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Green’s theorem tutorial

Green’s theorem-derived methods

Wavefield separation: no subsurface information required

- Predicting the reference wave and the scattered wavefield (the reflection data) from the total wavefield
- Predicting the source signature and radiation pattern
- Source and receiver deghosting
Wavefield prediction for migration: subsurface information required

- One-way-wave pre-stack Stolt FK migration
- Two-way-wave wave-equation-migration RTM
Green’s theorem and seismic processing
(wave separation or wave prediction)

By way of illustration, consider an inhomogeneous acoustic medium

\[ \left[ \nabla^2 + \frac{\omega^2}{c^2(\vec{r})} \right] P(\vec{r}, \vec{r}_s, \omega) = A(\omega)\delta(\vec{r} - \vec{r}_s) \]  \hspace{1cm} (1)

Characterize the velocity field in terms of a reference, \( c_0 \), and a perturbation, \( \alpha(\vec{r}) \).

\[ \frac{1}{c^2(\vec{r})} = \frac{1}{c_0^2}(1 - \alpha(\vec{r})) \]  \hspace{1cm} (2)

\[ k = \frac{\omega}{c_0} \]
Green’s theorem and seismic processing (wave separation or wave prediction)

\[
\left( \nabla^2 + \frac{\omega^2}{c_0^2} \right) P(\vec{r}, \vec{r}_s, \omega) = k^2 \alpha(\vec{r})P + A(\omega)\delta(\vec{r} - \vec{r}_s) \quad (3)
\]

Define \( \rho(\vec{r}, \omega) \equiv k^2 \alpha(\vec{r})P + A(\omega)\delta(\vec{r} - \vec{r}_s) \) and equation 2 becomes

\[
\left( \nabla^2 + \frac{\omega^2}{c_0^2} \right) P = \rho(\vec{r}, \omega) \quad (4)
\]
Introduce

\[(\nabla^2 + k^2) G_0 = \delta(\vec{r} - \vec{r}')\]  \hspace{1cm} (5)

Equation 4 can be solved in terms of the solution of equation 5

\[P(\vec{r}, \omega) = \int_{\text{Causal}} d\vec{r}' \rho(\vec{r}', \omega) G_0^+(\vec{r}, \vec{r}', \omega)\]  \hspace{1cm} (6)

\[\vec{r} \text{ in } \infty \text{ (anywhere)}\]
Green’s second identity

\[ \int_V d\mathbf{r}' (P \nabla^2 G_0 - G_0 \nabla^2 P) = \int_S dS \, \hat{n} \cdot (P \nabla G_0 - G_0 \nabla P) \]  

(7)

Substituting \( \nabla^2 P \) and \( \nabla^2 G_0 \) from equations 4 and 5 in equation 7
Green's theorem tutorial (continued)

\[ P(\mathbf{r}, \omega) = \int_V d\mathbf{r}' \rho G_0 + \oint_S dS \mathbf{n} \cdot (P \nabla G_0 - G_0 \nabla P) \quad (8) \]

Valid for any choice \( G_0 \) that satisfies equation 5.

- Different choices of solutions for \( G_0 \) will derive each of the Green's theorem applications we listed
- If we choose \( G_0 = G_0^+ \) then equation 8 becomes

\[ P_{\mathbf{r} \text{ in } V} = \int_V d\mathbf{r}' \rho G_0^+ + \oint_S dS \mathbf{n} \cdot (P \nabla G_0^+ - G_0^+ \nabla P) \quad (9) \]
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The earliest wave equation migration pioneers viewed the backpropagation region as an infinite hemispherical half space with known mechanical properties, whose upper plane surface corresponded to the measurement surface, as in, e.g., Schneider (1978) and Stolt (1978). See Fig. 1.
Figure 1: The infinite hemispherical migration model. The measurement surface is denoted by MS.
There are several problems with the infinite hemispherical migration model. That model assumes: (1) that all subsurface properties beneath the measurement surface (MS) are known, and (2) that an anticausal Green’s function (e.g., Schneider (1978)), with a Dirichlet boundary condition on the measurement surface, would allow measurements (MS) of the wave-field, $P$, on the upper plane surface of the hemisphere to determine the value of $P$ within the hemispherical volume, $V$. 
The first assumption leads to the contradiction that we have not allowed for anything that is unknown to be determined in our model, since everything within the closed and infinite hemisphere is assumed to be known. Within the infinite hemispherical model there is nothing and/or nowhere below the measurement surface where an unknown scattering point or reflection surface can serve to produce reflection data whose generating reflectors are initially unknown and being sought by the migration process.
The second assumption, in early infinite hemispherical wave equation migration, assumes that Green’s theorem with wave-field measurements on the upper plane surface and using an anticausal Green’s function satisfying a Dirichlet boundary condition can determine the wave-field within $V$. That conclusion assumes that the contribution from the lower hemispherical surface of $S$ vanishes as the radius of the hemisphere goes to infinity. That is not the case.
Models for migration

The finite model for migration assumes that we know or can adequately estimate earth medium properties (velocity) down to the reflector we seek to image. The finite volume model assumes that beneath the sought after reflector the medium properties are and remain unknown. The “finite volume model” corresponds to the volume within which we assume the earth properties are known and within which we predict the wave-field from surface measurements.
Models for migration

We have moved away from the two issues of the infinite hemisphere model, i.e., (1) the assumption we know the subsurface to all depths and (2) that the surface integral with an anticausal Green’s function has no contribution to the field being predicted in the earth. The finite volume model takes away both assumptions. However, we are now dealing with a finite volume $V$, and with a surface $S$, consisting of upper surface $S_U$, lower surface $S_L$ and walls, $S_W$ (Fig. 2). We only have measurements on $S_U$. 
Figure 2: A finite volume model
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Consider a 1D up-going plane wave-field \( P = Re^{-ikz} \) propagating upward through the 1D homogeneous volume without sources between \( z = a \) and \( z = b \) (Fig. 3). The wave \( P \) inside \( V \) can be predicted from

\[
P(z, \omega) = \left| \left. \int_{z' = a}^{b} \{ P(z', \omega) \frac{dG_0}{dz'}(z, z', \omega) - G_0(z, z', \omega) \frac{dP}{dz'}(z', \omega) \} \right|_{z'}
\]

(10)

with the Green’s function, \( G_0 \), that satisfies

\[
\left( \frac{d^2}{dz'^2} + k^2 \right) G_0(z, z', \omega) = \delta(z - z'),
\]

(11)

for \( z \) and \( z' \) in \( V \).
Figure 3: 1D up-going plane wave-field
We can easily show that for an upgoing wave, 
\[ P = Re^{-ikz}, \]
that if one chooses \( G_0 = G_0^+ \) (causal, 
\[ e^{ik|z-z'|}/(2ik) \]), the lower surface (i.e. \( z' = b \)) constructs \( P \) in \( V \) and the contribution from the upper surface vanishes. On the other hand, if we choose \( G_0 = G_0^- \) (anticausal solution \( e^{-ik|z-z'|}/(-2ik) \)), then the upper surface \( z = a \) constructs \( P = Re^{-ikz} \) in \( V \) and there is no contribution from the lower surface \( z' = b \).
This makes sense since information on the lower surface \( z' = b \) will move with the upwave into the region between \( a \) and \( b \), with a forward propagating causal Green’s function, \( G_0^+ \). At the upper surface \( z' = a \), the anticausal \( G_0^- \) will predict from an upgoing wave measured at \( z' = a \), where the wave was previously and when it was moving up and deeper than \( z' = a \).
Stolt FK migration from Green’s theorem

Since in exploration seismology the reflection data is typically upgoing, once it is generated at the reflector, and we only have measurements at the upper surface \( z' = a \), we choose an anticausal Green’s function \( G_0^- \) in one-way wave back propagation in the finite volume model. If in addition we want to rid ourselves of the need for \( dP/dz' \) at \( z' = a \) we can impose a Dirichlet boundary condition on \( G_0^- \), to vanish at \( z' = a \). The latter Green’s function is labeled \( G_0^{-D} \).

\[
G_0^{-D} = - \frac{e^{-ik|z-z'|}}{2ik} - \left( - \frac{e^{-ik|z_I-z'|}}{2ik} \right)
\]

(12)

where \( z_I \) is the image of \( z \) through \( z' = a \).
Stolt FK migration from Green's theorem

It is easy to see that \( z_I = 2a - z \) and that

\[
P(z) = -\frac{dG_0^-}{dz'}(z, z', \omega) \bigg|_{z'=a} \quad P(a) = e^{-ik(z-a)}P(a)
\]

in agreement with a simple Stolt FK phase shift for back propagating an up-field. Please note that

\[
P(z, \omega) = -dG_0^-/dz'(z, z', \omega) \bigg|_{z'=a} P(a, \omega) \quad \text{back propagates} \quad P(z' = a, \omega), \text{ not } G_0^-.
\]

The latter thinking that \( G_0^- \) back propagates data is a fundamental mistake/flaw in many seismic back propagation migration and inversion theories.
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The Green’s function

For one-way wave propagation the double downward continued data, $D$ is

$$D(\text{at depth}) = \int_{S_s} \frac{\partial G_0^{-D}}{\partial z_s} \int_{S_g} \frac{\partial G_0^{-D}}{\partial z_g} D \, dS_g \, dS_s, \quad (14)$$

where $D$ in the integrand $= D(\text{on measurement surface})$, $\partial G_0^{-D}/\partial z_s =$ anticausal Green’s function with Dirichlet boundary condition on the measurement surface, $s =$ shot, and $g =$ receiver.
The Green’s function

For two-way wave double downward continuation:

\[
D(\text{at depth}) = \int_{S_s} \left[ \frac{\partial G_{0DN}}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_{0DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_{0DN} \right\} dS_g \right] dS_s \tag{15}
\]

where \( D \) in the integrands = \( D(\text{on measurement surface}) \). \( G_{0DN} \) is neither causal nor anticausal. \( G_{0DN} \) is not an anticausal Green’s function; it is not the inverse or adjoint of any physical propagating Green’s function. It is the Green’s function needed for RTM. Details can be found in Weglein et al. (2011a,b).
The Green’s function

This Report is an attachment in this year’s Annual Report.

The first wave theory RTM, examples with a layered medium, predicting the source and receiver at depth and then imaging, providing the correct location and reflection amplitude at every depth location, and where the data includes primaries and all internal multiples

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Figure 4: *Imaging with primaries and internal multiples.*
References I

