The first *wave equation migration* RTM with data consisting of primaries and internal multiples: theory and 1D examples

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SUMMARY

Reverse time migration (RTM) is the cutting-edge imaging method used in seismic exploration. In earlier RTM publications, density was often chosen and used to balance a medium with velocity variation, such that the acoustic impedance – the product of velocity and density – stays constant. Thus, normal incidence reflections from sharp boundaries are avoided. In order to be more complete, consistent, realistic, and predictive, general velocity and density variations (not constrained by impedance matching) are intentionally included in our study so that we can test the impact of reflections on the Green's theorem-based wave-theory RTM algorithms.

The major objectives of this article are to advance our understanding and to provide concepts, added imaging capabilities, and new algorithms for RTM. Although our objective of extracting useful subsurface information from recorded data is not different from that of well-known previous RTM publications, our method is different.

Although all current methods utilize the wave equation, the imaging condition they call upon, the time and space coincidence of up- and down-going waves, ultimately results in an asymptotic or ray based algorithm. Current RTM application doesn't correspond to predicting a source and receiver experiment at depth at $t = 0$. That imaging principle is the defining property of wave equation migration (WEM). The method of this paper represents WEM for RTM.

In this paper, we also have some very early and very positive news on the first wave equation migration RTM imaging tests, with a discontinuous reference medium and images that have the correct depth and amplitude (that is, producing the reflection coefficient at the correctly located target) with primaries and multiples in the data. There is "no cross talk" or any other artifacts as reported by other methods that seek to migrate data with primaries and multiples. That is an implementation and analysis of Weglein et al. (2011a,b) with primaries and internal multiples in the data.

INTRODUCTION

One of the major early objectives of Reverse Time Migration (RTM) is to obtain a better image of salt flanks through diving waves than is obtained by one way migration imaging through the complex overburden. The key new capability of the RTM method compared with one-way migration algorithms is to allow two-way wave propagation in the imaging procedure. This article follows closely the idea established in Weglein et al. (2011a,b): achieving a Green's function with vanishing Dirichlet and Neumann boundary conditions at the deeper boundary, to eliminate the need for measurements at depth.

As stated in Whitmore (1983); Baysal et al. (1983); Luo and Schuster (2004); Fletcher et al. (2006); Liu et al. (2009) and Vigh et al. (2009), accurate medium properties above the target are required for the RTM procedure discussed in this article. The major difference is that in most RTM algorithms in the industry, a smoothed version of the velocity is used in the imaging procedure to avoid reflections from the velocity model itself, while the exact velocity models (often discontinuous) are used in all three examples in this article. We adopt the notations of the aforementioned articles as much as possible while introducing some minor modifications to allow smooth expansion/extension into new territory.

The major contributions of this article are:

- It provides two methods to calculate the Green's function with vanishing Dirichlet and Neumann boundary conditions for an arbitrary 1D medium.
- It incorporates the density variation for Green's theorem RTM.
- It provides the finite-difference scheme for calculating the Green's function that vanishes at the deeper boundary.
- It provides a two-way propagation and downward continuation of wave fields, by using Green's function with double vanishing boundary conditions.

The following notations are worth mentioning at the beginning: G_0^+ and G_0^- are used to denote causal and anti-causal Green's functions, respectively. G_0^{DN} is used to denote the Green's function with vanishing Dirichlet and Neumann boundary conditions at the deeper boundary. $k = \omega/c_0$ where c_0 is the constant velocity of the reference medium, and ω is the angular frequency.

THEORY

Green's theorem wave-field prediction with density variation

First, let us assume the wave propagation problem in a volume V bounded by a shallower depth A and deeper depth B:

$$\left\{\frac{\partial}{\partial z'}\frac{1}{\rho(z')}\frac{\partial}{\partial z'}+\frac{\omega^2}{\rho(z')c^2(z')}\right\}D(z',\omega)=0, \qquad (1)$$

where $A \le z' \le B$ is the depth, and $\rho(z')$ and $c(z')$ are the density and velocity fields, respectively. In exploration seismology, we let the shallower depth A be the measurement surface where the seismic acquisition takes place. The volume V is the finite volume defined in the "finite volume model" for migration, the details of which can be found in Weglein et al. (2011a). We measure D at the measurement surface $z' = A$, and the objective is to predict D anywhere between the shallower surface and another surface with greater depth, $z' = B$.

This can be achieved via the solution of the wave-propagation equation in the same medium by an idealized impulsive source or Green's function:

$$\left\{ \frac{\partial}{\partial z'} \frac{1}{\rho(z')} \frac{\partial}{\partial z'} + \frac{\omega^2}{\rho(z')c^2(z')} \right\} G_0(z,z',\omega) = \delta(z-z'), \quad (2)$$

where z is the location of the source, and $A < z' < B$ and z increase in a downward direction. Abbreviating $G_0(z,z',\omega)$ as G_0, the solution for D in the interval $A < z' < B$ is given by Green's theorem:

$$D(z,\omega) = \frac{1}{\rho(z')} \left\{ D(z',\omega) \frac{\partial G_0}{\partial z'} - G_0 \frac{\partial D(z',\omega)}{\partial z'} \right\} \Bigg|_{z'=A}^{z'=B}, \quad (3)$$

where A and B are the shallower and deeper boundaries, respectively, of the volume to which the Green's theorem is applied. It is identical to equation (43) of Weglein et al. (2011a), except for the additional density contribution to the Green's theorem.

Note that in equation (3), the field values on the closed surface of the volume V are necessary for predicting the field value inside V. The surface of V contains two parts: the shallower portion $z' = A$ and the deeper portion $z' = B$. In seismic exploration, the need for data at $z' = B$ is not available. For example, one of the significant artifacts of the current RTM procedures is caused by this phenomenon: there are events necessary for accurate wave-field prediction that reach $z' = B$ but never return to $z' = A$. The solution, based on Green's theorem without any approximation, was first published in Weglein et al. (2011a) and Weglein et al. (2011b), the basic idea can be summarized as the following.

Since the wave equation is a second-order differential equation, its general solution has a great deal of freedom/flexibility. In other words, for a wave equation with a specific medium property, there are an infinite number of solutions. This freedom in choosing the Green's function has been taken advantage of in many seismic-imaging procedures. For example, the most popular choice in wave-field prediction is the physical solution G_0^+. In downward continuing an up-going wave field to a subsurface, the anti-causal solution G_0^- is often used.

If both G_0 and $\partial G_0/\partial z'$ vanish at the deeper boundary $z' = B$, where measurement is not available, then only the data at the shallower surface (i.e., the actual measurement surface) is needed in the calculation. We use G_0^{DN} to denote the Green's function with vanishing Dirichlet and Neumann boundary conditions at the deeper boundary.

Downward continuation of both source and receiver

The original Green's theorem in equation (3) is derived to downward continue the wave field (i.e., receivers) to the subsurface over a source-free region. It can also be used to downward continue the sources down to the subsurface by taking advantage of reciprocity: the recording is the same after the source and receiver locations are exchanged.

Assuming we have data on the measurement surface: $D(z_g, z_s)$ (its ω dependency is ignored), we can use $G_0^{DN}(z, z_g)$ to downward continue it from the receiver depth z_g to the target depth z:

$$D(z,z_s) = \frac{\frac{\partial D(z_g,z_s)}{\partial z_g} G_0^{DN}(z,z_g) - D(z_g,z_s) \frac{\partial G_0^{DN}(z,z_g)}{\partial z_g}}{\rho(z_g)}. \quad (4)$$

Taking the $\frac{\partial}{\partial z_s}$ operation on equation (4), we have a similar procedure to downward continue $\frac{D(z_g,z_s)}{\partial z_s}$ to the subsurface:

$$\frac{\partial D(z,z_s)}{\partial z_s} = \frac{\frac{\partial^2 D(z_g,z_s)}{\partial z_g \partial z_s} G_0^{DN}(z,z_g) - \frac{\partial D(z_g,z_s)}{\partial z_s} \frac{\partial G_0^{DN}(z,z_g)}{\partial z_g}}{\rho(z_g)}. \quad (5)$$

With equations (4) and (5), we downward continue the data D and its partial derivative over z_s to the subsurface location z. According to reciprocity, $D(z,z_s) = E(z_s,z)$, where $E(z_s,z)$ is resulted from exchanging the source and receiver locations in the experiment to generate D at the subsurface. The predicted data $E(z_s,z)$ can be considered as the recording of receiver at z_s for a source located at z.

For this predicted experiment, the source is located at depth z, according to the Green's theorem which is derived for a source-free region, we can downward continue the recording at z_s to any depth $Z \leq z$.

In seismic migration, we downward continue $E(z_s,z)$ to the same subsurface depth z with $G_0^{DN}(z,z_s)$ to have an experiment with coincident source and receiver:

$$\begin{aligned} E(z,z) &= \frac{\frac{\partial E(z_s,z)}{\partial z_s} G_0^{DN}(z,z_s) - E(z_s,z) \frac{\partial G_0^{DN}(z,z_s)}{\partial z_s}}{\rho(z_s)}, \\ &= \frac{\frac{\partial D(z,z_s)}{\partial z_s} G_0^{DN}(z,z_s) - D(z,z_s) \frac{\partial G_0^{DN}(z,z_s)}{\partial z_s}}{\rho(z_s)}. \end{aligned} \quad (6)$$

If the $z_s < z_g$ and there is no heterogeneity above z_s, the $\frac{\partial}{\partial z_s}$ operation on $D(z_g,z_s)$ is equivalent to multiplying $-ik$, in this case, equation (6) can be further simplified:

$$E(z,z) = -\frac{\frac{\partial G_0^{DN}(z,z_s)}{\partial z_s} + ikG_0^{DN}(z,z_s)}{\rho(z_s)} D(z,z_s). \quad (7)$$

NUMERICAL EXAMPLES

As an example, for a 2-reflector model (with an ideal impulsive source located at z_s, the depth of receiver is $z_g > z_s$, the

Depth Range	Velocity	Density
$(-\infty, a_1)$	c_0	ρ_0
(a_1, a_2)	c_1	ρ_1
(a_2, ∞)	c_2	ρ_2

Table 1: The properties of an acoustic medium with two reflectors, at depth a_1 and a_2.

geological model is listed in Table 1), the data and its various derivatives can be expressed as:

$$\begin{aligned}
D(z_g,z_s) &= \frac{\rho_0 x^{-1}}{2ik}\left\{y+\alpha y^{-1}\right\},\\
\frac{\partial D(z_g,z_s)}{\partial z_g} &= \frac{\rho_0}{2}x^{-1}\left\{y-\alpha y^{-1}\right\}\\
\frac{\partial D(z_g,z_s)}{\partial z_s} &= -\frac{\rho_0}{2}x^{-1}\left\{y+\alpha y^{-1}\right\},\\
\frac{\partial^2 D(z_g,z_s)}{\partial z_g \partial z_s} &= \frac{\rho_0 k}{2i}x^{-1}\left\{y-\alpha y^{-1}\right\}.
\end{aligned} \tag{8}$$

where $x=e^{ikz_s}$, $y=e^{ikz_g}$, $\sigma=e^{ikz}$, $\alpha=e^{ik(2a_1)}\left(R_1+(1-R_1^2)\beta\right)$, and $\beta=\sum\limits_{n=0}^{\infty}(-1)^n R_1^n R_2^{n+1} e^{ik_1(2n+2)[a_2-a_1]}$. And $R_1=\frac{c_1\rho_1-c_0\rho_0}{c_1\rho_1+c_0\rho_0}$, and $R_2=\frac{c_2\rho_2-c_1\rho_1}{c_2\rho_2+c_1\rho_1}$ are the reflection coefficients from geological boundaries.

Above the first reflector

For $z<a_1$, the boundary values of the Green's function are:

$$\begin{aligned}
G_0^{DN}(z,z_g) &= \rho_0\frac{e^{ik(z-z_g)}-e^{ik(z_g-z)}}{2ik} &&= \rho_0\frac{\sigma y^{-1}-\sigma^{-1}y}{2ik},\\
G_0^{DN}(z,z_s) & &&= \rho_0\frac{\sigma x^{-1}-\sigma^{-1}x}{2ik},\\
\frac{\partial G_0^{DN}(z,z_s)}{\partial z_g} & &&= \rho_0\frac{\sigma y^{-1}+\sigma^{-1}y}{-2},\\
\frac{\partial G_0^{DN}(z,z_s)}{\partial z_s} & &&= \rho_0\frac{\sigma x^{-1}+\sigma^{-1}x}{-2}.
\end{aligned} \tag{9}$$

After applying equation (8) into equation (7), we have:

$$E(z,z)=\frac{1+e^{ik(2a_1-2z)}\left(R_1+(1-R_1^2)\beta\right)}{2ik/\rho_0}. \tag{10}$$

The result above can be Fourier transformed into the time domain to have:

$$\frac{E(z,z,t)}{-\rho_0 c_0/2}=\begin{matrix} H(t)+R_1 H(t-t_1)+(1-R_1^2)\times \\ \sum\limits_{n=0}^{\infty}(-1)^n R_1^n R_2^{n+1} H(t-t_1-(2n+2)t_2) \end{matrix} \tag{11}$$

where, $t_1=\frac{2a_1-2z}{c_0}$ and $t_2=\frac{(a_2-a_1)}{c_1}$. Balancing out the $-\frac{\rho_0 c_0}{2}$ factor, the data after removing the direct wave is denoted as $\hat{D}(z,t)\overset{\Delta}{=}\frac{-2}{\rho_0 c_0}E(z,z,t)-H(t)$:

$$\begin{aligned}
\hat{D}(z,t) &= R_1 H(t-t_1)\\
&+(1-R_1^2)\sum_{n=0}^{\infty}(-1)^n R_1^n R_2^{n+1} H(t-t_1-(2n+2)t_2).
\end{aligned} \tag{12}$$

If we use the $t=0$ imaging condition, we have:

$$\hat{D}(z,t)=\begin{cases} 0 & \texttt{if } (z<a_1)\\ R_1 & \texttt{if } (z=a_1)\end{cases} \tag{13}$$

In other words, we obtained the image of the first reflector at its actual depth a_1 with its correct reflection coefficient as amplitude.

Between the first and second reflectors

For $a_1<z<a_2$, we have:

$$\begin{aligned}
G_0^{DN}(z,z_g) &= \frac{(R_1\lambda-\lambda^{-1})\mu+(\lambda-R_1\lambda^{-1})\mu^{-1}}{2ik_1(1+R_1)/\rho_1},\\
\frac{\partial G_0^{DN}(z,z_g)}{\partial z_g} &= \frac{(R_1\lambda-\lambda^{-1})\mu-(\lambda-R_1\lambda^{-1})\mu^{-1}}{2ik_1(1+R_1)/\rho_1},
\end{aligned} \tag{14}$$

where $\lambda=e^{ik_1(z-a_1)}$, $\mu=e^{ik(z_g-a_1)}$. Using equations (14) and (8), we have:The final result can be Fourier transformed into the time domain as:

$$\frac{E(z,z,t)}{-\rho_1 c_1/2}=\begin{matrix} H(t)+2\sum\limits_{n=1}^{\infty}(-1)^n R_1^n R_2^n H\left(t-\frac{2n(a_2-a_1)}{c_1}\right)\\ +\sum\limits_{n=0}^{\infty}(-1)^{n+1}R_1^{n+1}R_2^n H\left(t-\frac{2z+2na_2-2(n+1)a_1}{c_1}\right)\\ +\sum\limits_{n=0}^{\infty}(-1)^n R_1^n R_2^{n+1} H\left(t-\frac{2(n+1)a_2-2na_1-2z}{c_1}\right)\end{matrix}$$

Balancing out the $-\frac{\rho_1 c_1}{2}$ factor, the data after removing the direct wave is denoted as $\hat{D}(z,t)\overset{\Delta}{=}\frac{-2}{\rho_1 c_1}E(z,z,t)-H(t)$:

$$\hat{D}(z,t)=\left\{\begin{matrix} 2\sum\limits_{n=1}^{\infty}(-1)^n R_1^n R_2^n H\left(t-\frac{2n(a_2-a_1)}{c_1}\right)\\ +\sum\limits_{n=0}^{\infty}(-1)^{n+1}R_1^{n+1}R_2^n H\left(t-\frac{2z+2na_2-2(n+1)a_1}{c_1}\right)\\ +\sum\limits_{n=0}^{\infty}(-1)^n R_1^n R_2^{n+1} H\left(t-\frac{2(n+1)a_2-2na_1-2z}{c_1}\right)\end{matrix}\right.$$

and after taking the $t=0$ imaging condition, we have:

$$\hat{D}(z,t)=\begin{cases} -R_1 & \texttt{if } (z=a_1)\\ 0 & \texttt{if } (a_1<z<a_2)\\ R_2 & \texttt{if } (z=a_2)\end{cases} \tag{15}$$

Note that in the previous section, i.e., to image above the first reflector at a_1, we obtain the amplitude R_1 when z approach a_1 from above. In this section we image below the first reflector at a_1, the amplitude of the image is $-R_1$ when z approaches a_1 from below, as it should.

Below the second reflector

For $z > a_1$, the boundary value of the Green's function is:

$$G_0^{DN}(z,z_g) = \frac{\begin{array}{c}\left[\nu^{-1}(R_2\lambda-\lambda^{-1})+R_1\nu(\lambda-R_2\lambda^{-1})\right]\mu \\ +\left[R_1\nu^{-1}(R_2\lambda-\lambda^{-1})+\nu(\lambda-R_2\lambda^{-1})\right]\mu^{-1}\end{array}}{2ik_2(1+R_1)(1+R_2)/\rho_2},$$

where $\lambda \equiv e^{ik_2(z-a_2)}$, $\mu \equiv e^{ik(z_g-a_1)}$, and $\nu \equiv e^{ik_1(a_2-a_1)}$, $k_1 = \omega/c_1$.

The final downward continuation result can be expressed as:

$$E(z,z) = \frac{\rho_2}{2ik_2}\left\{\begin{array}{l} 1-R_2e^{ik_2(2z-2a_2)}+(1-R_2^2)e^{ik_2(2z-2a_2)}\times \\ \sum_{n=0}^{\infty}(-1)^{n+1}R_1^{n+1}R_2^n e^{ik_1(2n+2)(a_2-a_1)} \end{array}\right. .$$

The time domain counterpart of the equation above is:

$$E(z,z,t) = -\frac{\rho_2c_2}{2}\left\{\begin{array}{l} H(t)-R_2H\left(t-\frac{2z-2a_2}{c_2}\right) \\ +(1-R_2^2)H\left(t-\frac{2z-2a_2}{c_2}-\frac{(2n+2)(a_2-a_1)}{c_1}\right) \end{array}\right.$$

Balancing out the $-\frac{\rho_2c_2}{2}$ factor, the data after removing the direct wave is denoted as $\hat{D}(z,t) \overset{\Delta}{=} \frac{-2}{\rho_2c_2}E(z,z,t)-H(t)$:

$$\hat{D}(z,t) = \left\{\begin{array}{l} -R_2H\left(t-\frac{2z-2a_2}{c_2}\right) \\ +(1-R_2^2)H\left(t-\frac{2z-2a_2}{c_2}-\frac{(2n+2)(a_2-a_1)}{c_1}\right) \end{array}\right.$$

and after taking the $t=0$ imaging condition, we have:

$$\hat{D}(z,t) = \left\{\begin{array}{ll} -R_2 & \texttt{if } (z=a_2) \\ 0 & \texttt{if } (a_2<z) \end{array}\right. \tag{16}$$

Note that in the previous section, i.e., to image between the first and second reflectors, we obtain the amplitude R_2 when z approach a_2 from above. In this section we image below the second reflector at a_2, the amplitude of the image is $-R_2$ when z approaches a_2 from below, as it should.

CONCLUSIONS

A general and efficient procedure to compute the Green's function with vanishing Dirichlet and Neumann boundary conditions has been derived for a 1D medium of arbitrary complexity, and its effectiveness has been demonstrated with numerical examples that accurately predict the up-going and down-going wave field at depth using only the data on the shallower measurement surface. The density contribution to the Green's theorem and Green's function is accurately studied to better understand its role in imaging. In order to generalize the idea in this paper to a multidimensional earth, a finite-difference scheme is derived and validated by comparison with an analytic benchmark.

We also have reported some very early and very positive news on the first wave equation migration RTM imaging tests, with a discontinuous reference medium and images that have the correct depth and amplitude (that is, producing the reflection coefficient at the correctly located target) with primaries and multiples in the data. That is an implementation and analysis of Weglein et al. (2011a,b) with primaries and multiples in the data. There are no artifacts, "cross-talk" or other problems reported in the literature with other methods for migrating primaries and multiples for imaging and/or illumination (Weglein, 2014).

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