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A comparison of the inverse scattering series direct non-linear inversion and the iterative linear inversion for parameter estimation across a single horizontal reflector

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ABSTRACT

The inverse scattering series (ISS) can achieve all seismic processing objectives directly without requiring any subsurface information. In addition, there are distinct isolated task-specific subseries that derived from the ISS, which can perform free-surface multiple removal, internal multiple removal, depth imaging, parameter estimation, and Q compensation, each without subsurface information. For a plane wave normal incident on an 1D acoustic medium, a single measured pressure wave is the input data. We examine the difference between the industry standard approach, the iterative linear inverse and the direct inverse represented by the ISS parameter estimation subseries. The iterative approach shown in this study is allowed to avoid all practical issues, e.g., the numerical issues and issues related to band-limited noisy data and the choice of different generalized inverses at each step. In addition, in this paper, the iterative inverse method has been given the advantage (not available in practice) of an analytic and exact Frechet derivative/sensitivity matrix. Providing the analytic and exact Frechet derivative allows the iterative linear inverse method in this comparison to avoid important issues that it faces in real world application. However, the ISS method performs as it does in practice with an analytic and unchanged inverse at every step. Numerical tests demonstrate that when the velocity contrast is small, both methods converge although the ISS inversion method converges faster than the iterative inversion method. When the velocity contrast increases, the iterative linear inverse can simply break down, while the ISS inverse always remains operational and converges to the correct result. Therefore, for the simplest situation, the iterative linear inversion is not equivalent to the ISS direct non-linear solution. In general, the differences are much greater, not just regarding the algorithms, but also on data and subsurface information requirements.

INTRODUCTION

The objective of seismic inversion is to estimate the medium properties of the subsurface from the recorded wavefield at the surface. Inversion methods can be classified as a direct method or an indirect method. A direct inversion method can solve an inverse problem (as its name suggests) directly depending on the algorithm and its data requirements without searching or model matching. On the other hand, an indirect inversion method seeks to solve an inverse problem through indirect approaches and paths. Among indicators, identifiers and examples of "indirect" inverse solutions (Weglein, 2015a) are: (1) model matching, (2) objective/cost functions, (3) searching algorithms, (4) iterative linear inversion, and (5) methods corresponding to necessary but not sufficient conditions, e.g., common image gather flatness as an indirect migration velocity analysis method. As a simple illustration, a quadratic equation $ax^2+bx+c=0$ can be solved through a direct method as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, or it can be solved by an indirect method searching for x such that, e.g., $(ax^2+bx+c)^2$ is a minimum.

A direct inverse solution for parameter estimation can be derived from an operator identity that relates the change in a medium's properties and the commensurate change in the wavefield. This operator identity is valid and can accommodate any model-type, for example, acoustic, elastic, anisotropic, heterogeneous, and inelastic earth models. That identity can be the basis of: (1) modeling methods and can provide (2) a firm and solid math-physics foundation and framework for direct inverse methods. For all multidimensional seismic applications, the direct inverse solution provided by that operator identity is in the form of a series, referred to as the inverse scattering series (Weglein et al., 2003). It can achieve all processing objectives within a single framework without requiring any subsurface information. There are distinct isolated-task inverse scattering subseries derived from the ISS, which can perform free-surface multiple removal, internal multiple removal, depth imaging, parameter estimation, and Q compensation, and each achieves its objective directly and without subsurface information. The direct inverse solution (e.g., Weglein et al., 2003, 2009) provides a solid framework and firm math-physics foundation that unambiguously defines both the data requirements and the distinct algorithms to solve all the tasks within the inverse problem, directly. There are many other issues that contribute to the gap between a direct parameter estimation inversion solution and e.g., conventional and industry standard amplitude-versus-offset (AVO) and full-waveform-inversion (FWI). However, starting with and employing a framework that provides confidence of the data and methods that are actually solving the problem of interest is a significant, fundamental, and practical contribution towards identifying the gap (Weglein, 2015b). Only a direct solution can provide that clarity, confidence and effectiveness. The current industry standard AVO and FWI, using variants of model-matching and iterative linear inverse, correspond to indirect approaches and procedures, and iteratively linearly updating P data or multicomponent data does not correspond to, and will not produce, a direct solution with its clarity, confidence and capability. If we seek the parameters of an elastic heterogeneous isotropic subsurface, then the differential operator in the operator identity is the differential operator that occurs in the elastic, heterogeneous isotropic wave equation. From forty years of AVO and amplitude analysis application in the petroleum industry, the elastic isotropic model is the base-line minimally realistic and acceptable earth model-type for amplitude analysis, for example, for AVO and FWI. Then taking the operator identity (called the Lippmann-Schwinger or scattering theory equation) for the elastic wave equation, we can obtain a direct inverse solution for the changes in elastic properties and density. The direct inverse solution specifies both the data required and the algorithm to achieve a direct parameter estimation solution. In this presentation we explain how this methodology differs from all current AVO and FWI methods, that are in fact forms of model matching (often, and in addition, with the wrong/innately inadequate/inappropriate model type and/or the less than necessary data) and are not direct solutions. This paper focuses on one specific task, parameter estimation, within the overall and broaden set of inversion objectives and tasks. Furthermore, many of the important and serious practical issues, such as, differences in the need for subsurface information, the impact of band-limited data and noise, are not being examined in this paper. The purpose is to isolate the purely algorithmic differences under perfect data conditions, assuming and providing perfect information above the target, so that once the purely algorithmic differences are examined, defined, and understood, it makes the next steps of attributing differences in the direct inverse and iterative linear inverse methods that are beyond purely algorithmic differences able to (once again) be able to be isolated and examined and defined.

In this paper, we focus on analyzing and examining the direct inverse solution that the ISS inversion subseries provides for parameter estimation. The distinct issues of: (1) data requirements, (2) model-type, and (3) inversion algorithm for the direct inverse are all important (Weglein, 2015b). For an elastic heterogeneous medium, we show that the direct inverse requires multi-component/PS (P-component and S-component) data and prescribes how that data are utilized for a direct parameter estimation solution (Zhang and Weglein, 2006). For an acoustic medium, if we consider a normal incident wave on a single horizontal reflector, a closed-form direct inverse solution exists. Therefore, we can isolate and focus on the algorithmic difference (between a direct inverse solution and an iterative linear method) when model-type agrees and there is a single reflector and acoustic P wave data. Under that very limited and focused circumstance, a direct comparison is realizable and for the iterative approach we will not consider the considerable practical issues, e.g., the impact of noise and the different generalized inverses at each step. The iterative linear approach will be provided an analytic Frechet derivative/sensitivity matrix courtesy of the ISS direct method, thus providing the iterative linear updating approach an advantage it would never benefit from on its own, in practice, to bend over backwards for (more than) a 'level-playing-field' and to avoid (within this study), a serious downside of the indirect linear approach. That allows us to focus on how the pure algorithmic differences between the iterative linear approach and the direct inverse method, under circumstances where we artificially arrange for certain practical issues and limitations of the former (that are not shared by the latter direct method) to not exist. However, the ISS method performs as it would in practice with an analytic and unchanged inverse at every step, where the inverse operation never is updated or changed and more importantly never depends on data, and its band-limited noisy nature. In this paper, we provide numerical and analytic comparison between the ISS direct non-linear parameter inversion and the iterative inversion for a normal incident wave on a 1D one parameter (constant density, velocity only changing) acoustic model with a single horizontal reflector, where the velocity is assumed to be known above the reflector and unknown below the reflector. Yang and Weglein (2015) provided an initial discussion and introduction to the analysis and conclusions of this paper.

In this paper, these two methods are compared including their respective convergence properties and the region and rates of convergence. In the ISS inversion subseries, each term of the series works purposefully towards the final goal. Sometimes when more terms in the series are included, the estimation may be temperately worse, but that apparent unhelpful behavior is in fact purposeful, necessary and essential in the overall goal and a required contribution towards convergence and the final goal of parameter estimation. This property has also been indicated by Carvalho (1992) and Weglein et al. (1997) in the free-surface multiple elimination subseries, e.g., what appears to make a second-order free-surface multiple larger with a first-order free-surface algorithm is actually preparing the second-order multiple to be removed by the higher-order terms in the free-surface multiple elimination subseries.

This extremely simple and transparent example and comparison is useful both for the unambiguous lesson it communicates and for the reference and lesson it will provide for future studies that examine both the much more complicated elastic multi-parameter generalization as well as the practical issues of band-limited noisy data. The iterative linear inverse methods will suffer under band-limited and noisy data conditions, since their matrices to be inverted will depend on that data. The inverse steps in the direct inverse method from ISS do not depend on data, and hence have no sensitivity to data limitations. That shortcoming of iterative linear inverse methods is circumvented in this paper.

The paper is arranged as follows: First, the ISS direct inversion method is discussed in general terms. Second, the direct inversion is presented in the 2D heterogeneous elastic medium. Third, the ISS direct inversion and the iterative linear inversion are examined and compared for a plane wave normal incident on a 1D acoustic velocity only changing medium. Finally, we offer a discussion and conclusions.

THEORY

Scattering theory is a perturbation theory that relates a change (or perturbation) in a medium to a change (or perturbation) in the associated wavefield. The direct inverse solution (Weglein et al., 2003; Zhang, 2006) is derived from the operator identity that relates the change in a medium's properties and the commensurate change in the wavefield within and exterior to the medium. Let L_0, L, G_0, and G be the differential operators and Green's functions for the reference and actual media, respectively, that satisfy:

$$L_0 G_0 = \delta \quad \text{and} \quad LG = \delta,$$

where δ is a Dirac δ-function. We define the perturbation operator, V, and the scattered wavefield, ψ_s, as follows:

$$V \equiv L_0 - L \quad \text{and} \quad \psi_s \equiv G - G_0.$$

The operator identity

The relationship (called the Lippmann-Schwinger or scattering theory equation)

$$G = G_0 + G_0 V G \tag{1}$$

is an operator identity that follows from

$$L^{-1} = L_0^{-1} + L_0^{-1}(L_0 - L)L^{-1},$$

and the definitions of L_0, L, and V. For forward modeling the wavefield, G, for a medium described by L is given by

$$L \rightarrow G \quad \text{or} \quad L_0, V \rightarrow G$$

where the second (perturbation) form has L entering the modeling algorithms in terms of L_0 and V. Modeling using scattering theory requires a complete and detailed knowledge of the earth model type and medium properties within the model type.

Direct forward series and direct inverse series

The operator identity equation (1) can be solved for G as

$$G = (1 - G_0V)^{-1}G_0, \tag{2}$$

or

$$G = G_0 + G_0VG_0 + G_0VG_0VG_0 + \cdots , \tag{3}$$

and is called the Born or Neumann series in scattering theory literature (see e.g., Taylor, 1972). Equation (3) has the form of a generalized geometric series

$$G - G_0 = S = ar + ar^2 + \cdots = \frac{ar}{1-r}, \tag{4}$$

where we identify $a = G_0$ and $r = VG_0$ in equation (3), and

$$S = S_1 + S_2 + S_3 + \cdots , \tag{5}$$

where the portion of S that is linear, quadratic, ... in r is:

$$S_1 = ar,$$

$$S_2 = ar^2,$$

$$\vdots$$

and the sum is

$$S = \frac{ar}{1-r}. \tag{6}$$

Solving equation (6) for r, produces the inverse geometric series,

$$r = \frac{S/a}{1 + S/a} = S/a - (S/a)^2 + (S/a)^3 + \cdots$$

$$= r_1 + r_2 + r_3 + \cdots , \text{ when } |S/a < 1|.$$

This is the simplest prototype of an inverse series, i.e., the inverse of the geometric series. For the seismic inverse problem, we associate S with the measured data

$$S = (G - G_0)_{ms} = \text{Data},$$

and the forward and inverse series follow from treating the forward solution as S in terms of V, and the inverse solution as V in terms of S. The inverse series assumes

$$V = V_1 + V_2 + V_3 + \cdots, \tag{7}$$

where V_n is the portion of V that is n^{th} order in measured data. Equation (3) is the forward series; and equation (7) is the inverse series. The identity (equation 1) provides a geometric forward series rather than a Taylor series. In general, a Taylor series doesn't have an inverse series; however, a geometric series has an inverse series. For example, solving the forward problem in an inverse sense,

$$\begin{aligned} S &= ar + ar^2 + \cdots + ar^n, \\ S - ar - ar^2 - \cdots - ar^n &= 0. \end{aligned} \tag{8}$$

There are n roots for equation (8). When n goes to infinity, the number of roots goes to infinity. However, from $S = \frac{ar}{1-r}$ (equation 6), we found that the direct inverse has only one real root $r = \frac{S/a}{1+S/a}$. All conventional current mainstream inversion methods, including iterative linear inversion and FWI, are based on a Taylor series concept and are solving a forward problem in an inverse sense (as a caveat see, e.g., equation 8). Solving a forward problem in an inverse sense is not the same as solving an inverse problem directly (Weglein, 2013).

In terms of the expansion of V in equation (7), and G_0, G, $D = (G - G_0)_{ms}$, the inverse scattering series (Weglein et al., 2003) can be obtained as

$$G_0 V_1 G_0 = D, \tag{9}$$

$$G_0 V_2 G_0 = -G_0 V_1 G_0 V_1 G_0, \tag{10}$$

$$\begin{aligned} G_0 V_3 G_0 = & - G_0 V_1 G_0 V_1 G_0 V_1 G_0 \\ & - G_0 V_1 G_0 V_2 G_0 - G_0 V_2 G_0 V_1 G_0, \end{aligned} \tag{11}$$

$$\vdots$$

The inverse scattering series provides a direct method for obtaining the subsurface properties contained within L, by inverting the series order-by-order to solve for the perturbation operator V, using only the measured data D and a reference Green's function G_0, for any assumed earth model type. We can imagine that a set of tasks need to be achieved to determine the subsurface properties, V, from recorded seismic data, D. The tasks that are within a direct inverse solution are: (1) free-surface multiple removal, (2) internal multiple removal, (3) depth imaging, (4) Q compensation, and (5) non-linear parameter estimation. Each of these five tasks has its own task-specific subseries from $V_1, V_2, \cdots$, and each of those tasks is achievable directly and without subsurface information (see, e.g., Weglein et al., 2012 and Lira, 2009). Equations (9)-(11) provide V in terms of $V_1, V_2, \cdots$, and each of the V_i is computable directly in terms of D and G_0. In the next section, we review the details of equations (9)-(11) for a 2D heterogeneous elastic medium.

The operator identity for a 2D heterogeneous elastic medium

We exemplify the method for a 2D elastic heterogeneous earth. The starting point for the 3D generalization is found in Stolt and Weglein (2012). The 2D elastic wave equation for a heterogeneous isotropic medium (Zhang, 2006) is

$$L\vec{u} = \begin{pmatrix} f_x \\ f_z \end{pmatrix} \quad \text{and} \quad \hat{L}\begin{pmatrix} \phi^P \\ \phi^S \end{pmatrix} = \begin{pmatrix} F^P \\ F^S \end{pmatrix}, \tag{12}$$

where $\vec{u}$, f_x, and f_z are the displacement and forces in displacement coordinates and ϕ_P, ϕ_S and F^P, F^S are the P and S waves and the force components in P and S coordinates, respectively. The operators L and L_0 in the actual and reference elastic media are

$$L = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x\gamma\partial_x + \partial_z\mu\partial_z & \partial_x(\gamma - 2\mu)\partial_z + \partial_z\mu\partial_x \\ \partial_z(\gamma - 2\mu)\partial_x + \partial_x\mu\partial_z & \partial_z\gamma\partial_z + \partial_x\mu\partial_x \end{pmatrix}\right],$$

$$L_0 = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0\partial_x^2 + \mu_0\partial_z^2 & (\gamma_0 - \mu_0)\partial_x\partial_z \\ (\gamma_0 - \mu_0)\partial_x\partial_z & \mu_0\partial_x^2 + \gamma_0\partial_z^2 \end{pmatrix}\right],$$

and the perturbation V is

$$V \equiv L_0 - L = \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2\partial_x a_\gamma\partial_x + \beta_0^2\partial_z a_\mu\partial_z & \partial_x(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_z + \beta_0^2\partial_z a_\mu\partial_x \\ \partial_z(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_x + \beta_0^2\partial_x a_\mu\partial_z & a_\rho\omega^2 + \alpha_0^2\partial_z a_\gamma\partial_z + \beta_0^2\partial_x a_\mu\partial_x \end{bmatrix},$$

where the quantities $a_\rho \equiv \rho/\rho_0 - 1$, $a_\gamma \equiv \gamma/\gamma_0 - 1$, and $a_\mu \equiv \mu/\mu_0 - 1$ are defined in terms of the bulk modulus, shear modulus and density (γ_0, μ_0, ρ_0, γ, μ, ρ) in the reference and actual media, respectively.

The forward problem is found from the identity equation (3) and the elastic wave equation (12) in PS coordinates as

$$\hat{G}-\hat{G}_0=\hat{G}_0\hat{V}\hat{G}_0+\hat{G}_0\hat{V}\hat{G}_0\hat{V}\hat{G}_0+\cdots,$$

$$\begin{pmatrix}\hat{D}^{PP} & \hat{D}^{PS}\\ \hat{D}^{SP} & \hat{D}^{SS}\end{pmatrix}=\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}^{PP} & \hat{V}^{PS}\\ \hat{V}^{SP} & \hat{V}^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}$$
$$+\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}^{PP} & \hat{V}^{PS}\\ \hat{V}^{SP} & \hat{V}^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}^{PP} & \hat{V}^{PS}\\ \hat{V}^{SP} & \hat{V}^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}+\cdots, \tag{13}$$

and the inverse solution, equations (9)-(11), for the elastic equation (12) is

$$\begin{pmatrix}\hat{D}^{PP} & \hat{D}^{PS}\\ \hat{D}^{SP} & \hat{D}^{SS}\end{pmatrix}=\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}_1^{PP} & \hat{V}_1^{PS}\\ \hat{V}_1^{SP} & \hat{V}_1^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix},$$

$$\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}_2^{PP} & \hat{V}_2^{PS}\\ \hat{V}_2^{SP} & \hat{V}_2^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}$$
$$=-\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}_1^{PP} & \hat{V}_1^{PS}\\ \hat{V}_1^{SP} & \hat{V}_1^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}\begin{pmatrix}\hat{V}_1^{PP} & \hat{V}_1^{PS}\\ \hat{V}_1^{SP} & \hat{V}_1^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^{P} & 0\\ 0 & \hat{G}_0^{S}\end{pmatrix}, \tag{14}$$

$$\vdots$$

where $\hat{V}^{PP}=\hat{V}_1^{PP}+\hat{V}_2^{PP}+\hat{V}_3^{PP}+\cdots$ and any one of the four matrix elements of V requires the four components of the data

$$\begin{pmatrix}\hat{D}^{PP} & \hat{D}^{PS}\\ \hat{D}^{SP} & \hat{D}^{SS}\end{pmatrix}.$$

In summary, from equation (13), $\hat{D}^{PP}$ can be determined in terms of the four elements of V. The four components $\hat{V}^{PP}$, $\hat{V}^{PS}$, $\hat{V}^{SP}$, and $\hat{V}^{SS}$ require the four components of D.

That's what the general relationship $G = G_0 + G_0VG$ requires, i.e., a direct non-linear inverse solution is a solution order-by-order in the four matrix elements of D (in 2D).

Direct inverse and indirect inverse

The direct inverse solution described above is not iterative linear inversion. Iterative linear inversion starts with equation (9). We solve for V_1 and change the reference medium iteratively. The new differential operator L_0' and the new reference medium G_0' satisfy

$$L_0' = L_0 - V_1 \quad \text{and} \quad L_0'G_0' = \delta. \tag{15}$$

Through the same equation (9) with different reference background

$$G_0'V_1'G_0' = D' = (G - G_0')_{ms}, \tag{16}$$

where V_1' is the portion of V linear in data $(G - G_0')_{ms}$. We can continually update L_0' and G_0', and hope to solve for the perturbation operator V. In contrast, the direct inverse solution equations (7) and (14) calls for a single unchanged reference medium, for computing $V_1, V_2, \ldots$. For a homogeneous reference medium, $V_1, V_2, \ldots$ are each obtained by a single unchanged analytic inverse. The inverse to find V_1 from data, is the same inverse to find $V_2, V_3, \ldots,$ from equations (9),(10),.... There are no numerical inverses, no generalized inverses, no inverses of matrices that are computed from and contain noisy band-limited data. The latter is an intrinsic characteristic of iterative linear inverse approaches and is not in any way a characteristic of the direct inverse method from the inverse scattering series.

The difference between iterative linear and the direct inverse of equation (14) is much more substantive and serious than merely a different way to solve $G_0V_1G_0 = D$ (equation

9), for V_1. If equation (9) is our entire basic theory, you can mistakenly think that $\hat{D}^{PP} = \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P$ is sufficient to update $\hat{D}^{PP} = \hat{G}_0'^P \hat{V}_1'^{PP} \hat{G}_0'^P$. This step loses contact with and violates the basic operator identity $G = G_0 + G_0 V G$ for the elastic wave equation. That's as serious as considering problems involving a right triangle and violating the Pythagorean theorem in your method. That is, iteratively updating PP data with an elastic model violates the basic relationship between changes in a medium, V and changes in the wavefield, $G - G_0$, for the simplest elastic earth model.

This direct inverse method provides a platform for amplitude analysis, AVO and FWI. It communicates when a "FWI" method should work, in principle. Iteratively inverting multi-component data has the correct data but doesn't corresponds to a direct inverse algorithm. To honor $G = G_0 + G_0 V G$, you need both the data and the algorithm that direct inverse prescribes. Not recognizing the message that an operator identity and the elastic wave equation unequivocally communicate is a fundamental and significant contribution to the gap in effectiveness in current AVO and FWI methods and application (equation 14). This analysis generalizes to 3D with P, S_h, and S_v data.

There's a role for direct and indirect methods in practical real world applications. Indirect methods are to be called upon for recognizing that the world is more complicated than the physics that we assume in our models and methods. For the part of the world that you are capturing in your model (and methods) nothing compares to direct methods for clarity and effectiveness. An optimal indirect method would seek to satisfy a cost function that derives from a property of the direct method. In that way the indirect and direct method would be aligned and cooperative for accommodating the part of the world described by your physical model and the part that is outside.

In the example in the next section, we consider an acoustic medium consisting of two different homogeneous half spaces. The upper half space is assumed to be known, and the lower half space is unknown. In this example, there is only one inverse task, parameter estimation, to determine the properties of the lower half space from reflection data. The parameter estimation subseries is the only relevant subseries for this example.

The operator identity for a 1D acoustic medium

For a normal incidence plane wave on a 1D acoustic medium (where only the velocity is assumed to vary), the model we consider here consists of two half-spaces with acoustic velocities c_0 and c_1 and an interface located at $z = a$ as shown in Figure 1. If we put the source and receiver on the surface, $z = 0$, the pressure wave

$$D(t) = R\delta(t - 2a/c_0) \tag{17}$$

will be recorded, where the reflection coefficient $R = \frac{c_1 - c_0}{c_1 + c_0}$. For this example, $D(t)$ is the only input to the direct ISS inverse and the iterative inversion methods. Since we will assume knowledge of the velocity in the upper half space, c_0, the location of the reflector at $z = a$ is not an issue. We will focus on only determining the change of velocity across the reflector at $z = a$. The operators L_0 and L in the reference and actual acoustic media are

$$L_0 = \frac{d^2}{dz^2} + \frac{\omega^2}{c_0^2} \quad \text{and} \quad L = \frac{d^2}{dz^2} + \frac{\omega^2}{c^2(z)}, \tag{18}$$

and we characterize the velocity perturbation as,

$$\alpha(z) \equiv 1 - \frac{c_0^2}{c^2(z)}. \tag{19}$$

The perturbation V (Weglein et al., 2003) can be expressed as

$$V(z) = L_0 - L = \frac{\omega^2}{c_0^2} - \frac{\omega^2}{c^2(z)} = k_0^2\alpha(z), \tag{20}$$

where ω is the angular frequency and $k_0 = \omega/c_0$. c_0 and $c(z)$ are the reference and local acoustic velocity. Therefore, the inverse series of V (equation 7) becomes

$$\alpha(z) = \alpha_1(z) + \alpha_2(z) + \alpha_3(z) + \cdots . \tag{21}$$

That is

$$V_1 = k_0^2 \alpha_1, \quad V_2 = k_0^2 \alpha_2, \quad \cdots . \tag{22}$$

From the inverse scattering series (Equations 9-11), Shaw et al. (2004) isolated the leading order imaging subseries and the direct non-linear inversion subseries.

In this section, we will focus on studying the convergence properties of the ISS inversion subseries. The inversion only terms isolated from the inverse scattering series (Zhang, 2006; Li, 2011) are

$$\alpha(z) = \alpha_1(z) - \frac{1}{2}\alpha_1^2(z) + \frac{3}{16}\alpha_1^3(z) + \cdots . \tag{23}$$

For a 1D normal incidence case, the linear equation (9) solves for α_1 in terms of the single trace data $D(t)$ (Shaw et al., 2004) as

$$\alpha_1(z) = 4\int_{-\infty}^{z} D(z')dz', \tag{24}$$

where $z' = c_0 t/2$. For a single reflector, inserting data D (equation 17) gives

$$\alpha_1 = 4RH(z-a), \tag{25}$$

where R is the reflection coefficient $R = \frac{c_1 - c_0}{c_1 + c_0}$ and H is the Heaviside function. When $z > a$, substituting α_1 into equation (23), the ISS direct non-linear inversion subseries in terms of R can be written as (where α is the magnitude of $\alpha(z)$ for $z > a$)

$$\alpha = 4R - 8R^2 + 12R^3 + \cdots = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n. \tag{26}$$

After solving for α, the inverted velocity $c(z)$ can be obtained through $c_1 = c_0(1-\alpha)^{-1/2}$ (equation 20).

Considering the convergence property of the series for α or the inversion subseries, we can calculate the ratio test,

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \left|\frac{(n+2)(-R)^{n+1}}{(n+1)(-R)^n}\right| = \left|\frac{n+2}{n+1}R\right|. \tag{27}$$

If $\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < 1$, this subseries converges absolutely. That is

$$|R| < \lim_{n\to\infty}\frac{n+1}{n+2} = 1. \tag{28}$$

Therefore, the ISS direct non-linear inversion subseries converges when the reflection coefficient $|R|$ is less than 1, which is always true. Hence, for this example, the ISS inversion subseries will converge under any velocity contrasts between the two media.

For the iterative linear inversion, we use the first linear estimate of $\alpha = \alpha_1^1$ to compute the first estimate of $c_1 = c_1^1$. Then we choose the first estimate of $c_1 = c_0(1-\alpha_1^1)^{-1/2} \equiv c_1^1$ as the new reference velocity, $c_0^1 = c_0(1-\alpha_1^1)^{-1/2}$, where $\alpha_1^1 = 4R_1$ and $R_1 = \frac{c_1-c_0}{c_1+c_0}$. Repeating the linear process with a new reflection coefficient R_2 (again exploiting the analytic inverse generously provided by ISS to benefit the iterative linear approach) gives

$$R_2 = \frac{c_1 - c_0^1}{c_1 + c_0^1}, \quad \alpha_1^2 = 4R_2 \text{ and } c_1^2 = c_0^1(1-\alpha_1^2)^{-1/2} = c_0^2, \tag{29}$$

$$\vdots \tag{30}$$

$$R_{n+1} = \frac{c_1 - c_0^n}{c_1 + c_0^n}, \quad \alpha_1^{n+1} = 4R_{n+1} \text{ and } c_1^n = c_0^{n-1}(1-\alpha_1^n)^{-1/2} = c_0^n, \tag{31}$$

where $\alpha_1^n = n^{th}$ estimate of α_1 and $c_1^n = n^{th}$ estimate of c_1. The questions are (1) under what conditions does c_1^n approach c_1, and (2) when it converges, what is its rate of convergence.

From the above analysis, we can see that the ISS method for α always converges and the resulting α can be used to find c_1. For the iterative linear inverse, there are values of α_1

such that you cannot compute a real c_1^1. When $\alpha_1^1 > 1$ and $4R > 1$, $R > 1/4$ and you cannot compute an updated reference velocity and the method simply shuts down and fails. The inverse scattering series never computes a new reference and doesn't suffer that problem, with the series for α always converging and then outputting c_1, the correct unknown velocity below the reflector.

NUMERICAL EXAMPLES FOR A 1D NORMAL INCIDENT WAVE ON AN ACOUSTIC MEDIUM

Numerical examples for a 1D normal incident wave on a acoustic medium are shown in this section. First, we examine and compare the convergence of the ISS direct inversion and iterative inversion. Second, the rate of convergence of the ISS inversion subseries is examined and studied using an analytic example, where the ISS method converges and the iterative linear method doesn't and where both methods converge.

The convergence of the ISS direct inversion and iterative inversion

In this section, we will examine and compare the convergence property of the ISS inversion (equation 26) and the iterative linear inversion for different velocity contrasts in the 1D acoustic case. In the 1D normal incident acoustic model (Figure 1), only one parameter (velocity) varies and a plane wave propagates into the medium. There is only a single reflector and we assume the velocity is known above the reflector and unknown below the reflector. We will compare the convergence of the perturbation α and the inversion results by using the ISS direct non-linear method and the iterative linear method.

With the reference velocity $c_0 = 1500m/s$, two analytic examples with different velocity contrasts for $c_1 = 2000m/s$ and $c_1 = 3000m/s$ are examined. Figure 2 shows the estimated α by the ISS method (green line) for $c_1 = 2000m/s$. The red line represents the actual α that is calculated from the model. The horizontal axis represents the order of the ISS inversion subseries. The vertical axis shows the value of α. The updated estimation of α using the iterative inversion method (blue line) is shown in figure 3. The horizontal axis represents the iteration numbers in the iterative inversion method. From the figures 2 and 3, we can see that at the small velocity contrast, the estimated α by ISS method becomes the actual α after about five orders calculation and the updated estimation of α by the iterative inversion method goes to zero as we expected, because after several iteration, the updated model is close to and approaching to the actual model. Figure 4 represents the velocity estimation. The green blue lines represent the estimated velocity by using the ISS inversion method and the iterative inversion method, respectively. We can see that at the small velocity contrast, both methods converge and produce correct velocity after five orders or iterations and the ISS inversion method converges faster than the iterative inversion method.

Figure 5 shows the estimated α by the ISS method (green line) for $c_1 = 3000m/s$. When the velocity contrast is larger, i.e., $R > 0.25$, the iterative inversion method can not be computable, but the ISS inversion method always converges (see green line in Figure 5) after the summation of more orders in computing α.

As we know, the reflection coefficient R is almost always less than 0.2 in practice, so that both the ISS method and the iterative method converge, but the ISS method converges faster than the iterative method. Moreover, for more complicated circumstances (e.g., the elastic non-normal incidence case), the difference between the ISS method and the iterative method is much greater, not just on the algorithms, but also on data requirements and on how the band-limited noisy nature of the seismic data impact the inverse operators in the iterative method but not in the ISS method.

The rate of convergence of the ISS inversion subseries

The rate of convergence of the estimated α or the ISS inversion subseries (equation 26) is analytically examined and studied. Since α is always convergent when $R < 1$, the summation of this subseries (Zhang, 2006) is

$$\alpha = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n = 4R\frac{1}{(1+R)^2}. \qquad (32)$$

If the error between the estimated and the actual α is monotonically decreasing, it means the subseries is a term-by-term added value improvement towards determining the actual medium properties. If this error is increasing before decreasing, it means that the estimate of α becomes worse before it gets better. The error for the first order and the error for the second order have the relation,

$$|\alpha - \alpha_1 - \alpha_2| > |\alpha - \alpha_1|, \qquad (33)$$

i.e.,

$$|4R\frac{3R^2+2R^3}{(1+R)^2}| > |4R\frac{-R^2-2R}{(1+R)^2}|. \qquad (34)$$

After simplification, it gives

$$R^2 + R - 1 > 0. \qquad (35)$$

We can solve it and obtain the reflection coefficient $R < \frac{-1-\sqrt{5}}{2} = -1.618$ or $R > \frac{-1+\sqrt{5}}{2} = 0.618$. Therefore, when $R > 0.618$, the error increases first. Similarly, if the error for the third order is greater than that for the second order, we get $R > 0.667$. If the error for the fourth order is greater than that for the third order, we obtain $R > 0.721$. In summary, when $R > 0.618$ the error increases and the estimated α gets worse before getting better. The sum of terms in the direct inverse ISS solution (for very large contrasts) requires certain partial sums to be temporally worse in order for the entire series to produce the correct velocity. The dashed green line in Figure 7 shows that when the reflection coefficient R is equal to 0.618, the error for the first order is equal to the error for the second order.

As the analytic calculation, when the reflection coefficient R is smaller than 0.618, this inversion subseries gives a monotonically term-by-term added value improvement towards determining c_1. When the reflection coefficient is larger than 0.618, the ISS inversion series still converges, but the estimation of α will become worse before it gets better. Each term in the series works towards the final goal. Sometimes when more terms in the series are included, the estimation looks temporally worse, but once it starts to improve the estimation at a specific order, the approximations never become worse again, every single term after that order will produce an improved estimation. The locally worse partial sum behavior is, in fact, purposeful and essential for convergence to and for computing the exact velocity. The direct inverse solution fulfills its commitment to always predict c_1, and not necessarily to having order-by-order improvement. The ISS direct inversion always converges in contrast to the iterative linear inverse method. This property has also been indicated by Carvalho (1992) in the free-surface multiple elimination subseries, e.g., what appears to make a second-order free-surface multiple larger with a first-order free-surface algorithm is actually helpful and necessary for preparing the second-order multiple to be removed by the higher-order terms.

CONCLUSIONS

In this paper, we discuss a direct inverse method, which is derived from the operator identity that relates the change in a medium's properties and the commensurate change in the wavefield. We describe the direct inversion algorithm for parameter estimation (ISS subseries) and its data requirements. In a specific 1D acoustic medium, we examine and compare the ISS inversion and the iterative inversion for parameter estimation across a single horizontal reflector, where the velocity is assumed to be known above the reflector and unknown below the reflector. Numerical results show that when the velocity contrast is small, i.e., the reflection coefficient is small, both inversion methods converge and the ISS inversion method converges faster than the iterative inversion method. When velocity contrast increases, the reflection coefficient gets larger, the iterative inversion method breaks down and the ISS inversion method always converges. Hence, for the simplest single horizontal reflector parameter estimation situation, the iterative linear inversion is not equivalent to the direct non-linear inverse solution provided by the inverse scattering series. For more complicated circumstances (e.g., the elastic non-normal incidence case), the difference is much greater, not just on the algorithms, but also on data requirements and on how the band-limited noisy nature of the seismic data impact the inverse operators in iterative linear inversion but not in the ISS direct inversion.

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Figure 1: 1D acoustic model with velocities c_0 over c_1

Figure 2: The estimated α at $R = 0.1429$: The horizontal axis is the order of the ISS suberies and the vertical axis shows the value of α. The red line shows the actual value of $\alpha = 0.4375$. The green line shows the estimation of α using the ISS inversion method order-by-order.

Figure 3: The updated α at $R = 0.1429$: The horizontal axis is the iteration numbers and the vertical axis shows the updated value of α. The blue line represents the updated estimation of α using the iterative inversion method.

Figure 4: The estimated velocity by using the ISS inversion method (green line) and the iterative inversion method (blue line).

Figure 5: The estimated α at $R = 0.3333$: The horizontal axis is the order of the ISS suberies and the vertical axis represents the value of α. The red line shows the actual value of $\alpha = 0.7500$. The green line shows the estimation of α using the ISS inversion method order-by-order.

Figure 6: The estimated velocity at $R = 0.3333$: The horizontal axis is the iteration numbers and the vertical axis shows the estimated velocity. Since $R > 0.25$, the iterative inversion method can not be computable.

Figure 7: The error (dashed green line) of estimated α at $R = 0.6180$ and $\alpha = 0.9443$.

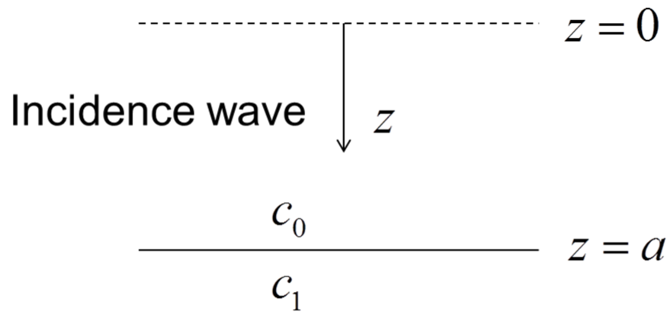


Figure 1: 1D acoustic model with velocities c_0 over c_1

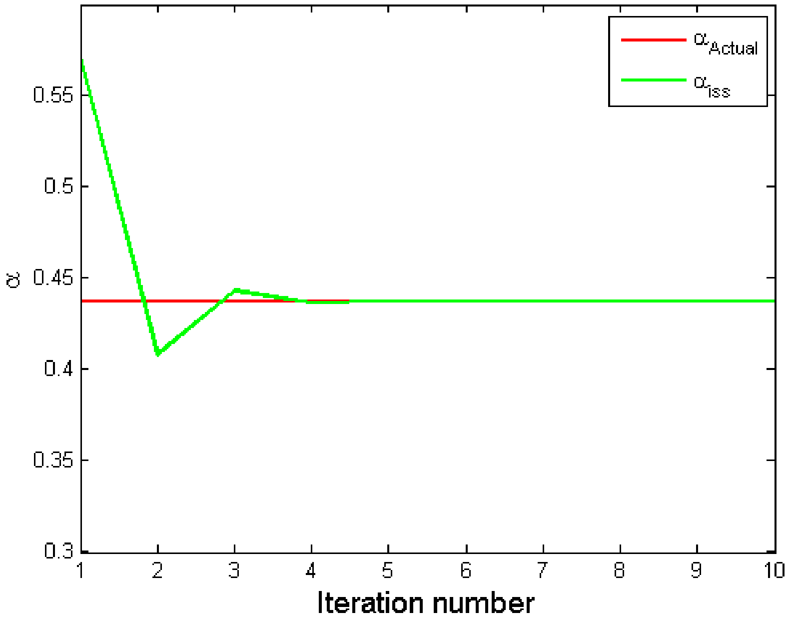
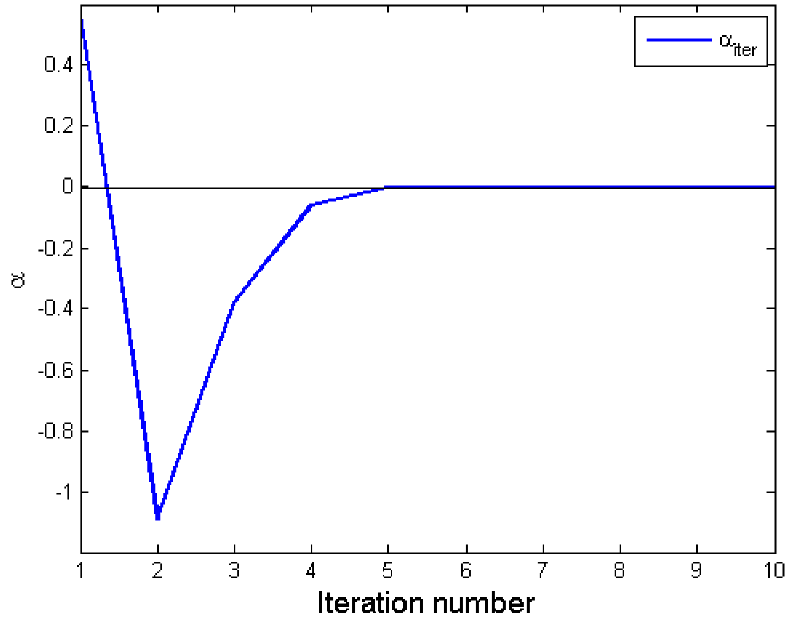


Figure 2: The estimated α at $R = 0.1429$: The horizontal axis is the order of the ISS suberies and the vertical axis shows the value of α. The red line shows the actual value of $\alpha = 0.4375$. The green line shows the estimation of α using the ISS inversion method order-by-order.



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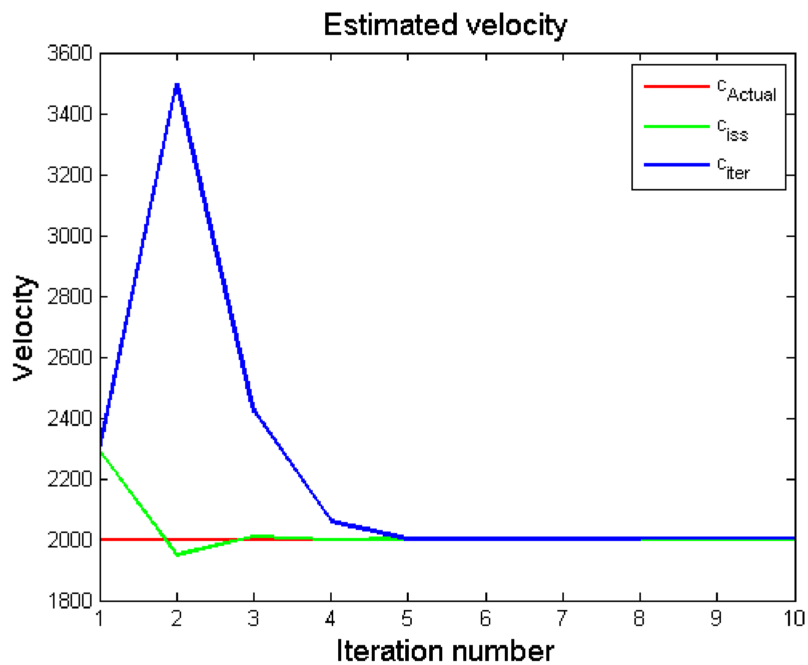
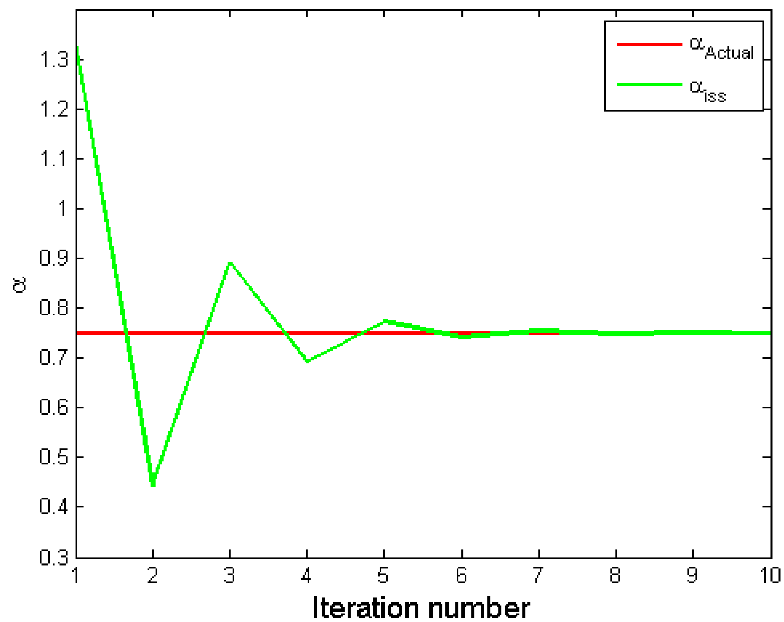


Figure 4: The estimated velocity by using the ISS inversion method (green line) and the iterative inversion method (blue line).



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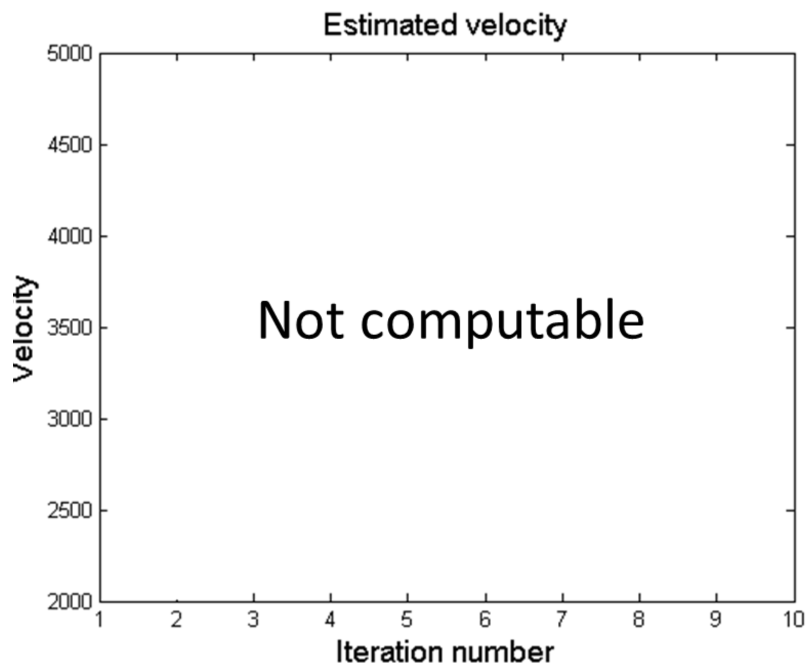
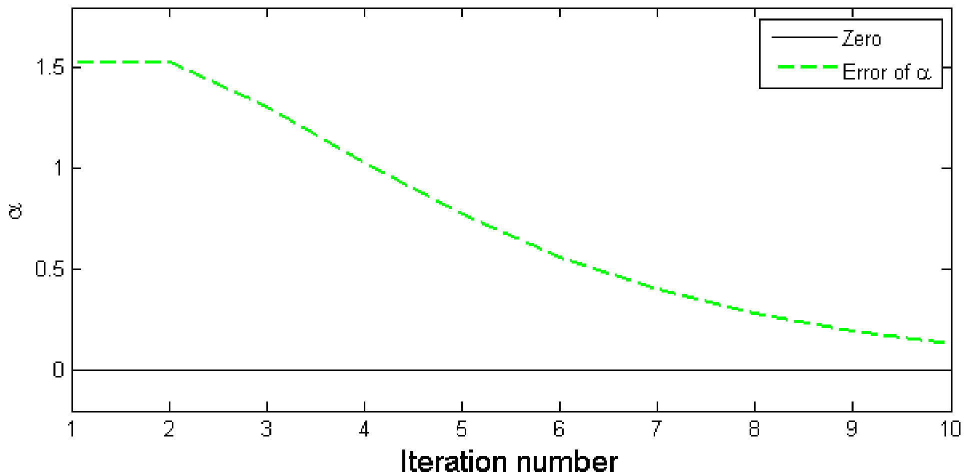


Figure 6: The estimated velocity at $R = 0.3333$: The horizontal axis is the iteration numbers and the vertical axis shows the estimated velocity. Since $R > 0.25$, the iterative inversion method can not be computable.



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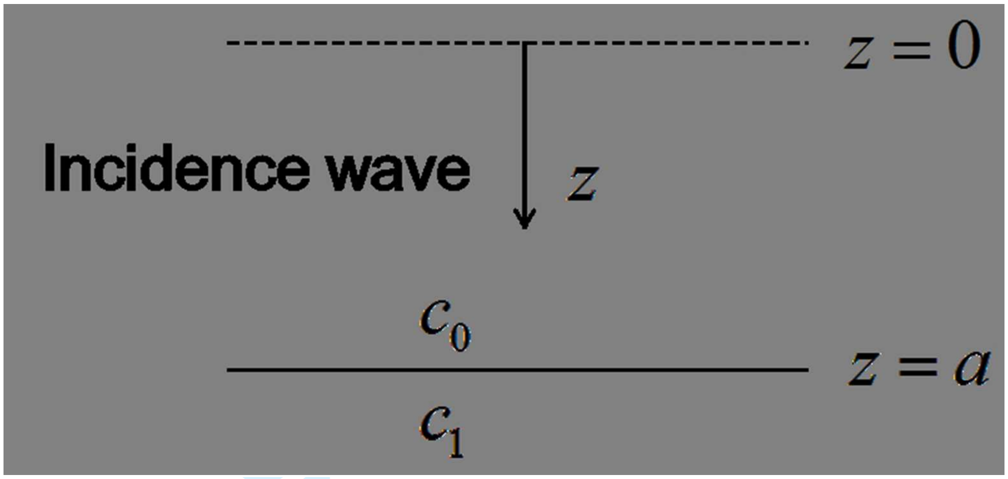
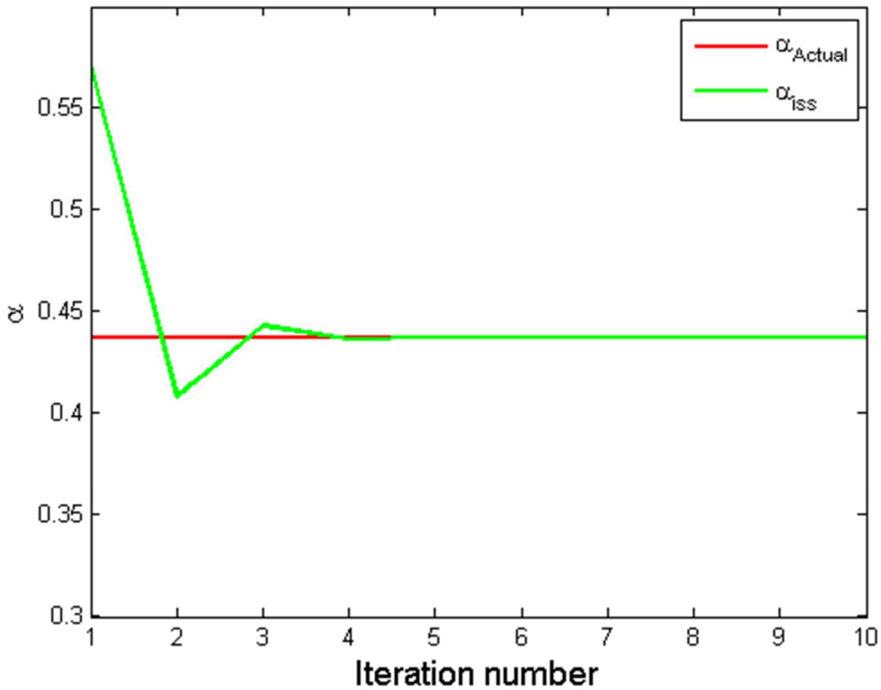
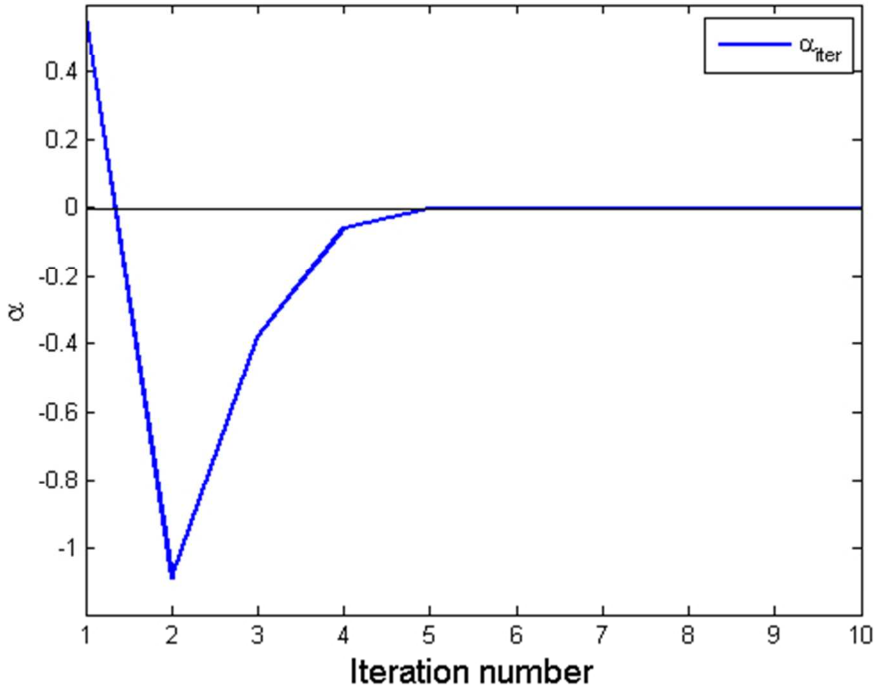
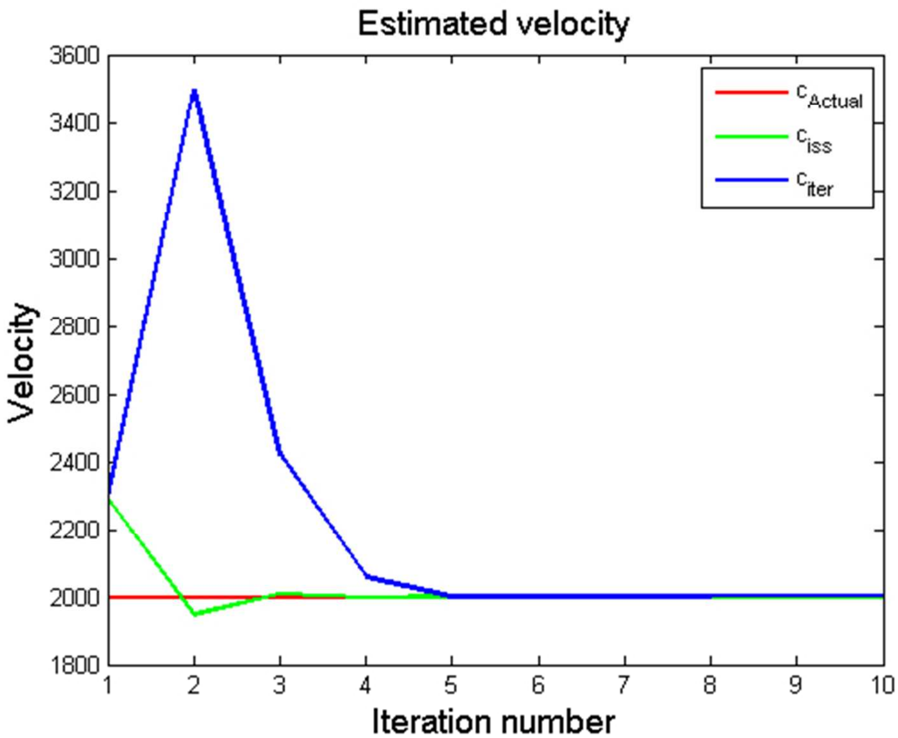

$z = 0$
Incidence wave
z
c_0
$z = a$
c_1

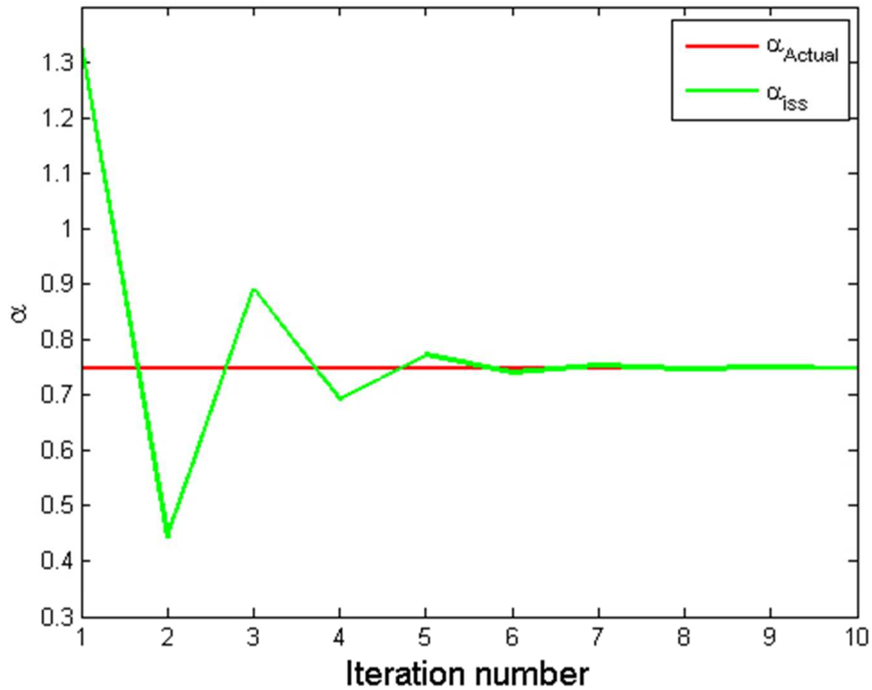

α
0.55
0.5
0.45
0.4
0.35
0.3
1
2
3
4
5
6
7
8
9
10
Iteration number
α_{Actual}
α_{iss}



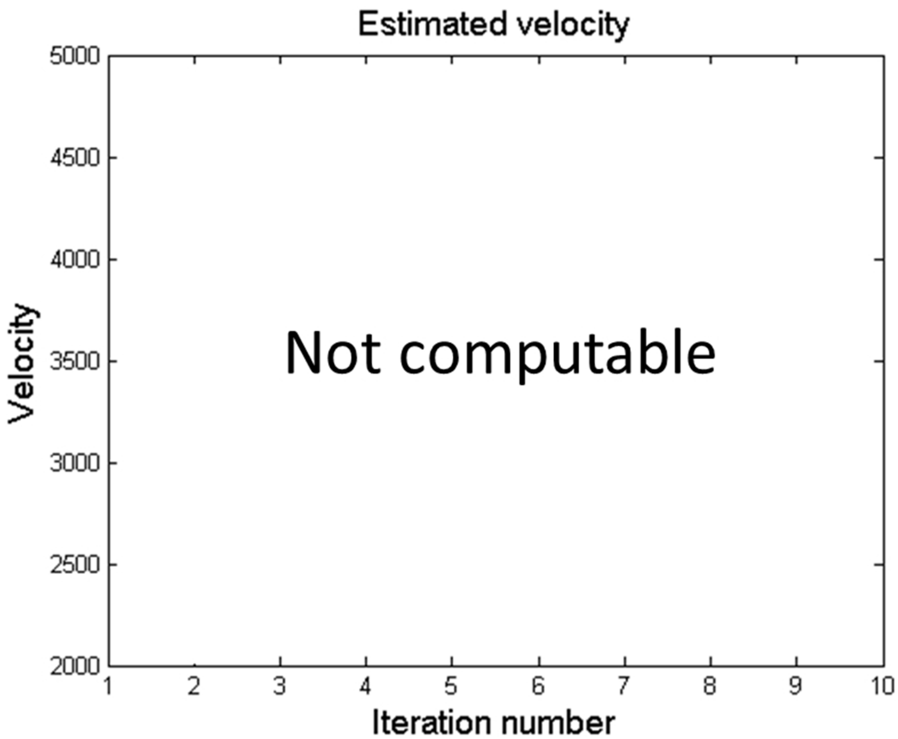
α_iter
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
α
1
2
3
4
5
6
7
8
9
10
Iteration number

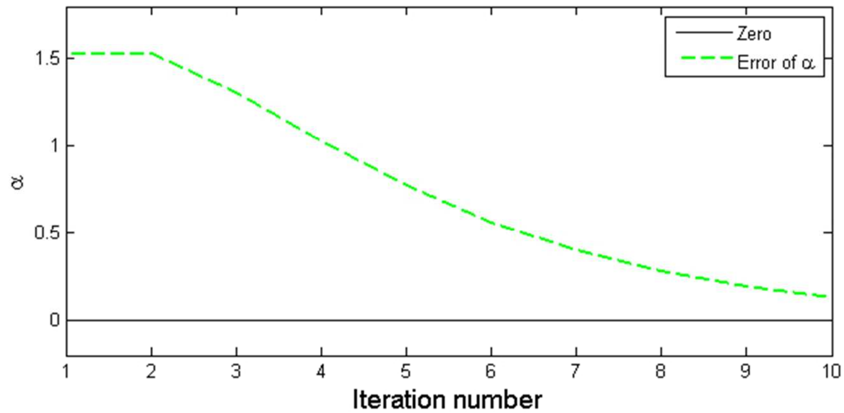


Estimated velocity
Velocity
Iteration number
3600
3400
3200
3000
2800
2600
2400
2200
2000
1800
1
2
3
4
5
6
7
8
9
10
c_{Actual}
c_{iss}
c_{iter}



α_{Actual}
α_{iss}
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
α
1
2
3
4
5
6
7
8
9
10
Iteration number


Estimated velocity
5000
4500
4000
3500
3000
2500
2000
Velocity
Not computable
1
2
3
4
5
6
7
8
9
10
Iteration number


Zero
Error of α
α
1.5
1
0.5
0
1
2
3
4
5
6
7
8
9
10
Iteration number