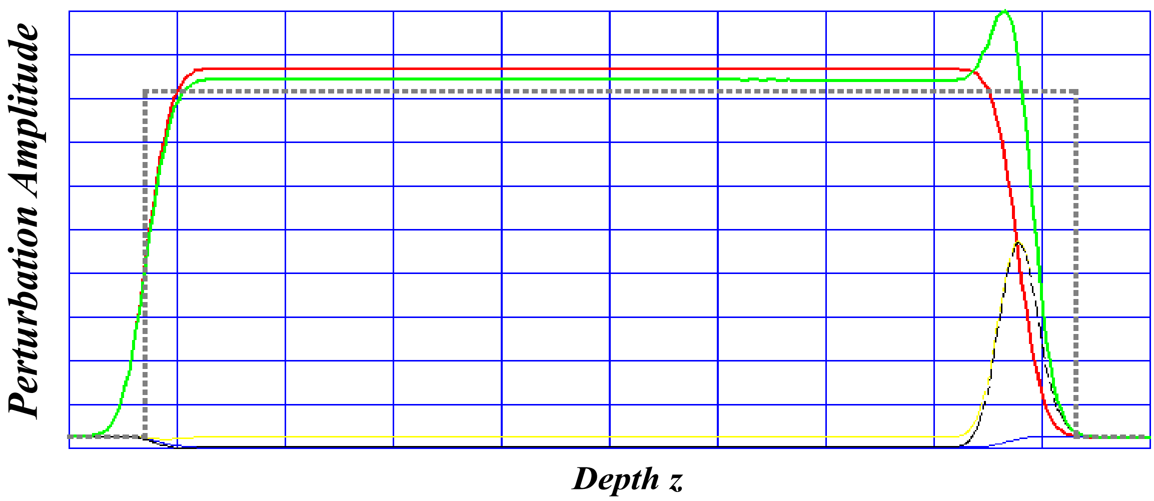


Figure 6: *Comparison of various terms near the edge of the left half. In red is the original* $\alpha_1$*, in green is* $\alpha_1+\alpha_2$*, in yellow is the imaging part of* $\alpha_2$*, in blue is the parameter inversion part of* $\alpha_2$*, and in black is the sum of the imaging and inversion part of* $\alpha_2$*. The dashed gray line shows the actual location of the first and second reflector. Both* $\alpha_1$ *and* $\alpha_1+\alpha_2$ *show the correct location of the first reflector, but* $\alpha_1+\alpha_2$ *successfully moves the second reflector towards its correct location. Compared with the previous figure, this portion of the model has a deeper high velocity zone, and the second reflector is further deflected up. The second term* $\alpha_2$ *has moved this part of the second reflector a greater distance as required. This encouraging result is our first numerical evidence of imaging in the presence of lateral medium variations without knowledge of the actual velocity field.*

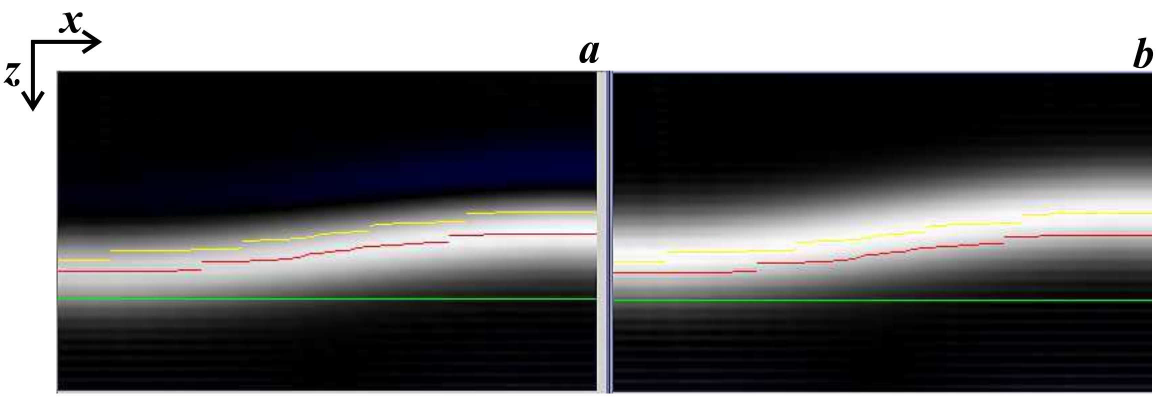


Figure 7: *Detail around the central part of the second reflector. (To make the image more like seismic data, the derivative over z is displayed.) (a) $\alpha_1 + \alpha_2$ is shown; (b) $\alpha_1$ is shown. In the central part of the model, which has maximal lateral variation, the second term moves our target toward its correct location. To calculate how far the second reflector has moved, we automatically pick the location of the second reflector using a maximum energy criterion, in green is the desired location, in yellow is the location picked from the original $\alpha_1$, and in red is the location picked from $\alpha_1 + \alpha_2$. To make the comparison easier, all 3 horizons are shown in both (a) and (b).*

# 6 Conclusions

We present a formalism for the nonlinear imaging and inversion of 2D wave field data over a medium with laterally-varying parameters. The development is motivated and follows closely the co-development of 1D (depth-varying) methods. We consider the second order term in the series for the wavespeed perturbation $\alpha(x, z)$, namely $\alpha_2(x, z)$.

We further present a form of this second order algorithm that directly mirrors the purposeful casting of the terms as those which correspond separately and exclusively to tasks of imaging and inversion. These can be seen to compare closely to their 1D brethren.

Numerically we see encouraging results given an unknown overburden with lateral variation.

This research represents the initial foray into the domains in which the inverse scattering series stands alone as a formalism for direct inversion seismic data: multiple dimensions. Future research, long and short term, involves the pursuit of more general physical models, greater dimensionality, and the inclusion of higher order terms.

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# Appendix A

In this appendix we illustrate the basic strategies behind the re-expression of $\gamma(z, z', z'', k_\lambda)$. We have

$$\gamma(z, z', z'', k_\lambda) = i \int_{-\infty}^{\infty} dk_z \frac{k_z^2 + k_m^2}{\text{sgn}(k_z)\sqrt{k_z^2 + k_m^2 - 4k_\lambda^2}} e^{i\left[z_1 k_z + z_2 \text{sgn}(k_z)\sqrt{k_z^2 + k_m^2 - 4k_\lambda^2}\right]}, \tag{32}$$

where

$$\begin{aligned} z_1 &= 0.5(z' + z'') - z, \\ z_2 &= 0.5|z' - z''|. \end{aligned} \tag{33}$$

We have seen that symmetry may be exploited to make computation more efficient. As a result, we may consider only specific domains of depth variables, in fact, we may drop the absolute value sign in the definition of $z_2$, such that $z_2 = 0.5(z' - z'')$. Furthermore, depending on whether $0.5k_m^2 - k_\lambda^2 > 0$ or $0.5k_m^2 - k_\lambda^2 < 0$, we may define

$$\pm a^2 = k_m^2 - 4k_\lambda^2, \tag{34}$$

such that

$$\gamma(z, z', z'', k_\lambda) = i \int_{-\infty}^{\infty} dk_z \frac{k_z^2 + k_m^2}{\text{sgn}(k_z)\sqrt{k_z^2 \pm a^2}} e^{i\left[z_1 k_z + z_2 \text{sgn}(k_z)\sqrt{k_z^2 \pm a^2}\right]}, \tag{35}$$

and with a further change of variables to $\kappa_z = k_z/a$, the integral becomes

$$\gamma(z, z', z'', k_\lambda) = i \int_{-\infty}^{\infty} d\kappa_z \frac{a^2\kappa_z^2 + k_m^2}{\text{sgn}(\kappa_z)\sqrt{\kappa_z^2 \pm 1}} e^{ia\left[z_1 \kappa_z + z_2 \text{sgn}(\kappa_z)\sqrt{\kappa_z^2 \pm 1}\right]}. \tag{36}$$

The phase term in the above expression is more suitably expressed using a further variable change:

$$\begin{aligned}\phi &= z_1\kappa_z + z_2\mathrm{sgn}(\kappa_z)\sqrt{\kappa_z^2 \pm 1},\\ \psi &= \mathrm{sgn}(\phi)\sqrt{\phi^2 \mp (z_2^2 - z_1^2)}.\end{aligned} \tag{37}$$

Using these new variables, we may re-express many of the terms in the integral. To wit:

$$\begin{aligned}&\kappa_z = \frac{-z_1\phi + z_2\psi}{z_2^2 - z_1^2},\\ &\mathrm{sgn}(\kappa_z)\sqrt{\kappa_z^2 \pm 1} = \frac{z_2\phi - z_1\psi}{z_2^2 - z_1^2},\\ &d\kappa_z = \frac{z_2\phi - z_1\psi}{\psi(z_2^2 - z_1^2)}d\phi,\\ &ie^{ia\phi}\frac{a\kappa_z^2 + k_m^2}{\mathrm{sgn}(\kappa_z)\sqrt{\kappa_z^2 \pm 1}}\\ &\quad = ie^{-ia\phi}d\phi\left\{\left[\frac{a}{z_1+z_2}\right]^2\phi + \frac{2a^2(z_1^2+z_2^2)}{(z_2^2-z_1^2)^2}\frac{\phi(\phi-\psi)}{\psi} + \left[k_m^2 \mp \frac{a^2z_2^2}{z_2^2-z_1^2}\right]\frac{1}{\psi}\right\}.\end{aligned} \tag{38}$$

The strategy is to utilize this integral as a normal Fourier transform, where the variable is now $\phi$ rather than $k_z$; however, the integral range of $\phi$ behaves in a more complicated manner, and requires some case-by-case analysis. (We find it to be truncated in some cases, and integrated multiply in others.)

Consider the first case $0 < k_m^2 - 4k_\lambda^2 = a^2$. This begets two subcases:

(1) $z_1 + z_2 > 0$, in which (as $k_z$ goes from $-\infty$ to $\infty$) $\phi$ ranges from $-\infty$ to $-z_2$, then jumps to the interval $z_2$ to $\infty$.

(2) $z_1 + z_2 < 0$, in which (again as $k_z$ goes from $-\infty$ to $\infty$) $\phi$ ranges from $\infty$ to $-z_2$, then jumps up to the interval $z_2$ to $-\infty$.

This integration range is summarized as follows:

$$\begin{aligned}L_1 + L_2 + L_3 + L_4 = &\left(\mathrm{sgn}(z_1+z_2)\int_{-\infty}^{\infty} - \int_{-z_2}^{z_2}\right) ie^{-ia\phi}d\phi\\ &\times\left\{\left[\frac{a}{z_1+z_2}\right]^2\phi + \frac{a^2(z_1^2+z_2^2)}{(z_2^2-z_1^2)^2}\frac{\phi(\phi-\psi)}{\psi} + \left[k_m^2 - \frac{a^2z_2^2}{z_2^2-z_1^2}\right]\frac{1}{\psi}\right\}.\end{aligned} \tag{39}$$

The component $L_1$ being

$$
\begin{aligned}
L_1 &= \operatorname{sgn}(z_1+z_2)\int_{-\infty}^{\infty} d\phi i e^{-ia\phi}\left(\frac{a}{z_1+z_2}\right)^2\phi \\
&= \frac{\operatorname{sgn}(z_1+z_2)}{(z_1+z_2)^2}a^2\int_{-\infty}^{\infty} d\phi i\phi e^{-ia\phi} \\
&= \frac{\operatorname{sgn}(z_1+z_2)}{(z_1+z_2)^2}a^2\delta'(a) \\
&= \delta'(z_1+z_2).
\end{aligned}
\tag{40}
$$

The last step is justified by making use of the identity $\operatorname{sgn}(w)/w^2\delta'(x) = \delta'(wx)$, and the fact that $a$ is always positive. We thus get a sense of the efficiency associated with this way of expressing the integral: the sifting aspect of these delta-like quantities produces simple results.

The second integral $L_2$ is:

$$
\begin{aligned}
L_2 &= -\int_{-z_2}^{z_2} d\phi i e^{-ia\phi}\left(\frac{a}{z_1+z_2}\right)^2\phi \\
&= -\frac{2[az_2\cos az_2 - \sin az_2]}{(z_1+z_2)^2}.
\end{aligned}
\tag{41}
$$

The remaining parts of the integral are likewise computable. For example, for the third term $L_3$, we have:

$$
L_3 = \left(\operatorname{sgn}(z_1+z_2)\int_{-\infty}^{\infty} - \int_{-z_2}^{z_2}\right) i e^{-ia\phi} d\phi \frac{a^2(z_1^2+z_2^2)}{(z_2^2-z_1^2)^2}\frac{\phi(\phi-\psi)}{\psi}. \tag{42}
$$

The integral above can be simplified by making the following changes of variables: $z_e = z_1^2 - z_2^2$, $z_f = z_1^2 + z_2^2$, $\nu = \frac{\phi}{\sqrt{|z_e|}}$, and notice the fact that in this case: $\psi = \operatorname{sgn}\sqrt{\phi^2 - (z_2^2 - z_1^2)}$, we have:

$$
L_3 = \left(\operatorname{sgn}(z_1+z_2)\int_{-\infty}^{\infty} - \int_{-\frac{z_2}{\sqrt{|z_e|}}}^{\frac{z_2}{\sqrt{|z_e|}}}\right) i e^{-ia\sqrt{|z_e|}\nu} d\nu \frac{a^2 z_f}{|z_e|}\frac{\nu(\nu - \operatorname{sgn}\sqrt{\nu^2\pm 1})}{\operatorname{sgn}\sqrt{\nu^2\pm 1}}. \tag{43}
$$

where the plus or minus sign in the equation above depends on whether $z_1^2 - z_2^2 > 0$ or $z_1^2 - z_2^2 < 0$. The integral above can be easily expressed by our pre-defined functions, in the case of $z_2^2 < z_1^2$, we have

$$
L_3 = -2\pi\frac{a^2 z_f}{|z_e|}(\operatorname{sgn}(z_1+z_2)) \times H_{K2}(a\sqrt{|z_e|}, \text{trunc} = \operatorname{sgn}(z_1+z_2)\frac{z_2}{\sqrt{|z_e|}}) \tag{44}
$$

In the case of $z_2^2 > z_1^2$, we have

$$L_3 = -2\pi \frac{a^2 z_f}{|z_e|} (\mathrm{sgn}(z_1+z_2)) \times H_{N2}(a\sqrt{|z_e|}, \mathrm{trunc} = \mathrm{sgn}(z_1+z_2)\frac{z_2}{\sqrt{|z_e|}}) \tag{45}$$

For the fourth term $L_4$, we have:

$$L_4 = \left(\mathrm{sgn}(z_1+z_2)\int_{-\infty}^{\infty} - \int_{-z_2}^{z_2}\right) ie^{-ia\phi} d\phi \left(k_m^2 - \frac{a^2 z_2^2}{z_2^2 - z_1^2}\right)\frac{1}{\psi}. \tag{46}$$

Using the same change-of-variables used for $L_3$, we calculate $L_4$.

$$L_4 = \left(\mathrm{sgn}(z_1+z_2)\int_{-\infty}^{\infty} - \int_{-\frac{z_2}{\sqrt{|z_e|}}}^{\frac{z_2}{\sqrt{|z_e|}}}\right) ie^{-ia\sqrt{|z_e|}\nu} d\nu \left(k_m^2 - \frac{a^2 z_2^2}{z_2^2 - z_1^2}\right)\frac{1}{\sqrt{\nu^2 \pm 1}}. \tag{47}$$

where the plus or minus sign in the equation above depends on whether $z_1^2 - z_2^2 > 0$ or $z_1^2 - z_2^2 < 0$. The integral above is exactly the integral we defined in Appendix B, so, in the case of $z_2^2 < z_1^2$ :

$$L_4 = 2\pi \frac{4k_\lambda^2 z_2^2 - k_m^2 z_1^2}{z_e} (\mathrm{sgn}(z_1+z_2)) \times H_K(a\sqrt{|z_e|}, \mathrm{trunc} = \mathrm{sgn}(z_1+z_2)\frac{z_2}{\sqrt{|z_e|}}). \tag{48}$$

In the case of $z_2^2 > z_1^2$, we have

$$L_4 = 2\pi \frac{4k_\lambda^2 z_2^2 - k_m^2 z_1^2}{z_e} (\mathrm{sgn}(z_1+z_2)) \times H_N(a\sqrt{|z_e|}, \mathrm{trunc} = \mathrm{sgn}(z_1+z_2)\frac{z_2}{\sqrt{|z_e|}}) \tag{49}$$

Next, we consider the second case $0 > k_m^2 - 4k_\lambda^2 = -a^2$. In this case, within the finite interval: $-a < k_z < a$, $\sqrt{k_z^2 - a^2}$ will be imaginary. We introduce specifically defined functions to express this portion of integral:

$$L_{\mathrm{sc}} = i\int_{-a}^{a} dk_z \frac{k_z^2 + k_m^2}{\mathrm{sgn}(k_z)\sqrt{k_z^2 - a^2}} e^{i\left[z_1 k_z + z_2 \mathrm{sgn}(k_z)\sqrt{k_z^2 - a^2}\right]}. \tag{50}$$

After changing the integration variable: $\kappa_z = k_z/a$, we have:

$$L_{\mathrm{sc}} = i\int_{-1}^{1} d\kappa_z \frac{a^2\kappa_z^2 + k_m^2}{\mathrm{sgn}(\kappa_z)\sqrt{\kappa_z^2 - 1}} e^{ia\left[z_1 \kappa_z + z_2 \mathrm{sgn}(\kappa_z)\sqrt{\kappa_z^2 - 1}\right]}. \tag{51}$$

Because $\sqrt{\kappa_z^2 - 1} = i\sqrt{1 - \kappa_z^2}$, we have:

$$L_{\mathrm{sc}} = \int_{-1}^{1} d\kappa_z \frac{a^2\kappa_z^2 + k_m^2}{\mathrm{sgn}(\kappa_z)\sqrt{1-\kappa_z^2}} e^{iaz_1\kappa_z} e^{-az_2\mathrm{sgn}(\kappa_z)\sqrt{1-\kappa_z^2}}. \tag{52}$$

Using the fact that in this case, $a^2 = 4k_\lambda^2 - k_m^2$, we have:

$$\begin{aligned} L_{\mathrm{sc}} &= 4k_\lambda^2 \int_{-1}^{1} d\kappa_z \frac{\kappa_z^2}{\mathrm{sgn}(\kappa_z)\sqrt{1-\kappa_z^2}} e^{iaz_1\kappa_z} e^{-az_2\mathrm{sgn}(\kappa_z)\sqrt{1-\kappa_z^2}} \\ &+ k_m^2 \int_{-1}^{1} d\kappa_z \mathrm{sgn}(\kappa_z)\sqrt{1-\kappa_z^2} e^{iaz_1\kappa_z} e^{-az_2\mathrm{sgn}(\kappa_z)\sqrt{1-\kappa_z^2}} \\ &= 2\pi i \times \left(4k_\lambda^2 H_{\mathrm{J2}}(az_1, \mathrm{fluc} = az2) + 4k_m^2 H_{\mathrm{J3}}(az_1, \mathrm{fluc} = az_2)\right). \end{aligned} \tag{53}$$

The remaining parts can be handled as before:

(1) $z_1 + z_2 > 0$, in which (as $k_z$ goes from $-\infty$ to $-a$, then jumps to the interval $a$ to $\infty$) $\phi$ ranges from $-\infty$ to $-z_1$, then jumps to the interval $z_1$ to $\infty$.

(2) $z_1 + z_2 < 0$, in which (again as $k_z$ goes from $-\infty$ to $-a$, then jumps to the interval $a$ to $\infty$) $\phi$ ranges from $\infty$ to $z_1$, then jumps up to the interval $-z_1$ to $-\infty$.

This integration range is summarized as follows:

$$\begin{aligned} L_1 + L_2 + L_3 + L_4 = &\left(\mathrm{sgn}(z_1+z_2)\int_{-\infty}^{\infty} - \int_{-z_1}^{z_1}\right) ie^{-ia\phi} d\phi \\ &\times \left\{ \left[\frac{a}{z_1+z_2}\right]^2 \phi + \frac{a^2(z_1^2+z_2^2)}{(z_2^2-z_1^2)^2}\frac{\phi(\phi-\psi)}{\psi} + \left[k_m^2 + \frac{a^2 z_2^2}{z_2^2 - z_1^2}\right]\frac{1}{\psi}\right\}. \end{aligned} \tag{54}$$

The first integral $L_1$ can be derived the same way as in the previous case:

$$\begin{aligned} L_1 &= \mathrm{sgn}(z_1+z_2)\int_{-\infty}^{\infty} d\phi i e^{-ia\phi}\left(\frac{a}{z_1+z_2}\right)^2 \phi \\ &= \frac{\mathrm{sgn}(z_1+z_2)}{(z_1+z_2)^2} a^2 \int_{-\infty}^{\infty} d\phi i\phi e^{-ia\phi} \\ &= \frac{\mathrm{sgn}(z_1+z_2)}{(z_1+z_2)^2} a^2 \delta'(a) \\ &= \delta'(z_1+z_2). \end{aligned} \tag{55}$$

The second integral $L_2$ is:

$$L_2 = -\int_{-z_1}^{z_1} d\phi i e^{-ia\phi} \left( \frac{a}{z_1 + z_2} \right)^2 \phi$$
$$= -\frac{2[az_1 \cos az_1 - \sin az_1]}{(z_1 + z_1)^2}. \tag{56}$$

The remaining parts of the integral can be computed similarly. For the third term $L_3$, we have:

$$L_3 = \left( \text{sgn}(z_1 + z_2) \int_{-\infty}^{\infty} - \int_{-z_1}^{z_1} \right) i e^{-ia\phi} d\phi \frac{a^2(z_1^2 + z_2^2)}{(z_2^2 - z_1^2)^2} \frac{\phi(\phi - \psi)}{\psi}. \tag{57}$$

The integral above can be simplified by making the following transforms: $z_e = z_1^2 - z_2^2$, $z_f = z_1^2 + z_2^2$, $\nu = \frac{\phi}{\sqrt{|z_e|}}$, and, noticing that, in this case, $\psi = \text{sgn}\sqrt{\phi^2 + (z_2^2 - z_1^2)}$, we have:

$$L_3 = \left( \text{sgn}(z_1 + z_2) \int_{-\infty}^{\infty} - \int_{-\frac{z_1}{\sqrt{|z_e|}}}^{\frac{z_1}{\sqrt{|z_e|}}} \right) i e^{-ia\sqrt{|z_e|}\nu} d\nu \frac{a^2 z_f}{|z_e|} \frac{\nu(\nu - \text{sgn}\sqrt{\nu^2 \mp 1})}{\text{sgn}\sqrt{\nu^2 \mp 1}}, \tag{58}$$

where the minus or plus sign in the equation above depends on whether $z_1^2 - z_2^2 > 0$ or $z_1^2 - z_2^2 < 0$. The integral above can be easily expressed by our pre-defined functions; in the case of $z_2^2 < z_1^2$, we have

$$L_3 = -2\pi \frac{a^2 z_f}{|z_e|} (\text{sgn}(z_1 + z_2)) \times H_{N2}(a\sqrt{|z_e|}, \text{trunc} = \text{sgn}(z_1 + z_2) \frac{z_1}{\sqrt{|z_e|}}). \tag{59}$$

In the case of $z_2^2 > z_1^2$, we have

$$L_3 = -2\pi \frac{a^2 z_f}{|z_e|} (\text{sgn}(z_1 + z_2)) \times H_{K2}(a\sqrt{|z_e|}, \text{trunc} = \text{sgn}(z_1 + z_2) \frac{z_1}{\sqrt{|z_e|}}). \tag{60}$$

For the fourth term $L_4$, we have:

$$L_4 = \left( \text{sgn}(z_1 + z_2) \int_{-\infty}^{\infty} - \int_{-z_1}^{z_1} \right) i e^{-ia\phi} d\phi \left( k_m^2 + \frac{a^2 z_2^2}{z_2^2 - z_1^2} \right) \frac{1}{\psi}. \tag{61}$$

Changing variables as we did to solve for $L_3$, we can similarly calculate $L_4$.

$$L_4 = \left( \mathrm{sgn}(z_1+z_2) \int_{-\infty}^{\infty} - \int_{-\frac{z_1}{\sqrt{|z_e|}}}^{\frac{z_1}{\sqrt{|z_e|}}} \right) ie^{-ia\sqrt{|z_e|}\nu} d\nu \left( k_m^2 + \frac{a^2 z_2^2}{z_2^2 - z_1^2} \right) \frac{1}{\sqrt{\nu^2 \mp 1}}, \tag{62}$$

where the minus or plus sign in the equation above again depends on whether $z_1^2 - z_2^2 > 0$ or $z_1^2 - z_2^2 < 0$. The integral above is exactly the integral we defined in Appendix B, so

In the case of $z_2^2 < z_1^2$, we have therefore

$$L_4 = 2\pi \frac{4k_\lambda^2 z_2^2 - k_m^2 z_1^2}{z_e} (\mathrm{sgn}(z_1+z_2)) \times H_N(a\sqrt{|z_e|}, \mathrm{trunc} = \mathrm{sgn}(z_1+z_2)\frac{z_1}{\sqrt{|z_e|}}), \tag{63}$$

and in the case of $z_2^2 > z_1^2$, we have

$$L_4 = 2\pi \frac{4k_\lambda^2 z_2^2 - k_m^2 z_1^2}{z_e} (\mathrm{sgn}(z_1+z_2)) \times H_K(a\sqrt{|z_e|}, \mathrm{trunc} = \mathrm{sgn}(z_1+z_2)\frac{z_1}{\sqrt{|z_e|}}). \tag{64}$$

# Appendix B

In this appendix we itemize some Fourier integrals used in the body of the paper. We have:

$$H_J(x, \mathrm{fluc} = y) = \frac{1}{2\pi} \int_{-1}^{1} d\omega \frac{1}{i\mathrm{sgn}(\omega)\sqrt{1-\omega^2}} e^{i\omega x} e^{-y\mathrm{sgn}(\omega)\sqrt{1-\omega^2}} \tag{65}$$

$$H_{J2}(x, \mathrm{fluc} = y) = \frac{1}{2\pi} \int_{-1}^{1} d\omega \frac{\omega^2}{i\mathrm{sgn}(\omega)\sqrt{1-\omega^2}} e^{i\omega x} e^{-y\mathrm{sgn}(\omega)\sqrt{1-\omega^2}} \tag{66}$$

$$\begin{aligned} H_{J3}(x, \mathrm{fluc} = y) &= \frac{1}{2\pi} \int_{-1}^{1} d\omega \frac{\sqrt{1-\omega^2}}{i\mathrm{sgn}(\omega)} e^{i\omega x} e^{-y\mathrm{sgn}(\omega)\sqrt{1-\omega^2}} \\ &= H_J(x, \mathrm{fluc} = y) - H_{J2}(x, \mathrm{fluc} = y) \end{aligned} \tag{67}$$

$$H_K(x, \mathrm{trunc} = d_1) = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} - \mathrm{sgn}(d_1) \int_{-|d_1|}^{|d_1|} \right) \frac{e^{i\omega x}}{i\mathrm{sgn}(\omega)\sqrt{\omega^2+1}} d\omega \tag{68}$$

$$H_{K2}(x, \mathrm{trunc} = d_1) = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} - \mathrm{sgn}(d_1) \int_{-|d_1|}^{|d_1|} \right) \frac{e^{i\omega x}}{i\mathrm{sgn}(\omega)} \left( \frac{\omega^2}{\sqrt{\omega^2+1}} - |\omega| \right) d\omega \tag{69}$$

$$H_N(x, \text{trunc} = d_1) = \frac{1}{2\pi}\left(\int_{|\omega|>1} - \text{sgn}(d_1)\int_{|d_2|>|\omega|>1}\right)\frac{e^{i\omega x}}{i\text{sgn}(\omega)\sqrt{\omega^2-1}}d\omega \quad (70)$$

$$H_{N2}(x, \text{trunc} = d_1) = \frac{1}{2\pi}\left(\int_{|\omega|>1} - \text{sgn}(d_1)\int_{|d_2|>|\omega|>1}\right)\frac{e^{i\omega x}}{i\text{sgn}(\omega)}\left(\frac{\omega^2}{\sqrt{\omega^2-1}} - |\omega|\right)d\omega \quad (71)$$

.

# Appendix C

In this appendix we demonstrate the reduction of the general derivation of $\alpha_2(x, z)$ to the previously developed and tested 1D form $\alpha_2(z)$.

The data may be expressed as

$$\begin{aligned}
&D(k_g = 0.5k_m, z_g, k_s = -0.5k_m, z_s, \omega) \\
&= \int_{-\infty}^{\infty} dx_g e^{-ik_g x_g} \int_{-\infty}^{\infty} dx_s e^{ik_s x_s} D(x_g, z_g, x_s, z_s, \omega) \\
&= \int_{-\infty}^{\infty} dx_g e^{-i(k_m/2)x_g} \int_{-\infty}^{\infty} dx_s e^{-i(k_m/2)x_s} D(x_g, z_g, x_s, z_s, \omega) \\
&= \int_{-\infty}^{\infty} dx_g e^{-i(k_m/2)x_g} \int_{-\infty}^{\infty} dx_s e^{-i(k_m/2)x_s} D(z_g, z_s, x_g - x_s\omega),
\end{aligned} \quad (72)$$

in which we make use of the fact that in 1D the data depends only on the offset coordinate and not the midpoint coordinate (that is, all shot-record like experiments are identical). The Fourier transform with respect to midpoint is therefore a delta-function:

$$D(k_g = 0.5k_m, z_g, k_s = -0.5k_m, z_s, \omega) = 2\pi\delta(k_m)\int_{-\infty}^{\infty} dx_h D(z_g, z_s, x_h, \omega). \quad (73)$$

Recognizing $\int_{-\infty}^{\infty} D(z_g, z_s, x_h, \omega)dx_h$ as $D(z_g, z_s, \omega)$, the data used in Shaw et al. (2002) and Zhang and Weglein (2002), the latter at normal incidence and for invariant density. The delta function itself sifts out the component $k_m = 0$:

$$
\begin{aligned}
\widetilde{\alpha}_1(k_m, z) &= -\frac{c_0^2}{2\pi\rho_r}\int_{-\infty}^{\infty} dk_z \frac{k_z^2}{\omega^2} e^{-ik_z[z-0.5(z_g+z_s)]} D\left(\frac{k_m}{2}, z_g, -\frac{k_m}{2}, z_s, \omega\right) \\
&= -\frac{c_0^2}{2\pi\rho_r}\int_{-\infty}^{\infty} dk_z \frac{k_z^2}{\omega^2} e^{-ik_z[z-0.5(z_g+z_s)]} 2\pi\delta(k_m) D(z_g, z_s, \omega) \\
&= -\frac{c_0^2\delta(k_m)}{\rho_r}\int_{-\infty}^{\infty} dk_z \frac{k_z^2}{\omega^2} e^{-ik_z[z-0.5(z_g+z_s)]} D(z_g, z_s, \omega) \\
&= -\frac{c_0^2\delta(k_m)}{\rho_r}\int_{-\infty}^{\infty} dk_z \frac{4}{c_0^2} e^{-ik_z[z-0.5(z_g+z_s)]} D(z_g, z_s, \omega) \\
&= -\frac{4\delta(k_m)}{\rho_r}\int_{-\infty}^{\infty} dk_z e^{-ik_z[z-0.5(z_g+z_s)]} D(z_g, z_s, \omega) \\
&= 2\pi\delta(k_m)\alpha_1(z).
\end{aligned}
\tag{74}
$$

Here $\alpha_1(z)$ is recognizable as the 1D linear perturbation. The second term is then computed using equation (74):

$$
\begin{aligned}
\widetilde{\alpha}_2(k_m, z) &= \frac{1}{16\pi^2}\int_{-\infty}^{\infty} dk_\lambda \int_0^{\infty} dz' \widetilde{\alpha}_1(0.5k_m - k_\lambda, z') \\
&\times \left\{\int_0^{\infty} dz'' \widetilde{\alpha}_1(k_\lambda + 0.5k_m, z'')\right\} \gamma(z, z', z'', k_\lambda) \\
&= \frac{1}{16\pi^2}\int_{-\infty}^{\infty} dk_\lambda \int_0^{\infty} dz' 2\pi\delta(0.5k_m - k_\lambda)\widetilde{\alpha}_1(z') \\
&\times \int_0^{\infty} dz'' 2\pi\delta(k_\lambda + 0.5k_m)\widetilde{\alpha}_1(z'')\gamma(z, z', z'', k_\lambda) \\
&= \frac{1}{4}\int_{-\infty}^{\infty} dk_\lambda \int_0^{\infty} dz' \delta(0.5k_m - k_\lambda)\widetilde{\alpha}_1(z') \\
&\times \int_0^{\infty} dz'' \delta(k_\lambda + 0.5k_m)\widetilde{\alpha}_1(z'')\gamma(z, z', z'', k_\lambda) \\
&= \frac{1}{4}\int_0^{\infty} dz' \widetilde{\alpha}_1(z') \int_0^{\infty} dz'' \widetilde{\alpha}_1(z'') \\
&\times \int_{-\infty}^{\infty} dk_\lambda \delta(0.5k_m - k_\lambda)\delta(k_\lambda + 0.5k_m)\gamma(z, z', z'', k_\lambda).
\end{aligned}
\tag{75}
$$

We simplify by recognizing that since

$$
(k_z^2 + k_m^2)\delta(k_m) = k_z^2\delta(k_m), \tag{76}
$$

we have

$$
\begin{aligned}
\gamma(z, z', z'', 0.5k_m) &= i \int_{-\infty}^{\infty} dk_z \frac{k_z^2}{k_z} e^{i(z_1+z_2)k_z} \\
&= 2\pi\delta'(z_1 + z_2) \\
&= 2\pi\delta'(0.5(z' + z'') - z + 0.5|z' - z''|).
\end{aligned} \tag{77}
$$

This simplified $\gamma$ may then be substituted into equation (75):

$$
\begin{aligned}
\widetilde{\alpha}_2(k_m, z) &= \frac{\delta(k_m)}{4} \int_0^{\infty} dz'\alpha_1(z') \int_0^{\infty} dz''\alpha_1(z'') \\
&\times 2\pi\delta'(0.5(z' + z'') - z + 0.5|z' - z''|) \\
&= 2\pi \frac{\delta(k_m)}{4} \int_0^{\infty} dz'\alpha_1(z') \int_0^{\infty} dz''\alpha_1(z'') \\
&\times 2\pi\delta'(0.5(z' + z'') - z + 0.5|z' - z''|) \\
&= 2\pi\delta(k_m)\frac{1}{4} \int_0^{\infty} dz'\alpha_1(z') \\
&\times \left\{ \int_0^{z'} dz''\alpha_1(z'')\delta'(z' - z) + \int_{z'}^{\infty} dz''\alpha_1(z'')\delta'(z'' - z) \right\} \\
&= 2\pi\delta(k_m) \left(-\frac{1}{2}\right) \left[\alpha_1^2(z) + \alpha_1'(z) \int_0^z dz'\alpha_1(z')\right].
\end{aligned} \tag{78}
$$

Defining $\widetilde{\alpha}_2(k_m, z) = 2\pi\delta(k_m)\alpha_2(z)$, as in the linear term, we have

$$
\alpha_2(z) = -\frac{1}{2} \left[\alpha_1^2(z) + \alpha_1'(z) \int_0^z dz'\alpha_1(z')\right]. \tag{79}
$$

This is the expression for $\alpha_2(z)$ in the 1D case, as desired.