Wave theory RTM

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By way of illustration, consider an inhomogeneous acoustic medium

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\vec{r})}\right] P(\vec{r}, \vec{r}_s, \omega) = A(\omega)\delta(\vec{r} - \vec{r}_s) \qquad (1)$$

Characterize the velocity field in terms of a reference, c_0, and a perturbation, $\alpha(\vec{r})$.

$$\frac{1}{c^2(\vec{r})} = \frac{1}{c_0^2}(1 - \alpha(\vec{r})) \qquad (2)$$

$$k = \frac{\omega}{c_0}$$

Green's theorem and seismic processing (wave separation or wave prediction)

$$\left[\nabla^2 + \frac{\omega^2}{c_0^2}\right] P(\vec{r}, \vec{r}_s, \omega) = k^2 \alpha(\vec{r}) P + A(\omega)\delta(\vec{r} - \vec{r}_s) \quad (3)$$

Define $\rho(\vec{r}, \omega) \equiv k^2 \alpha(\vec{r}) P + A(\omega)\delta(\vec{r} - \vec{r}_s)$ and equation 2 becomes

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right) P = \rho(\vec{r}, \omega) \quad (4)$$

Green's theorem tutorial (continued)

Introduce

$$(\nabla^2 + k^2) G_0 = \delta(\vec{r} - \vec{r}\,') \tag{5}$$

Equation 4 can be solved in terms of the solution of equation 5

$$\underset{Causal}{P(\vec{r}, \omega)} = \int_{\infty} d\vec{r}\,' \rho(\vec{r}\,', \omega) G_0^+(\vec{r}, \vec{r}\,', \omega) \tag{6}$$

$\vec{r}$ in ∞ (anywhere)

Green's theorem tutorial (continued)

Green's second identity

$$\int_V d\vec{r}\,'(P\nabla^2 G_0 - G_0\nabla^2 P) = \oint_S dS\,\hat{n}\cdot(P\nabla G_0 - G_0\nabla P) \tag{7}$$

Substituting $\nabla^2 P$ and $\nabla^2 G_0$ from equations 4 and 5 in equation 7

Green's theorem tutorial (continued)

$$\underset{\text{for } \vec{r} \text{ in } V}{P(\vec{r},\omega)} = \int_V d\vec{r}\,'\rho G_0 + \oint_S dS\,\hat{n}\cdot(P\nabla G_0 - G_0\nabla P) \quad (8)$$

Valid for any choice G_0 that satisfies equation 5.

- Different choices of solutions for G_0 will derive each of the Green's theorem applications we listed
- If we choose $G_0 = G_0^+$ then equation 8 becomes

$$\underset{\vec{r} \text{ in } V}{P} = \int_V d\vec{r}\,'\rho G_0^+ + \oint_S dS\,\hat{n}\cdot(P\nabla G_0^+ - G_0^+\nabla P) \quad (9)$$

Models for migration

Models for migration

The earliest wave equation migration pioneers viewed the backpropagation region as an infinite hemispherical half space with known mechanical properties, whose upper plane surface corresponded to the measurement surface, as in, *e.g.*, Schneider (1978) and Stolt (1978). See Fig. 1.

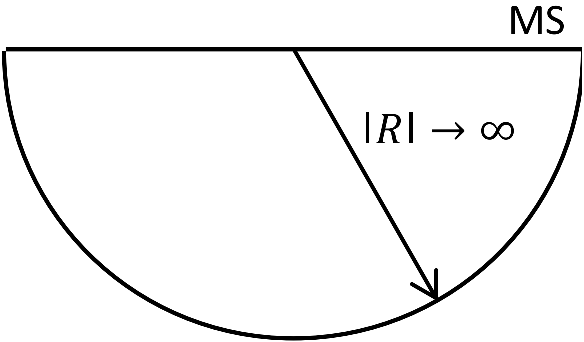

Figure 1: *The infinite hemispherical migration model. The measurement surface is denoted by MS.*

Models for migration

There are several problems with the infinite hemispherical migration model. That model assumes:
(1) that all subsurface properties beneath the measurement surface (MS) are known, and
(2) that an anticausal Green's function (*e.g.*, Schneider (1978)), with a Dirichlet boundary condition on the measurement surface, would allow measurements (MS) of the wave-field, P, on the upper plane surface of the hemisphere to determine the value of P within the hemispherical volume, V.

Models for migration

The first assumption leads to the contradiction that we have not allowed for anything that is unknown to be determined in our model, since everything within the closed and infinite hemisphere is assumed to be known. Within the infinite hemispherical model there is nothing and/or nowhere below the measurement surface where an unknown scattering point or reflection surface can serve to produce reflection data whose generating reflectors are initially unknown and being sought by the migration process.

Models for migration

The second assumption, in early infinite hemispherical wave equation migration, assumes that Green's theorem with wave-field measurements on the upper plane surface and using an anticausal Green's function satisfying a Dirichlet boundary condition can determine the wave-field within V. That conclusion assumes that the contribution from the lower hemispherical surface of S vanishes as the radius of the hemisphere goes to infinity. That is not the case.

Models for migration

The finite model for migration assumes that we know or can adequately estimate earth medium properties (velocity) down to the reflector we seek to image. The finite volume model assumes that beneath the sought after reflector the medium properties are and remain unknown. The "finite volume model" corresponds to the volume within which we assume the earth properties are known and within which we predict the wave-field from surface measurements.

Models for migration

We have moved away from the two issues of the infinite hemisphere model, *i.e.*, (1) the assumption we know the subsurface to all depths and (2) that the surface integral with an anticausal Green's function has no contribution to the field being predicted in the earth. The finite volume model takes away both assumptions. However, we are now dealing with a finite volume V, and with a surface S, consisting of upper surface S_U, lower surface S_L and walls, S_W (Fig. 2). We only have measurements on S_U.

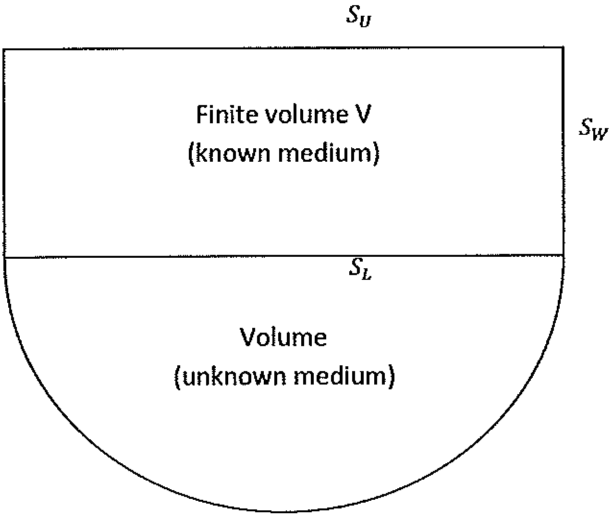

Figure 2: *A finite volume model*

Stolt FK migration from Green's theorem

Stolt FK migration from Green's theorem

Consider a 1D up-going plane wave-field $P = Re^{-ikz}$ propagating upward through the 1D homogeneous volume without sources between $z = a$ and $z = b$ (Fig. 3). The wave P inside V can be predicted from

$$P(z,\omega) = \Big|_{z'=a}^{b} \{P(z',\omega)\ \frac{dG_0}{dz'}(z,z',\omega) - G_0(z,z',\omega)\frac{dP}{dz'}(z',\omega)\}, \qquad (10)$$

with the Green's function, G_0, that satisfies

$$\left(\frac{d^2}{dz'^{\,2}} + k^2\right)\ G_0(z,z',\omega)\ =\ \delta(z-z'), \qquad (11)$$

for z and z' in V.

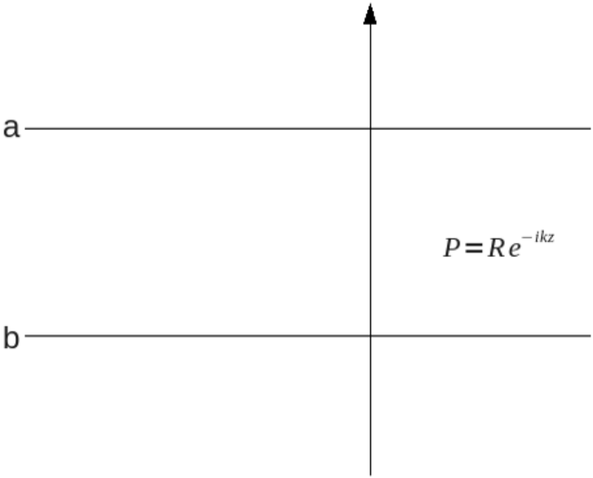

Figure 3: *1D up-going plane wave-field*

Stolt FK migration from Green's theorem

We can easily show that for an upgoing wave, $P = Re^{-ikz}$, that if one chooses $G_0 = G_0^+$ (causal, $e^{ik|z-z'|}/(2ik)$), the lower surface (i.e. $z' = b$) constructs P in V and the contribution from the upper surface vanishes. On the other hand, if we choose $G_0 = G_0^-$ (anticausal solution $e^{-ik|z-z'|}/(-2ik)$), then the upper surface $z = a$ constructs $P = Re^{-ikz}$ in V and there is no contribution from the lower surface $z' = b$.

Stolt FK migration from Green's theorem

This makes sense since information on the lower surface $z' = b$ will move with the upwave into the region between a and b, with a forward propagating causal Green's function, G_0^+. At the upper surface $z' = a$, the anticausal G_0^- will predict from an upgoing wave measured at $z' = a$, where the wave was previously and when it was moving up and deeper than $z' = a$.

Stolt FK migration from Green's theorem

Since in exploration seismology the reflection data is typically upgoing, once it is generated at the reflector, and we only have measurements at the upper surface $z' = a$, we choose an anticausal Green's function G_0^- in one-way wave back propagation in the finite volume model. If in addition we want to rid ourselves of the need for dP/dz' at $z' = a$ we can impose a Dirichlet boundary condition on G_0^-, to vanish at $z' = a$. The latter Green's function is labeled G_0^{-D}.

$$G_0^{-D} = -\frac{e^{-ik|z-z'|}}{2ik} - \left(-\frac{e^{-ik|z_I-z'|}}{2ik}\right) \tag{12}$$

where z_I is the image of z through $z' = a$.

Stolt FK migration from Green's theorem

It is easy to see that $z_I = 2a - z$ and that

$$P(z) \;=\; -\frac{dG_0^{-D}}{dz'}(z,z',\omega)\Big|_{z'=a} P(a) \;=\; e^{-ik(z-a)}P(a) \qquad (13)$$

in agreement with a simple Stolt FK phase shift for back propagating an up-field. Please note that $P(z,\omega) \;=\; -dG_0^{-D}/dz'(z,z',\omega)|_{z'=a}P(a,\omega)$ back propagates $P(z'=a,\omega)$, not G_0^{-D}. The latter thinking that G_0^{-D} back propagates data is a fundamental mistake/flaw in many seismic back propagation migration and inversion theories.

The Green's function

The Green's function

For one-way wave propagation the double downward continued data, D is

$$D(\text{at depth}) = \int_{S_s} \frac{\partial G_0^{-D}}{\partial z_s} \int_{S_g} \frac{\partial G_0^{-D}}{\partial z_g} D \, dS_g \, dS_s \quad , \quad (14)$$

where D in the integrand $= D(\text{on measurement surface})$, $\partial G_0^{-D}/\partial z_s =$ anticausal Green's function with Dirichlet boundary condition on the measurement surface, $s =$ shot, and $g =$ receiver.

The Green's function

For two-way wave double downward continuation:

$$D(\text{at depth}) = \int_{S_s} \left[\frac{\partial G_0^{DN}}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_0^{DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_0^{DN} \right\} dS_g \right.$$
$$\left. + \quad G_0^{DN} \frac{\partial}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_0^{DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_0^{DN} \right\} dS_g \right] dS_s \qquad (15)$$

where D in the integrands $= D$(on measurement surface). G_0^{DN} is *neither* causal nor anticausal. G_0^{DN} is not an *anti*causal Green's function; it is not the inverse or adjoint of any physical propagating Green's function. It is the Green's function needed for RTM.

The Green's function

G_0^{DN} is the Green's function for the model of the finite volume that vanishes along with its normal derivative on the lower surface and the walls. If we want to use the anticausal Green's function of the two-way propagation with Dirichlet boundary conditions at the measurement surface then we can do that, but we will need measurements at depth and on the vertical walls. To have the Green's function for two-way propagation that doesn't need data at depth and on the vertical sides/walls, that requires a non-physical Green's function that vanishes along with its derivative on the lower surface and walls.

Predicting a 2-way propagating wave from Green's theorem: A simple example

Predicting the receiver at depth

Green's theorem in 3D in the $(\mathbf{r}, \omega)$ domain to determine a wave-field, $P(\mathbf{r}, \omega)$ for $\mathbf{r}$ in V is given by

$$P(\mathbf{r}, \omega) = \int_V d\mathbf{r}' \, \rho(\mathbf{r}', \omega) G_0(\mathbf{r}, \mathbf{r}', \omega) + \oint_S dS' \, \mathbf{n} \cdot \left(P(\mathbf{r}', \omega) \nabla' G_0(\mathbf{r}, \mathbf{r}', \omega) - G_0(\mathbf{r}, \mathbf{r}', \omega) \nabla' P(\mathbf{r}', \omega) \right). \tag{16}$$

Predicting the receiver at depth for a 1D earth

In 1D in the slab $a \leq z \leq b$, (16) becomes

$$P(z,\omega) = \int_a^b dz'\,\rho(z',\omega)G_0(z,z',\omega) \tag{17}$$
$$+ \Big|_a^b \left(P(z',\omega)\frac{dG_0}{dz'}(z,z',\omega) - G_0(z,z',\omega)\frac{dP}{dz'}(z',\omega)\right).$$

Propagation for RTM in a 1D earth

Assuming no sources in the slab, the 1D homogeneous wave equation is

$$\left(\frac{d^2}{dz'^2}+k^2\right)P(z',\omega)=0,\ \text{for } z<z'<b \tag{18}$$

with general solution

$$P(z',\omega)=Ae^{ikz'}+Be^{-ikz'}\ \text{for } z<z'<b \tag{19}$$

where $k=\omega/c$. Given the conventions positive z' increasing downward and time dependence $e^{-i\omega t}$ in Fourier transforming from ω to t, the first term in (19) is a downgoing wave and the second term is an upgoing wave.

Propagation for RTM in a 1D earth

The equation for the corresponding Green's function is

$$\left(\frac{d^2}{dz'^{\,2}} + k^2\right) G_0(z, z', \omega) = \delta(z - z'), \qquad (20)$$

with causal and anticausal solutions

$$G_0^+(z, z', \omega) = \frac{1}{2ik} e^{ik|z-z'|}, \qquad (21)$$

$$G_0^-(z, z', \omega) = -\frac{1}{2ik} e^{-ik|z-z'|}. \qquad (22)$$

Propagation for RTM in a 1D earth

When the wave in the volume is not a one-way wave, but two-way, as is allowed in RTM then the choice of $G_0 = G_0^-$ will not remove the need for the lower boundary. In fact, no contribution of causal and anti-causal will eliminate the lower boundary contribution.

$$\left(\frac{d^2}{dz^2} + k^2\right) G_0 = \delta(z - z')$$

$$G_G^{In} = G_G^{Homogen.} + G_P^{In}$$

Propagation for RTM in a 1D earth

Eq. (17) suggests that the Green's function we need is such that it and its derivative vanish at $z' = b$. Such a Green's function removes the need for measurements at $z' = b$. Eq. (20) is an inhomogeneous differential equation with general solution $A_1 e^{ikz'} + B_1 e^{-ikz'} + G_0(z, z', \omega)$ where the first two terms are the general solution to the homogeneous differential equation and the third term is any particular solution to the inhomogeneous differential equation.

Propagation for RTM in a 1D earth

The choice $G_0(z, z', \omega) = G_0^+(z, z', \omega)$ gives the following general solution of (20):

$$G_0(z, z', \omega) = A_1 e^{ikz'} + B_1 e^{-ikz'} + \frac{1}{2ik} e^{ik|z-z'|}. \qquad (23)$$

Its derivative is

$$\frac{dG_0}{dz'}(z, z', \omega) = A_1 e^{ikz'} ik + B_1 e^{-ikz'}(-ik) + \frac{1}{2ik} e^{ik|z-z'|} ik \ \operatorname{sgn}(z - z')(-1). \qquad (24)$$

Now we impose boundary conditions in order to find A_1 and B_1.

Propagation for RTM in a 1D earth

The requirement that (23) and (24) vanish at $z' = b$ gives

$$0 = A_1 e^{ikb} + B_1 e^{-ikb} + \frac{1}{2ik} e^{ik \overbrace{|z-b|}^{b-z}}$$

$$0 = A_1 e^{ikb} ik + B_1 e^{-ikb}(-ik) + \frac{1}{2ik} e^{ik \overbrace{|z-b|}^{b-z}} ik \underbrace{\operatorname{sgn}(z-b)}_{-1}(-1)$$

$$A_1 e^{ikb} + B_1 e^{-ikb} = -\frac{1}{2ik} e^{ik(b-z)}$$

$$A_1 e^{ikb} - B_1 e^{-ikb} = -\frac{1}{2ik} e^{ik(b-z)}$$

$$2A_1 e^{ikb} = -2\frac{1}{2ik} e^{ik(b-z)}$$

Propagation for RTM in a 1D earth

$$A_1 = -\frac{1}{2ik} e^{-ikz}, \qquad (25)$$

$$2B_1 e^{-ikb} = 0$$

$$B_1 = 0. \qquad (26)$$

Substituting (25) and (26) into (23) gives

$$\begin{aligned} G_0(z, z', \omega) &= -\frac{1}{2ik} e^{-ikz} e^{ikz'} + \frac{1}{2ik} e^{ik|z-z'|} \\ &= -\frac{1}{2ik} \left(e^{-ik(z-z')} - e^{ik|z-z'|} \right). \qquad (27) \end{aligned}$$

Propagation for RTM in a 1D earth

Note the following about (27):

1. When $z' = b$, $G_0(z, b, \omega)$ vanishes:

$$G_0(z, b, \omega) = -\frac{1}{2ik}\left(e^{-ik(z-b)} - e^{ik\overbrace{|z-b|}^{b-z}}\right)$$
$$= -\frac{1}{2ik}\underbrace{\left(e^{-ik(z-b)} - e^{-ik(z-b)}\right)}_{0}.$$

2. When $a < z' < b$, $G_0(z, z', \omega)$ is neither causal nor anticausal due to the presence of the term $-1/(2ik)\, e^{-ik(z-z')}$.

Propagation for RTM in a 1D earth

3. When $z' = a$, $G_0(z,a,\omega)$ is the sum of anticausal and causal terms, but not in general or at any other depth.

$$
\begin{aligned}
G_0(z,a,\omega) &= -\frac{1}{2ik}\left(e^{-ik\overbrace{(z-a)}^{|z-a|}} - e^{ik|z-a|}\right) \\
&= \underbrace{-\frac{1}{2ik}e^{-ik|z-a|}}_{anticausal} + \underbrace{\frac{1}{2ik}e^{ik(z-a)}}_{causal}.
\end{aligned}
$$

Propagation for RTM in a 1D earth

4. Normally one uses Dirichlet or Neumann or Robin boundary conditions on the surface S (in our 1D case at both a and b). Constructing the Green's function (27) has enabled us to use both Dirichlet and Neumann boundary conditions on part of the surface S (in our 1D case only at a).

Propagation for RTM in a 1D earth

- The Green's function for two-way propagation that will eliminate the need for data at the lower surface of the closed Green's theorem surface is found by finding a general solution to the Green's function for the medium in the finite volume model and imposing both Dirichlet and Neumann boundary conditions at the lower surface.
- We confirm that the Green's function (27), when used in Green's theorem, will produce a two-way wave for $a < z < b$ with only measurements on the upper surface.
- Substituting (19), (27), and their derivatives into (17) gives $P(z, \omega) = Ae^{ikz} + Be^{-ikz}$, i.e., we recover the original two-way wave-field.
- The details are in Appendix A of Weglein et al. (2011).

Propagation for RTM in a 1D earth

A and B can be derived from the measured data $P(a)$ and $P'(a)$:

$$\begin{aligned}
P(a) &= Ae^{ika} + Be^{-ika} \\
P'(a) &= Ae^{ika}ik + Be^{-ika}(-ik) \\
\frac{P'(a)}{ik} &= Ae^{ika} - Be^{-ika} \\
2Ae^{ika} &= P(a) + \frac{P'(a)}{ik} \\
A &= e^{-ika}\frac{ikP(a) + P'(a)}{2ik}, \qquad (28) \\
2Be^{-ika} &= P(a) - \frac{P'(a)}{ik} \\
B &= e^{ika}\frac{ikP(a) - P'(a)}{2ik}. \qquad (29)
\end{aligned}$$

Propagation for RTM in a 1D earth

In a homogeneous medium the 3D equivalent of (18) is

$$(\nabla'^{2} + k^2)P(x', y', z', \omega) = 0,$$

where $k = \omega/c$. Fourier transforming over x' and y' gives

$$\left(\frac{d^2}{dz'^{2}} \underbrace{-k_{x'}^2 - k_{y'}^2 + \frac{\omega^2}{c^2}}_{\equiv k_{z'}^2} \right) P(k_{x'}, k_{y'}, z', \omega) = 0,$$

which looks like the 1D problem

$$\left(\frac{d^2}{dz'^{2}} + k_{z'}^2 \right) P(k_{x'}, k_{y'}, z', \omega) = 0,$$

Propagation for RTM in a 1D earth

with general solution

$$P(k_{x'}, k_{y'}, z', \omega) = Ae^{ik_{z'}z'} + Be^{-ik_{z'}z'}.$$

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