Direct Inversion and FWI
A key-note address

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March 30 - April 1, 2015
• Modeling
• Inversion

Direct

Indirect
Direct

\[ ax^2 + bx + c = 0 \]

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Indirect

Search for \( x \) such that \((ax^2 + bx + c)^2\) is a minimum
Indicators of “indirect”

• model matching

• objective/cost functions

• search algorithm

• iterative linear “inversion”

• necessary and not sufficient conditions, e.g., CIG flatness
There’s a role for direct and indirect methods in practical real world application.
Direct Forward and Direct Inverse

\[ L_0 G_0 = \delta \quad L G = \delta \]

\[ V = L_0 - L \quad \psi_s = G - G_0 \]

Relationship

\[ G = G_0 + G_0 VG \quad (1) \]

An operator identity that follows from

\[ L^{-1} = L_0^{-1} + L_0^{-1} (L_0 - L) L^{-1} \]

Modeling

\[ L \rightarrow G \quad L_0, V \rightarrow G \]
Direct Forward and Direct Inverse

Relationship
\[ G = G_0 + G_0 VG \]  \hspace{1cm} (1)

Modeling
\[ G = G_0 + G_0 VG_0 + G_0 VG_0 VG_0 + \cdots \]  \hspace{1cm} (2)

Form
\[ G - G_0 = S = ar + ar^2 + \cdots = \frac{ar}{1-r} \]

with
\[ a = G_0 \]
\[ r = VG_0 \]
\[ S = S_1 + S_2 + S_3 + \cdots \]
Direct Forward and Direct Inverse

\[ S = S_1 + S_2 + S_3 + \cdots \]

\[ S = \frac{ar}{1 - r} \]

Solve for \( r \)

\[ r = \frac{S/a}{1 + S/a} = S/a - (S/a)^2 + (S/a)^3 + \cdots \]

\[ = r_1 + r_2 + r_3 + \cdots \]
Direct Forward and Direct Inverse

\[ S = (G - G_0)_{ms} = Data \]

Forward \( S \) in terms of \( V \), inverse \( V \) in terms of \( S \)

\[ V = V_1 + V_2 + \cdots \quad (3) \]

where \( V_n \) is the portion of \( V \), \( n \)-th order in the data

(2) is the forward series;

(3) is the inverse series.
• The relationship (2) provides a Geometric forward series rather than a Taylor series.

• In general, a Taylor series doesn’t have an inverse series; however, a Geometric series has an inverse series.

• All conventional current mainstream inversion, including iterative linear inversion and FWI, are based on a Taylor series concept.

• Solving a forward problem in an inverse sense is not the same as solving an inverse problem directly.
• The $r_1$, $r_2$, … equations generalize

$$
\begin{align*}
  r &= S_{a} - \left( S_{a} \right)^2 + \left( S_{a} \right)^3 + \cdots \\
  &= r_1 + r_2 + r_3 + \cdots \\

  G_0 V_1 G_0 &= D \\
  G_0 V_2 G_0 &= -G_0 V_1 G_0 V_1 G_0 \\
  G_0 V_3 G_0 &= -G_0 V_1 G_0 V_1 G_0 V_1 G_0 + \cdots
\end{align*}
$$
(2D) Elastic (isotropic)

\[
L\tilde{u} = \begin{pmatrix} f_x \\ f_z \end{pmatrix} \quad \hat{L} \begin{pmatrix} \phi^p \\ \phi^s \end{pmatrix} = \begin{pmatrix} F^p \\ F^s \end{pmatrix}
\]

where

\[
L = \rho \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x \gamma \partial_x + \partial_z \mu \partial_z & \partial_x (\gamma - 2 \mu) \partial_z + \partial_z \mu \partial_x \\ \partial_z (\gamma - 2 \mu) \partial_x + \partial_x \mu \partial_z & \partial_z \gamma \partial_z + \partial_x \mu \partial_x \end{pmatrix}
\]

\[
L_0 = \rho_0 \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0 \partial_x^2 + \mu_0 \partial_z^2 & (\gamma_0 - \mu_0) \partial_x \partial_z \\ (\gamma_0 - \mu_0) \partial_x \partial_z & \mu_0 \partial_x^2 + \gamma_0 \partial_z^2 \end{pmatrix}
\]

\[
V \equiv L_0 - L
\]

\[
\begin{pmatrix} a_\rho \omega^2 + \alpha_0^2 \partial_x a_\gamma \partial_x + \beta_0^2 \partial_z a_\mu \partial_z & \partial_x (\alpha_0^2 a_\gamma - 2 \beta_0^2 a_\mu) \partial_z + \beta_0^2 \partial_z a_\mu \partial_x \\ \partial_z (\alpha_0^2 a_\gamma - 2 \beta_0^2 a_\mu) \partial_x + \beta_0^2 \partial_x a_\mu \partial_z & a_\rho \omega^2 + \alpha_0^2 \partial_z a_\gamma \partial_z + \beta_0^2 \partial_x a_\mu \partial_x \end{pmatrix}
\]

\[
a_\rho \equiv \frac{\rho}{\rho_0} - 1, a_\gamma \equiv \frac{\gamma}{\gamma_0} - 1, a_\mu \equiv \frac{\mu}{\mu_0} - 1
\]
The forward problem

\[ \hat{G} - \hat{G}_0 = \hat{G}_0 \hat{V} \hat{G} = \hat{G}_0 \hat{V} \hat{G}_0 + \hat{G}_0 \hat{V} \hat{G}_0 \hat{V} \hat{G}_0 + \cdots \] (4)

\[
\begin{pmatrix}
\hat{D}^{PP} & \hat{D}^{PS} \\
\hat{D}^{SP} & \hat{D}^{SS}
\end{pmatrix} =
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\begin{pmatrix}
\hat{V}^{PP} & \hat{V}^{PS} \\
\hat{V}^{SP} & \hat{V}^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\begin{pmatrix}
\hat{V}^{PP} & \hat{V}^{PS} \\
\hat{V}^{SP} & \hat{V}^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hat{V}^{PP} & \hat{V}^{PS} \\
\hat{V}^{SP} & \hat{V}^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\]

\[+ \cdots \]
The inverse problem (solving for $r$ in terms of $S$)

\[ \hat{V}^{PP} = \hat{V}_1^{PP} + \hat{V}_2^{PP} + \hat{V}_3^{PP} + \cdots \]
\[ \hat{V}^{PS} = \hat{V}_1^{PS} + \hat{V}_2^{PS} + \hat{V}_3^{PS} + \cdots \]
\[ \hat{V}^{SP} = \hat{V}_1^{SP} + \hat{V}_2^{SP} + \hat{V}_3^{SP} + \cdots \]
\[ \hat{V}^{SS} = \hat{V}_1^{SS} + \hat{V}_2^{SS} + \hat{V}_3^{SS} + \cdots \]
\[ \hat{V}^{PP} = -\nabla^2 a_\gamma - \frac{\omega^2}{\alpha_0^2} (a_\rho \partial_x^2 + \partial_z a_\rho \partial_z) \frac{1}{\nabla^2} - \left[ -2\partial_z^2 a_{\mu u} \partial_x^2 - 2\partial_x^2 a_{\mu u} \partial_z^2 + 4\partial_x^2 \partial_z a_{\mu u} \partial_z \right] \frac{1}{\nabla^2} \]

\[ \hat{V}^{PS} = \frac{\alpha_0^2}{\beta_0^2} \left[ \frac{\omega^2}{\alpha_0^2} (\partial_x a_\rho \partial_z - \partial_z a_\rho \partial_x) + 2\partial_x \partial_z a_{\mu u} (\partial_z^2 - \partial_x^2) - 2(\partial_z^2 - \partial_x^2) a_{\mu u} \partial_z \partial_x \right] \frac{1}{\nabla^2} \]

\[ \hat{V}^{SP} = -\left[ \frac{\omega^2}{\alpha_0^2} (\partial_x a_\rho \partial_z - \partial_z a_\rho \partial_x) + 2\partial_x \partial_z a_{\mu u} (\partial_z^2 - \partial_x^2) - 2(\partial_z^2 - \partial_x^2) a_{\mu u} \partial_z \partial_x \right] \frac{1}{\nabla^2} \]

\[ \hat{V}^{SS} = -\frac{\alpha_0^2}{\beta_0^2} \left[ \frac{\omega^2}{\alpha_0^2} (a_\rho \partial_x^2 + \partial_z a_\rho \partial_z) + (\partial_z^2 - \partial_x^2) a_{\mu u} (\partial_z^2 - \partial_x^2) + 4\partial_x \partial_z a_{\mu u} \partial_x \partial_z \right] \frac{1}{\nabla^2} \]

\[ a_{\mu u} = \frac{\mu - \mu_0}{\gamma_0} = \frac{\beta_0^2}{\alpha_0^2} a_{\mu} \]
\[
\begin{pmatrix}
\hat{D}^{PP} & \hat{D}^{PS} \\
\hat{D}^{SP} & \hat{D}^{SS}
\end{pmatrix}
= \begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\begin{pmatrix}
\hat{V}_1^{PP} & \hat{V}_1^{PS} \\
\hat{V}_1^{SP} & \hat{V}_1^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\]

\[
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\begin{pmatrix}
\hat{V}_2^{PP} & \hat{V}_2^{PS} \\
\hat{V}_2^{SP} & \hat{V}_2^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\]

\[
= - \begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\begin{pmatrix}
\hat{V}_1^{PP} & \hat{V}_1^{PS} \\
\hat{V}_1^{SP} & \hat{V}_1^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\begin{pmatrix}
\hat{V}_1^{PP} & \hat{V}_1^{PS} \\
\hat{V}_1^{SP} & \hat{V}_1^{SS}
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0^P & 0 \\
0 & \hat{G}_0^S
\end{pmatrix}
\]

\[
\vdots
\]

Hence, for \( \hat{V}^{PP} = \hat{V}_1^{PP} + \hat{V}_2^{PP} + \hat{V}_3^{PP} + \cdots \) any one of the four matrix elements of \( V \) requires

\[
\begin{pmatrix}
\hat{D}^{PP} & \hat{D}^{PS} \\
\hat{D}^{SP} & \hat{D}^{SS}
\end{pmatrix}
\]
• $\hat{D}^{PP}$ can be determined independently in terms of
  \[
  \begin{pmatrix}
  \hat{V}^{PP} & \hat{V}^{PS} \\
  \hat{V}^{SP} & \hat{V}^{SS}
  \end{pmatrix}
  \]
• but $\hat{V}^{PP}$ or $\hat{V}^{PS}, \hat{V}^{SP}, \hat{V}^{SS}$ requires
  \[
  \begin{pmatrix}
  \hat{D}^{PP} & \hat{D}^{PS} \\
  \hat{D}^{SP} & \hat{D}^{SS}
  \end{pmatrix}
  \]
• That’s what the general relationship
  \[
  G = G_0 + G_0 VG
  \]
  requires.
• A direct non-linear solution
  order by order in the data matrix
  \[
  \begin{pmatrix}
  \hat{D}^{PP} & \hat{D}^{PS} \\
  \hat{D}^{SP} & \hat{D}^{SS}
  \end{pmatrix}
  \]
• ISS is not iterative linear inversion.

• Iterative linear starts with \( G_0 V_1 G_0 = D \) (7)

  solves for \( V_1 \), changes the reference medium, finds a new \( L_0 \) and \( G_0 \) (and require generalized inversions of noisy bandlimited data dependent operators).

• To find \( V_1' \), \( G_0' V_1' G_0' = D' = (G - G_0')_{ms} \)

\[
L_0' = L_0 - V_1 \\
L_0' G_0' = \delta
\]
• The problem is much more serious than a different approach to solve $G_0V_1G_0 = D$ (7) for $V_1$.

• If (7) is our entire basic theory, you can mistakenly think that
  
  $\hat{D}^{PP} = \hat{G}^P_0\hat{V}_1^{PP}\hat{G}^P_0$

  is sufficient to update
  
  $\hat{D}^{PP'} = \hat{G}^{P'}_0\hat{V}_1^{PP'}\hat{G}^{P'}_0$

• That step loses contact with and violates

  $G = G_0 + G_0VG$

for the elastic wave equation.
1. That is, it violates the basic relationship between changes in a medium, $V$ and changes in the wavefield, $G-G_0$ for the simplest elastic model.

2. This direct inverse method gives you a platform for FWI and communicates when a “FWI” method should work in principle.

3. Iteratively inverting multi-component data has the correct data but doesn’t corresponds to a direct inverse algorithm.

4. To honor $G=G_0+G_0VG$, you need both the data and the algorithm that direct inverse prescribes.

A.B. Weglein and Jinlong Yang

There’s a role for direct and indirect methods in practical real world application.
4D Application
Discrimination between pressure and fluid saturation using direct non-linear inversion method: an application to time-lapse seismic data

Haiyan Zhang, Arthur B. Weglein, Robert Keys, Douglas Foster and Simon Shaw
Statement of the problem

• Distinguishing pressure changes from reservoir fluid changes is difficult with conventional seismic time-lapse attributes.

• Pressure changes or fluid changes?
  – shear modulus sensitive to pressure changes
  – Vp/Vs sensitive to fluid changes

• A direct non-linear inversion method may be useful for accomplishing this goal.
### Introduction of the method

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<th>Time-lapse seismic monitoring</th>
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<td>Initial reservoir condition</td>
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<td>Actual medium $L$</td>
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<td>Earth property changes in space $V = L_0 - L$</td>
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<td>Reference wave field $G_0$</td>
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Conclusion

• Comparing the first and second order algorithms in estimating shear modulus and Vp/Vs contrasts.

• Second order direct inverse was able to distinguish pressure changes from fluid changes.
References

Thank you to the organizers of the Abu - Dhabi 2015 SEG Workshop on FWI for this opportunity to present and participate.
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