A direct inverse method for subsurface properties: the conceptual and practical benefit and added-value in comparison with all current indirect methods, for example, AVO and FWI

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ABSTRACT

Direct inverse methods solve the problem of interest, and in addition they communicate whether the problem of interest is the problem that we (the seismic industry) need to be interested in. When a direct solution doesn't result in an improved drill success rate, we know that the problem we have chosen to solve is not the right problem — since the solution is direct and cannot be the issue. On the other hand with an indirect method, if the result is not an improved drill success rate, then the issue can be either the chosen problem, or the particular choice within the plethora of indirect solution methods, or both. The inverse scattering series (ISS) is the only direct inversion method for a multidimensional subsurface. Solving a forward problem in an inverse sense is not equivalent to a direct inverse solution.

All current methods for parameter estimation, e.g., AVO and FWI, are solving a forward problem in an inverse sense and are indirect inversion methods. The direct ISS method for determining earth material properties, defines both the precise data required and the algorithms that directly output earth mechanical properties. For an elastic model of the subsurface the required data is a matrix of multi-component data, and a complete set of shot records, with only primaries. With indirect methods any data can be matched: one trace, one or several shot records, one component, multi-component, with primaries only or primaries and multiples. Added to that are the innumerable choices of cost functions, generalized inverses, and local and global search engines. Direct and indirect parameter inversion are compared. The direct ISS method has more rapid convergence and a broader region of convergence. The difference in effectiveness increases as subsurface circumstances become more realistic and complex and in particular with band-limited noisy data.

INTRODUCTION

Seismic processing is an inverse problem to determine the properties of a medium from measurements of a wavefield exterior to the medium. The ultimate inversion objective of seismic processing in seismic exploration is to use recorded reflection data to extract useful subsurface information that is relevant to the location and production of hydrocarbons. There is typically a coupled chain of intermediate steps and processing that takes place towards that objective, and I refer to each of those intermediate steps, stages and tasks as objectives "associated with inversion" or inverse tasks towards the ultimate subsurface information extraction goal and objective. All seismic processing methods used to extract subsurface information make assumptions and have prerequisites .

A seismic method will be effective when those assumptions/conditions/requirements are satisfied. When those assumptions are not satisfied the method can have difficulty and/or will fail. That failure can and will contribute to processing and interpretation difficulties with subsequent dry-hole exploration well drilling or drilling suboptimal appraisal and development wells.

Challenges in seismic processing and seismic exploration and production derive from the violation of assumptions/requirements behind seismic processing methods. Advances in seismic processing effectiveness are measured in terms of whether the new capability results in/contributes to more successful plays and better-informed decisions and an increased rate of successful drilling .

The purpose of seismic research is to identify and address seismic challenges and to thereby add more effective options to the seismic processing toolbox. These new options can be called upon when indicated, appropriate and necessary as circumstances dictate.

No tool box option is the appropriate choice under all circumstances. For example, the most effective method, from a technical perspective, might be more than is necessary and needed, under a given circumstance, and a less effective and often less costly option could be the appropriate and indicated choice. Under other more complex and daunting circumstances, the more effective and (perhaps) more costly option will be the only possible choice that's able to achieve the objective of that processing task and interpretation goal. The objective is to expand the number of options in the seismic toolbox to allow a capable response to a larger number of circumstances. As I will point out below, "identify the problem" is the first, the essential and sometimes the most difficult (and often ignored and/or underappreciated) aspect of seismic research.

Identifying and delineating the violation of assumptions behind seismic processing methods is an absolutely essential first step in a strategy and plan for developing a response to prioritizing and pressing seismic exploration challenges. This paper provides a new insight, and advance for the first and critical step of addressing seismic processing challenges: problem identification.

I explain in detail and exemplify why only a direct inversion method can help us decide whether the problem we (the seismic industry) are interested in addressing is, in fact, the problem we need to address.

Seismic processing methods can be classified as based on either statistical models and principles or wave theory concepts and approaches. Wave theory concepts used in seismic processing can be further catalogued as modeling and inversion.

In the next section, I describe these two wave theory approaches to seismic processing, that is, modeling and inversion, and will further distinguish between direct and indirect inversion methods. That clarification represents a central theme and objective of this paper.

MODELING AND INVERSION

Modeling, as a seismic processing tool, starts with a prescribed wavefield source mechanism and a model type (e.g., acoustic, elastic, anisotropic, anelastic, . . .) and then properties are defined within the model type for a given medium (e.g., velocities, density, attenuation Q, . . .). The modeling procedure then provides the seismic wave field that the energy source produces at all points inside and outside the medium.

Inversion also starts with an assumed known and prescribed energy source outside the medium. In addition, the wavefield outside the medium is assumed to be recorded and known. The objective of seismic inversion is to use the latter source description and wavefield measurement information to make inferences about the subsurface medium that are relevant to the location and production of hydrocarbons.

DIRECT AND INDIRECT INVERSION

Inversion methods can be classified as direct or indirect. A direct inversion method solves an inverse problem (as its name suggests) directly. On the other hand, an indirect inversion method seeks to solve an inverse problem circuitously through indirect approaches that often call up assumed aligned objectives or conditions. There are times when the indirect approach will seek to satisfy necessary (but typically not sufficient) conditions, and properties, and is often mistakenly considered and treated as though it was equivalent to a direct method and solution. Indirect methods come in many varieties, some are obvious and others are more subtle and harder to identify as indirect. Among indicators, identifiers and examples of "indirect" inverse solutions (Weglein, 2015a) are: (1) model matching, (2) objective/cost functions, (3) local and global search algorithms, (4) iterative linear inversion, (5) methods corresponding to necessary but not sufficient conditions, e.g., common image gather flatness as an indirect migration velocity analysis method and (6) solving a forward problem in an inverse sense, e.g., AVO and FWI. Regarding the last indirect indicator, item (6), I will show that solving a forward problem in an inverse sense is not equivalent to a direct inverse solution for those same objectives.

As a simple illustration, a quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

can be solved through a direct method as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{2}$$

or it can be solved by an indirect method searching for x such that, e.g., some functional of

$$(ax^2 + bx + c)^2 \tag{3}$$

is a minimum.

In the next section, this example will be further discussed and examined as a way to introduce and develop fundamental concepts in a simple and transparent context. The lesson gleaned from that simple example will later (in this paper) be extended and applied to the more complicated and relevant seismic inverse formulations and methods. In Weglein (2013) there is an introduction to the subject of direct and indirect inverse solutions, that provides a useful background reference for this paper.

THE IMPORTANT QUADRATIC EQUATION EXAMPLE

The direct quadratic formula solution equation 2 explicitly and directly outputs the exact roots, (for all values of a, b and c) when the roots are real and distinct, a real double root, and imaginary and complex roots. The quadratic equation and quadratic solution provides a very simple and insightful example. How would a search algorithm know after a double root is found that it is the only root and to not keep looking and searching forever for a second nonexistent root? How would a search algorithm know to search for only real or for real and complex roots? How would a search algorithm accurately locate an irrational root like $\sqrt{3} \cong 1.732\ldots$ as $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ would directly and precisely and immediately produce? Indirect methods like model matching and seeking and searching and determining roots as in equation 3 are ad hoc, and do not derive from a firm framework and foundation and never provide the confidence that we (the seismic industry) are actually solving the problem of interest.

WHAT'S THE POINT IN DISCUSSING THE QUADRATIC FORMULA? AND WHAT'S THE PRACTICAL BIG DEAL ABOUT A DIRECT SOLUTION?

How can this example and discussion of the quadratic equation possibly be relevant to exploration seismology? Please imagine for a moment that equation 1 $ax^2 + bx + c = 0$ was an equation whose inverse and solution for x given by equation 2 $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ had seismic exploration drill location prediction consequence. And furthermore suppose that this direct solution for x did not lead to successful and/or improved drilling decisions. Under the latter circumstance, we (the seismic industry) could not blame or question the method of solution of equation 1, since equation 2 is direct and unquestionably solving equation 1. If equation 2 was not producing useful and beneficial results we know that our starting equation 1 is the issue, and we have identified the problem. The problem we thought we need to solve, equation 1, is not the problem we need to solve. In contrast, with equation 3, an indirect method, any lack of drilling prediction improvement and added-value or other negative exploration consequences could be due to either the equation you are seeking to invert and/or the boundless, unlimited selection, and plethora of indirect methods using either partial or full recorded wavefields.

That lack of clarity and definitiveness within indirect methods obfuscates the underlying issue and makes identification of the problem (and what's behind a seismic challenge) considerably more difficult to identify and to define. Indirect methods with search engines such as equation 3, lead to "workshops" for solving equation 1 and grasping at mega HPC (High Performance Computing) straws (and capital expenditure investment for buildings full of HPC) that are required to search, seek and locate "solutions". The more HPC we invest in, and is required, the more we are literally "buying-in," and as stake-holders we become committed and therefore convinced of the unquestioned validity of the starting point and our indirect thinking and methodology.

Therefore, beyond the benefit of a direct method like equation 2 providing assurance that we are actually solving the problem of interest, equation 1, there is the unique problem location and identification benefit of a direct inverse when a seismic analysis, processing and interpretation produces unsatisfactory E&P results.

To bring this (quadratic equation example) closer to seismic experience, please imagine hypothetically, that we are not satisfied (in terms of improved drill location and success rate) with a direct inverse of the elastic isotropic equation for amplitude analysis. Since we were employing a direct inversion solution, we know we need to go to a different starting point, perhaps with a more complete and realistic model of wave propagation , since we can exclude the direct inverse solution method as the problem and issue. That's an example of determining that a problem of interest is not the same problem we need to be interested in.

HOW TO DISTINGUISH BETWEEN THE "PROBLEM OF INTEREST" AND THE PROBLEM WE NEED TO BE INTERESTED IN

Direct inverse methods provide value for knowing that you have actually solved the problem of interest. Furthermore, with direct inverse solutions there is the enormous additional value of determining whether our starting point, the "problem of interest", is in fact the problem we need to be interested in.

SCATTERING THEORY AND THE FORWARD AND INVERSE SCATTERING SERIES: THE BASIS OF DIRECT INVERSION THEORY AND ALGORITHMS

Scattering theory is a form of perturbation theory. It provides a direct inversion method for all seismic processing objectives realized by distinct isolated task subseries of the inverse scattering series (ISS) (Weglein et al., 2003). Each term in the inverse scattering series (and the distinct and specific collection of terms that achieve different specific inversion associated tasks) is computable (1) directly and (2) in terms of recorded reflection data and without any subsurface information known, estimated or determined before, during or after the task is performed and the specific processing objective is achieved.

For certain distinct tasks, and subseries, e.g. free-surface multiple elimination and internal multiple attenuation, the algorithms not only do not require subsurface information but in addition possess the absolutely remarkable property of being independent of earth model type (Weglein et al., 2003). That is, the distinct ISS free-surface and internal multiple algorithms are unchanged, without a single line of code having the slightest change for acoustic, elastic, anisotropic and anelastic earth models (Weglein et al., 2003; Wu and Weglein, 2014). For those who subscribe to indirect inversion methods as, e.g., the "be all and end all"of inversion with various model matching approaches, it would be a useful exercise for them to consider how they would formulate a model-type-independent model-matching scheme for free-surface and internal multiple removal. It is not conceivable, let alone realizable, to have a model-type-independent model matching.

For the specific topic and focus of this paper, the inversion task of parameter estimation, there is an obvious need to specify model type and what parameters are to be determined. Hence, it is for that parameter estimation/medium property objective, and that model-type-specific ISS subseries, that the difference between "the problem of interest" and the problem that we need to be interested in, is both relevant, central and significant. Only direct inversion methods for earth mechanical properties provide that assumed earth model-type evaluation, clarity and distinction.

THE BASIC OPERATOR IDENTITY THAT RELATES A CHANGE IN A MEDIUM AND THE CHANGE IN THE WAVEFIELD

A direct inverse solution for parameter estimation can be derived from an operator identity that relates the change in a medium's properties and the commensurate change in the wavefield. That operator identity is general and can accommodate any seismic model type, for example, acoustic, elastic, anisotropic, heterogeneous, and inelastic earth models. That operator identity can be the starting point and basis of both: (1) perturbative scattering-theory modeling methods and (2) a firm and solid math-physics foundation and framework for direct inverse methods.

THEORY

Let's consider an energy source that generates a wave in a medium with prescribed properties. With the same energy source, let's consider a change in the medium and the resulting change in the wavefield inside and outside the medium. Scattering theory is a form of perturbation theory that relates a change (or perturbation) in a medium to a corresponding change (or perturbation) in the original wavefield. When the medium changes the resulting wavefield changes. The direct inverse solution (Weglein et al., 2003; Zhang, 2006) for determining earth mechanical properties is derived from the operator identity that relates the change in a medium's properties and the commensurate change in the wavefield both within and exterior to the medium. Let L_0, L, G_0, and G be the differential operators and Green's functions for the reference and actual media, respectively, that satisfy:

$$L_0 G_0 = \delta \quad \text{and} \quad LG = \delta,$$

where δ is a Dirac delta function. I define the perturbation operator, V, and the scattered wavefield, ψ_s, as follows:

$$V \equiv L_0 - L \quad \text{and} \quad \psi_s \equiv G - G_0.$$

The operator identity

The relationship (called the Lippmann-Schwinger or scattering theory equation)

$$G = G_0 + G_0 V G \tag{4}$$

is an operator identity that follows from

$$L^{-1} = L_0^{-1} + L_0^{-1}(L_0 - L)L^{-1},$$

and the definitions of L_0, L, and V.

Direct forward series and direct inverse series

The operator identity equation 4 [for a fixed source function] is the exact relationship between changes in a medium and changes in the wavefield; it is a relationship between those quantities and not a solution. However the operator identity equation 4 can be solved for G as

$$G = (1 - G_0 V)^{-1} G_0, \tag{5}$$

and expanded as

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots . \tag{6}$$

The forward modeling of the wavefield, G, from equation 6 for a medium described by L is given in terms of the two parts of L, that is, L_0 and V. L_0 enters through G_0 and V

enters as V itself. Equation 6 communicates that modeling using scattering theory requires a complete and detailed knowledge of the earth model type and medium properties within the model type. Equation 6 communicates that any change in medium properties, V, will lead to a change in the wavefield, $G - G_0$, that is always non-linearly related to the medium property change, V. Equation 6 is called the Born or Neumann series in scattering theory literature (see, e.g., Taylor, 1972). Equation 6 has the form of a generalized geometric series

$$G - G_0 = S = ar + ar^2 + \cdots = \frac{ar}{1-r} \text{ for } |r| < 1, \tag{7}$$

where I identify $a = G_0$ and $r = VG_0$ in equation 6, and

$$S = S_1 + S_2 + S_3 + \cdots , \tag{8}$$

where the portion of S that is linear, quadratic, ... in r is:

$$S_1 = ar,$$
$$S_2 = ar^2,$$
$$\vdots$$

and the sum is

$$S = \frac{ar}{1-r}, \text{ for } |r| < 1. \tag{9}$$

Solving equation 9 for r, in terms of S/a produces the inverse geometric series,

$$\begin{aligned} r &= \frac{S/a}{1+S/a} = S/a - (S/a)^2 + (S/a)^3 + \cdots \\ &= r_1 + r_2 + r_3 + \cdots , \text{ when } |S/a| < 1, \end{aligned} \tag{10}$$

where r_n is the portion of r that is nth order in S/a. When S is a geometric power series in r, then r is a geometric power series in S. The former is the forward series and the latter is the inverse series. That is exactly what the inverse series represents, the inverse geometric series of the forward series equation 6. This is the simplest prototype of an inverse series for r, i.e., the inverse of the forward geometric series for S.

For the seismic inverse problem, I associate S with the measured data (see e.g. Weglein et al., 2003)

$$S = (G - G_0)_{ms} = \text{Data},$$

and the forward and inverse series follow from treating the forward solution as S in terms of V, and the inverse solution as V in terms of S (where S corresponds to the measured values of $G - G_0$). The inverse series is the analog of equation 10 where r_1, r_2, $\cdots$ are simply replaced with V_1, V_2, $\cdots$.

$$V = V_1 + V_2 + V_3 + \cdots , \tag{11}$$

where V_n is the portion of V that is nth order in measured data, D. Equation 6 is the forward scattering series; and equation 11 is the inverse scattering series. The identity [equation 4] provides a generalized geometric forward series, a very special case of a Taylor series. A Taylor series of a function, $S(r)$,

$$\begin{aligned} S(r) &= S(0) + S'(0)r + \frac{S''(0)r^2}{2} + \cdots \\ \text{and } s(r) = S(r) - S(0) &= S'(0)r + \frac{S''(0)r^2}{2} + \cdots \end{aligned} \tag{12}$$

whereas the geometric series is

$$S(r) - \underbrace{S(0)}_{a} = ar + ar^2 + \cdots . \tag{13}$$

The Taylor series equation 12 reduces to the special case of a geometric series equation 13 if

$$S(0) = S'(0) = \frac{S''(0)}{2} = \cdots = a.$$

The geometric series equation 13 has an inverse series whereas the Taylor series equation 12 does not. In general, a Taylor series doesn't have an inverse series. That's the reason that inversionists committed to a Taylor series starting point adopt the indirect linear updating approach, where a linear approximate Taylor series is inverted. They attempt through updating to make the linear form an ever more accurate approximate — and its premise and justification is entirely indirect and hence ad hoc — in the sense that some sort of iterative linear updating of a reference medium and model matching seek to satisfy a property that a solution might "reasonably" satisfy.

The relationship (6) provides a geometric forward series that honors equation 4 in contrast to a truncated Taylor series that doesn't.

All conventional current mainstream parameter estimation inversion, including iterative linear inversion, AVO and FWI, are based on a forward Taylor series description of a given data (where the chosen data can often be fundamentally and intrinsically inadequate from a direct inversion perspective), that doesn't honor and remain consistent with the identity equation 6.

SOLVING A FORWARD PROBLEM IN AN INVERSE SENSE IS NOT THE SAME AS SOLVING AN INVERSE PROBLEM DIRECTLY

I will show that in general solving a forward problem in an inverse sense is not the same as solving an inverse problem directly. The exception is when the exact direct inverse is linear, as e.g. in the theory of wave equation migration (see, e.g. Claerbout, 1971; Stolt, 1978; Stolt and Weglein, 2012; Weglein et al., 2016). For wave equation migration, given a velocity model the migration and structure map output is a linear function of the input recorded reflection data.

To explain the latter statement, if I assume $S = ar$ (that is, that there is an exact linear forward relationship between S and r) then $r = S/a$ is solving the inverse problem directly. In that case, solving the forward problem in an inverse sense is the same as solving the inverse problem directly, that is, it provides a direct inverse solution.

However, if the forward exact relationship is non-linear, for example

$$S_n = ar + ar^2 + \cdots + ar^n$$
$$S_n - ar - ar^2 - \cdots - ar^n = 0 \tag{14}$$

and solving the forward problem (14) in an inverse sense for r will have n roots, $r_1, r_2, \ldots, r_n$. As $n \to \infty$, the number of roots $\to \infty$. However, from the direct nonlinear forward problem $S = \frac{ar}{1-r}$, I found the direct inverse solution $r = \frac{S}{a+S}$ has one real root.

This discussion above provides an extremely simple, transparent and compelling illustration of how solving a forward problem in an inverse sense is not the same as solving the inverse problem directly when there is a non-linear forward and non-linear inverse problem. The difference between solving a forward problem in an inverse sense (for example using equation 6 to solve for V) and solving an inverse problem directly (for example, equations 15-17 is much more serious, substantive and practically significant the further I move away from a scalar single component acoustic framework. For example, it is hard to overstate the differences when examining the direct and indirect inversion of the elastic heterogeneous wave equation for earth mechanical properties, and the consequences for structural and amplitude analysis and interpretation. This is a central flaw in many inverse approaches, including AVO and FWI (please see Weglein, 2013).

The expansion of V in equation 11, in terms of G_0 and $D = (G - G_0)_{ms}$, the inverse scattering series (Weglein et al., 2003) can be obtained as

$$G_0 V_1 G_0 = D, \tag{15}$$

$$G_0 V_2 G_0 = - G_0 V_1 G_0 V_1 G_0, \tag{16}$$

$$\begin{aligned} G_0 V_3 G_0 = &- G_0 V_1 G_0 V_1 G_0 V_1 G_0 \\ &- G_0 V_1 G_0 V_2 G_0 - G_0 V_2 G_0 V_1 G_0, \end{aligned} \tag{17}$$

$$\vdots$$

To illustrate how to solve equations 15, 16, 17, ... for V_1, V_2, V_3, ... consider the marine case with L_0 corresponding to a homogeneous reference medium of water. G_0 is the Green's function for propagation in water. D is the data measured for example, with towed streamer acquisition, G is the total field the hydrophone receiver records on the measurement surface, and G_0 is the field the reference wave (due to L_0) would record at the receiver. V then represents the difference between earth properties L and water properties L_0. The solution for V is found using

$$V = V_1 + V_2 + V_3 + \cdots, \tag{18}$$

where V_n is the portion of V that is nth order in the data, D. Substituting equation 18 into the forward series equation 6, then evaluating equation 6 on the measurement surface and setting terms that are equal order in the data equal I find equations 15, 16, 17, ... Solving equation 15 for V_1 involves the data D and G_0 (water speed propagator) and solving for V_1 is analytic, and corresponds to a prestack water-speed Stolt FK migration of the data D.

Hence, solving for V_1 involves an analytic water speed FK migration of the data D.

Solving for V_2 from equation 16 involves the same water-speed analytic Stolt FK migration of $-G_0V_1G_0V_1G_0$, a quantity that depends on V_1 and G_0, where V_1 depends on data and water speed, and G_0 is the water speed Green's function. Each term in the series produces V_n as an analytic Stolt FK migration of a new "effective data", where the effective data, the right-hand side of equations 15-17, are multiplicative combinations of factors that only depend on the data, D, and G_0. Hence, every term in the ISS is directly computed in terms of data and water speed. That's the direct non-linear inverse solution.

There are closed form inverse solutions for a 1D earth and a normal incident plane wave (see, e.g., Ware and Aki, 1969) but the inverse scattering series is the only direct inverse method for a multi-dimensional subsurface.

The inverse scattering series provides a direct method for obtaining the subsurface properties contained within the differential operator L, by inverting the series order-by-order to solve for the perturbation operator V, using only the measured data D and a reference Green's function G_0, for any assumed earth model type. Equations 15-17 provide V in terms of $V_1, V_2, \cdots$, and each of the V_i is computable directly in terms of D and G_0. There is one equation, equation 15, that exactly produces V_1, and V_1 is the exact portion of V that is linear in the measured data, D. The inverse operation to determine V_1, V_2, $V_3, \ldots$ is analytic, and never is updated with a bandlimited data, D. The band-limited nature of D never enters an updating process as occurs in iterative linear inversion, non-linear AVO and FWI.

THE INVERSE SCATTERING SERIES AND ISOLATED TASK SUBSERIES

I can imagine that a set of tasks need to be achieved to determine the subsurface properties, V, from recorded seismic data, D. These tasks are achieved within equations 15, 16, 17, ... The inverse tasks (and processing objectives) that are within a direct inverse solution are: (1) free-surface multiple removal, (2) internal multiple removal, (3) depth imaging, (4) Q compensation without Q, and (5) non-linear direct parameter estimation. Each of these five tasks has its own task-specific subseries from the ISS for $V_1, V_2, \cdots$, and each of those tasks is achievable directly and without subsurface information (see, e.g., Weglein et al., 2003; Weglein et al., 2012; Innanen and Lira, 2010). In Appendix A, I review the details of equations 15-17 for a 2D heterogeneous isotropic elastic medium.

Direct inverse and indirect inverse

Since iterative linear inversion is the concept and thinking behind many inverse approaches I thought to make explicit the difference between that approach and a direct inverse method. The direct 2D elastic isotropic inverse solution described in Appendix A is not iterative linear inversion. Iterative linear inversion starts with equation 15. In that approach, I solve for V_1 and then change the reference medium iteratively. The new differential operator L_0' and the new reference medium G_0' satisfy

$$L_0' = L_0 - V_1 \quad \text{and} \quad L_0' G_0' = \delta. \tag{19}$$

In the indirect iterative linear approach, all steps basically relate to the linear relationship equation 15 with a new reference background medium, with differential operator L_0' and a new reference Green's function G_0' where in terms of the new updated reference, L_0', equation 15 becomes

$$G'_0 V'_1 G'_0 = D' = (G - G'_0)_{ms}, \tag{20}$$

where V'_1 is the portion of V linear in data $(G - G'_0)_{ms}$. We can continue to update L'_0 and G'_0, and hope that indirect procedure is solving for the perturbation operator V. In contrast, the direct inverse solution equations 11 and A-3 calls for a single unchanged reference medium, for computing $V_1, V_2, \ldots$. For a homogeneous reference medium, $V_1, V_2, \ldots$ are each obtained by a single unchanged analytic inverse. We remind ourselves that the inverse to find V_1 from data, is the same exact unchanged analytic inverse operation to find $V_2, V_3, \ldots,$ from equations 15,16,..., which is completely distinct and different from equations 19 and 20 and higher iterates.

For ISS direct inversion, there are no numerical inverses, no generalized inverses, no inverses of matrices that are computed from and contain noisy band-limited data. The latter issue is terribly troublesomeand difficult and a serious practical problem which simply doesn't exist or occur with direct ISS methods. The inverse of operators that contain and depend on band-limited noisy data is a central and intrinsic characteristic and practical pitfall of indirect methods, model matching, updating, iterative linear inverse approaches (e.g. AVO and FWI).

Are there any circumstances where the indirect iterative linear inversion and the direct ISS parameter estimation would be equivalent?

Are there any circumstances where the ISS direct parameter inversion subseries would be equivalent to and correspond to the indirect iterative linear approach? Let's consider the simplest acoustic single reflector model, and a normal incident plane wave reflection data experiment with ideal full band-width perfect data. Let the upper half space have velocity c_0 and lower half space have velocity c_1 and then analyze these two methods (direct ISS parameter estimation and indirect iterative linear inversion) to use the reflected data event to determine the velocity of the lower half space, c_1. Yang and Weglein (2015) examined and analyzed this problem and compared the results of the direct ISS method and the indirect iterative linear inversion. They showed that the direct ISS inversion to estimate c_1 converged to c_1 under all circumstances and all values of c_0 and c_1. In contrast, the indirect linear iterative inversion had a limited range of values of c_0 and c_1 where it converged to c_1, and in that range it converged much slower than the direct ISS parameter estimation for c_1. The iterative linear inverse simply shut down and failed when the reflection coefficient, R, was greater than 1/4. See Appendix B and Yang (2014).

The direct ISS parameter estimation method converged to c_1 for any value of the reflection coefficient R. Hence, under the simplest possible circumstance, and providing the iterative linear method with an analytic Frechet derivative, as a courtesy from and gift delivered to the linear iterative from the ISS direct inversion method, the ranges of usefulness, validity and relative effectiveness were never equivalent or comparable. With band-limited data and more complex earth models (e.g., elastic multiparameter) this gap in the range of validity, usefulness and effectiveness will necessarily widen (please see Zhang, 2006 and Weglein, 2013). The indirect iterative linear inversion and the direct ISS parameter-estimation method are never equivalent, and there are absolutely no simple or complicated circumstances where they are equally effective. The distinct ISS free-surface-multiple elimination subseries and internal-multiple attenuation subseries are not only not dependent on subsurface properties, they are precisely the same unchanged algorithms for any earth model type.

There was an earlier time when free-surface multiples were modeled and subtracted. Multiple removal methods have moved on. Parameter estimation methods continue to be firmly connected to model matching and subtraction. That stark and immense difference between iterative linear updating model matching and the direct inversion inverse scattering methods is an essential point to consider and comprehend for those interested in understanding these methodologies and their seismic processing and interpretation consequences and value. It is not conceivable to even formulate an iterative linear model matching method that is not dependent on a specified model type — let alone to compare it with ISS model-type-independent algorithms.

Direct ISS parameter Inversion: A time lapse application

The direct inverse ISS elastic parameter estimation method [equation A-3] was successfully applied (Zhang et al., 2006) in a time lapse sense to discriminate between pressure and fluid saturation changes. Traditional time-lapse estimation methods were unable to predict and match that direct inversion ISS discrimination.

FURTHER SUBSTANTIVE DIFFERENCES BETWEEN ITERATIVE LINEAR MODEL MATCHING INVERSION AND DIRECT INVERSION FROM THE LIPPMANN-SCHWINGER EQUATION AND THE INVERSE SCATTERING SERIES

The difference between iterative linear and the direct inverse of equation A-3 is much more substantive and serious than merely a different way to solve $G_0V_1G_0 = D$ [equation 15], for

V_1. If equation 15 is someone's entire basic theory, you can mistakenly think that

$$\hat{D}^{PP} = \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \tag{21}$$

is sufficient to update

$$\hat{D}'^{PP} = \hat{G}_0'^P \hat{V}_1'^{PP} \hat{G}_0'^P \tag{22}$$

(generalizing equations 19 and 20). Please note that ˆ indicates variables are transformed to PS space. This step loses contact with and violates the basic operator identity $G = G_0 + G_0VG$ for the elastic wave equation. The fundamental identity $G = G_0 + G_0VG$ for the elastic wave equation is a non-linear multiplicative matrix relationship. For the forward and inverse series the input and output variables are matrices. The inverse solution for a change in an earth mechanical property has a nonlinear coupled dependence on <u>all</u> the data components

$$\begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix}$$

in 2D and the P, SH, SV 3×3 generalization in 3D.

A unique expansion of VG_0 in orders of measurement values of $(G - G_0)$ is

$$VG_0 = (VG_0)_1 + (VG_0)_2 + \dots \tag{23}$$

The scattering-theory equation allows that forward series form the opportunity to find a direct inverse solution. Substituting equation 23 into equation 6 and setting the terms of equal order in the data to be equal, I have $D = G_0V_1G_0$, where the higher order terms are $V_2, V_3, \dots$, as given in Weglein et al. (2003) page R33 equations 7-14.

For the elastic equation, V is a matrix and the relationship between the data and V_1 is

$$\begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix} = \begin{pmatrix} G_0^P & 0 \\ 0 & G_0^S \end{pmatrix} \begin{pmatrix} V_1^{PP} & V_1^{PS} \\ V_1^{SP} & V_1^{SS} \end{pmatrix} \begin{pmatrix} G_0^P & 0 \\ 0 & G_0^S \end{pmatrix}$$

$$V_1 = \begin{pmatrix} V_1^{PP} & V_1^{PS} \\ V_1^{SP} & V_1^{SS} \end{pmatrix}$$

$$V = \begin{pmatrix} V^{PP} & V^{PS} \\ V^{SP} & V^{SS} \end{pmatrix}$$

$$V = V_1 + V_2 + \dots$$

where V_1, V_2 are linear, quadratic contributions to V in terms of the data,

$$D = \begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix}.$$

The changes in elastic properties and density are contained in

$$V = \begin{pmatrix} V^{PP} & V^{PS} \\ V^{SP} & V^{SS} \end{pmatrix},$$

and that leads to direct and explicit solutions for the changes in mechanical properties in orders of the data,

$$D = \begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix},$$

$$\frac{\Delta\gamma}{\gamma} = (\frac{\Delta\gamma}{\gamma})_1 + (\frac{\Delta\gamma}{\gamma})_2 + \dots$$

$$\frac{\Delta\mu}{\mu} = (\frac{\Delta\mu}{\mu})_1 + (\frac{\Delta\mu}{\mu})_2 + \dots$$

$$\frac{\Delta\rho}{\rho} = (\frac{\Delta\rho}{\rho})_1 + (\frac{\Delta\rho}{\rho})_2 + \dots$$

where γ, μ and ρ are the bulk modulus, shear modulus and density, respectively.

The ability of the forward series to have a direct inverse series derives from (1) the identity among G, G_0, V provided by the scattering equation and then (2) the recognition that the forward solution can be viewed as a geometric series for the data, D, in terms of VG_0. The latter derives the direct inverse series for VG_0 in terms of the data.

Viewing the forward problem and series as the Taylor series

$$D(m) = D(m_0) + D'(m_0)\Delta m + \frac{D''(m_0)}{2}\Delta m^2 + \ldots, \tag{24}$$

in which the derivatives are Frechet derivatives, in terms of Δm does not offer a direct inverse series, and hence there is no choice but to solve the forward series in an inverse sense. It is that fact that results in all current AVO and FWI methods being modeling methods that are solved in an inverse sense. Among references that solve a forward problem in an inverse sense in P-wave AVO are Beylkin and Burridge (1990), Boyse and Keller (1986), Burridge et al. (1998), Castagna and Smith (1994), Clayton and Stolt (1981), Foster et al. (2010), Goodway (2010), Goodway et al. (1997), Shuey (1985), Smith and Gidlow (2000), Stolt (1989), and Stolt and Weglein (1985). The intervention of the explicit relationship among G, G_0, and V (the scattering equation) in a Taylor series-like form produces a geometric series and a direct inverse solution.

The linear equations are:

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \tag{25}$$

$$\hat{D}^{PP} = \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \tag{26}$$

$$\hat{D}^{PS} = \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \tag{27}$$

$$\hat{D}^{SP} = \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \tag{28}$$

$$\hat{D}^{SS} = \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \tag{29}$$

$$\begin{aligned} \tilde{D}^{PP}(k_g,0;-k_g,0;\omega) &= -\frac{1}{4}\left(1-\frac{k_g^2}{\nu_g^2}\right)\tilde{a}_\rho^{(1)}(-2\nu_g) \\ &-\frac{1}{4}\left(1+\frac{k_g^2}{\nu_g^2}\right)\tilde{a}_\gamma^{(1)}(-2\nu_g)+\frac{2k_g^2\beta_0^2}{(\nu_g^2+k_g^2)\alpha_0^2}\tilde{a}_\mu^{(1)}(-2\nu_g) \end{aligned} \tag{30}$$

$$\begin{aligned} \tilde{D}^{PS}(\nu_g,\eta_g) &= -\frac{1}{4}\left(\frac{k_g}{\nu_g}+\frac{k_g}{\eta_g}\right)\tilde{a}_\rho^{(1)}(-\nu_g-\eta_g) \\ &-\frac{\beta_0^2}{2\omega^2}k_g(\nu_g+\eta_g)\left(1-\frac{k_g^2}{\nu_g\eta_g}\right)\tilde{a}_\mu^{(1)}(-\nu_g-\eta_g) \end{aligned} \tag{31}$$

$$\begin{aligned} \tilde{D}^{SP}(\nu_g,\eta_g) &= \frac{1}{4}\left(\frac{k_g}{\nu_g}+\frac{k_g}{\eta_g}\right)\tilde{a}_\rho^{(1)}(-\nu_g-\eta_g) \\ &+\frac{\beta_0^2}{2\omega^2}k_g(\nu_g+\eta_g)\left(1-\frac{k_g^2}{\nu_g\eta_g}\right)\tilde{a}_\mu^{(1)}(-\nu_g-\eta_g) \text{ and} \end{aligned} \tag{32}$$

$$\begin{aligned} \tilde{D}^{SS}(k_g,\eta_g) &= \frac{1}{4}\left(1-\frac{k_g^2}{\eta_g^2}\right)\tilde{a}_\rho^{(1)}(-2\eta_g) \\ &-\left[\frac{\eta_g^2+k_g^2}{4\eta_g^2}-\frac{2k_g^2}{\eta_g^2+k_g^2}\right]\tilde{a}_\mu^{(1)}(-2\eta_g), \end{aligned} \tag{33}$$

where $a_\gamma^{(1)}$, $a_\mu^{(1)}$, and $a_\rho^{(1)}$ are the linear estimates of the changes in bulk modulus, shear modulus, and density, respectively. k_g is the Fourier conjugate to the receiver position x_g and ν_g and η_g are the vertical wavenumbers for the P and S reference waves, respectively, where

$$\nu_g^2 + k_g^2 = \frac{\omega^2}{\alpha_0^2}$$
$$\eta_g^2 + k_g^2 = \frac{\omega^2}{\beta_0^2}$$

and α_0 and β_0 are the P and S velocities in the reference medium, respectively. The direct quadratic non-linear equations are

$$\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}$$
$$= - \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}, \tag{34}$$

$$\hat{G}_0^P \hat{V}_2^{PP} \hat{G}_0^P = -\hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P - \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \tag{35}$$

$$\hat{G}_0^P \hat{V}_2^{PS} \hat{G}_0^S = -\hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S - \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S, \tag{36}$$

$$\hat{G}_0^S \hat{V}_2^{SP} \hat{G}_0^P = -\hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P - \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \tag{37}$$

$$\hat{G}_0^S \hat{V}_2^{SS} \hat{G}_0^S = -\hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S - \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S. \tag{38}$$

Because $\hat{V}_1^{PP}$ relates to $\hat{D}^{PP}$, $\hat{V}_1^{PS}$ relates to $\hat{D}^{PS}$, and so on, the four components of the data will be coupled in the nonlinear elastic inversion. I cannot perform the direct nonlinear inversion without knowing all components of the data. Thus, the direct nonlinear solution determines the data needed for a direct inverse. That, in turn, defines what a linear estimate means. That is, a linear estimate of a parameter is an estimate of a parameter that is linear in data that can directly invert for that parameter. Since D^{PP}, D^{PS}, D^{SP}, and D^{SS} are needed to determine a_γ, a_μ, and a_ρ directly, a linear estimate for any one of these quantities requires simultaneously solving equations 30-33. See, e.g., Weglein et al. (2009) for further detail.

Those direct nonlinear formulas are like the direct solution for the quadratic equation mentioned above and solve directly and nonlinearly for changes in the velocities, α, β and the density ρ in a 1D elastic Earth. Stolt and Weglein (2012), present the linear equations for a 3D Earth that generalize equations 30-33. Those formulas prescribe precisely what data you need as input, and they dictate how to compute those sought-after mechanical properties, given the necessary data. There is no search or cost function, and the unambiguous and unequivocal data needed are full multicomponent data — PP, PS, SP, and SS — for all traces in each of the P and S shot records. The direct algorithm determines first the data needed and then the appropriate algorithms for using those data to directly compute the sought-after changes in the Earth's mechanical properties. Hence, any method that calls itself inversion (let alone full-wave inversion) for determining changes in elastic properties, and in particular the P-wave velocity α, and that inputs only P-data, is more off base, misguided, and lost than the methods that sought two or more functions of depth from a single trace. You can model-match P-data until the cows come home, and that takes a lot of computational effort and people with advanced degrees in math and physics computing Frechet derivatives, and requires sophisticated L_P norm cost functions and local or global search engines, so it must be reasonable, scientific, and worthwhile. Why can't I use just PP-data to invert for changes in V_P, V_S, and density, because Zoeppritz says that I can model PP from those quantities, and because I have, using PP-data with angle variation, enough dimension? As stated above, data dimension is good, but not good enough for a direct inversion of those elastic properties.

Adopting equations 21 and 22 as in AVO and FWI, there is a violation of the fundamental relationship between changes in a medium and changes in a wavefield, $G = G_0 + G_0VG$, which is as serious as considering problems involving a right triangle and violating the Pythagorean theorem . That is, iteratively updating PP data with an elastic model violates the basic relationship between changes in a medium, V, and changes in the wavefield, $G-G_0$, for the simplest elastic earth model.

This direct inverse method for parameter estimation provides a platform for amplitude analysis, and a solid framework and direct methodology for the goals and objectives of indirect methods like AVO and FWI. A direct method for the purposes of amplitude analysis provides a method that derives from, respects and honors the fundamental identity and relationship $G = G_0 + G_0VG$. Iteratively inverting multi-component data has the correct data but doesn't correspond to a direct inverse algorithm. To honor $G = G_0 + G_0VG$, you need both the data and the algorithm that direct inverse prescribes. Not recognizing the message that an operator identity and the elastic wave equation unequivocally communicate is a fundamental and significant contribution to the gap in effectiveness in current AVO and FWI methods and application [equation A-3]. This analysis generalizes to 3D with P, SH, and SV data.

The role of direct and indirect methods

There's a role for direct and indirect methods in practical real-world applications. In our view, indirect methods are to be called upon for recognizing that the world is more complicated than the physics that we assume in our models and methods. For the part of the world that you are capturing in your model and physics, nothing compares to direct methods for clarity and effectiveness. An optimal indirect method would seek to satisfy a cost function that derives from a property of the direct method. In that way the indirect and direct methods would be aligned, consistent and cooperative for accommodating both the part of the world described by your physical model (with a direct inverse method) and the part that is outside (with an indirect method).

The indirect method of model matching primaries and multiples (so-called FWI)

All model matching inverse approaches are indirect methods. Iterative linear inversion model matching is an indirect search methodology, and ad-hoc, and without a firm and solid foundation and theoretical and conceptual framework. Never the less, we can imagine and understand that model matching primaries and multiples, rather than only primaries, could improve upon matching only primaries. However, model matching primaries and multiples remains ad hoc and indirect and always on much shakier footing than direct inversion for the same inversion goals and objectives. Direct ISS inversion for parameter estimation only requires and inputs primaries.

For all multidimensional seismic applications, the only direct inverse solution is provided by the operator identity equation 4 and is in the form of a series equations 15-17, the inverse scattering series (Weglein et al., 2003). It can achieve all processing objectives within a single framework and a single set of equations 15-17 without requiring any subsurface information. There are distinct isolated-task inverse scattering subseries derived from the ISS, which can perform free-surface multiple removal (Carvalho et al., 1992; Weglein et al., 1997), internal multiple removal (Araújo et al., 1994; Weglein et al., 2003), depth imaging (e.g. Shaw, 2005;

Liu, 2006; Weglein et al., 2012), parameter estimation (Zhang, 2006; Liang, 2013; Li, 2011; Yang and Weglein, 2015), and Q compensation without needing, estimating or determining Q (Innanen and Weglein, 2007; Innanen and Lira, 2010; Lira, 2009), and each achieves its objective directly and without subsurface information. The direct inverse solution (e.g., Weglein et al., 2003, 2009) provides a framework and firm math-physics foundation that unambiguously defines both the data requirements and the distinct algorithms to perform each and every associated task within the inverse problem, directly and without subsurface information.

Having an ad hoc indirect method as the starting point places a cloud over issue identification when less than satisfactory results arise with field data. In addition, we saw that direct inversion parameter estimation has a significantly less dependence on the low frequency data components in comparison with indirect methods like nonlinear AVO and FWI.

Only a direct solution can provide algorithmic clarity, confidence and effectiveness. The current industry standard AVO and FWI, using variants of model-matching and iterative linear inverse, are indirect methods , and iteratively linearly updating P data or multicomponent data (with or without multiples) does not correspond to, and will not produce, a direct solution.

All direct inverse methods for structural determination and amplitude analysis require only primaries

In Weglein (2016) the role of primaries and multiples in imaging are examined and analyzed. The most capable and interpretable migration method derives from predicting a source and receiver experiment at depth. For data consisting of primaries and multiples, a discontinuous velocity model is needed to achieve that predicted experiment at depth. With that discontinuous velocity model, free-surface and internal multiples play no role in the migration and the exact same image results with or without multiples (see Weglein, 2016). For a smooth velocity model, multiples will result in false and misleading images and must be removed before the migration and migration-inversion of primaries.

In Weglein et al. (2003), the ISS direct depth imaging (without a velocity model or subsurface information) removes free-surface and internal multiples prior to the distinct sub-series that input primaries and perform depth imaging and amplitude analysis, respectively, each directly and without subsurface information, and only using and requiring primaries.

Hence, all direct inversion methods, both those with and those without subsurface/velocity information, require only primaries for complete structural determination and amplitude analysis. Methods that seek to use multiples to address issues from less than a complete acquisition of primaries, are seeking an appropriate image of an unrecorded primary.

Indirect methods are ad hoc without a clear or firm math physics foundation and framework, and they start without knowing whether "the indirect solution" is in fact a solution. A more complete or fuller data set being matched between model data and field data each with primaries and multiples could at times improve upon matching only primaries, but the entire approach is indirect and ad hoc with or without multiples, and lacks the benefits of a direct method. With indirect methods there is no framework and theory to rely on, and to have confidence that a solution is forthcoming under any circumstances.

If I seek the parameters of an elastic heterogeneous isotropic subsurface, then the differential operator in the operator identity is the differential operator that occurs in the elastic, heterogeneous isotropic wave equation. From forty years of AVO and amplitude analysis application in the petroleum industry, the elastic isotropic model is the base-line minimally realistic and acceptable earth model-type for amplitude analysis, for example, for AVO and FWI. Then taking the operator identity (called the Lippmann-Schwinger or scattering theory equation) for the elastic wave equation, I can obtain a direct inverse solution for the changes in elastic properties and density. The direct inverse solution specifies both the data required and the algorithm to achieve a direct parameter estimation solution. In this paper I explain how this methodology differs from all current AVO and FWI methods, that are in fact forms of model matching . Multicomponent data consisting of only primaries are needed for a direct inverse solution for subsurface properties. This paper focuses on one specific inverse task, parameter estimation, within the overall and broader set of inversion objectives and tasks. Furthermore, the impact of band-limited data and noise, are discussed and compared for the direct ISS parameter estimation and indirect (AVO and FWI) inversion methods.

In this paper, I focused on analyzing and examining the direct inverse solution that the ISS inversion subseries provides for parameter estimation. The distinct issues of: (1) data requirements, (2) model-type, and (3) inversion algorithm for the direct inverse are all important (Weglein, 2015b). For an elastic heterogeneous medium, I show that the direct inverse requires multi-component/PS (P-component and S-component) data and prescribes how that data are utilized for a direct parameter estimation solution (Zhang and Weglein, 2006).

CONCLUSIONS

In this paper, I describe, illustrate and analyze the considerable conceptual, substantive and practical benefit and added-value that a direct parameter inversion from the inverse scattering series provides in comparison with all current indirect inverse methods (e.g., AVO and FWI) for amplitude analysis goals and objectives. A direct method provides: (1) a solution that we (the seismic industry) can have confidence is in fact solving the defined problem of interest and (2) in addition, when the method doesn't improve drilling decisions, then we know that the issue is that the problem of interest is not the problem that we need to be interested in. On the other hand, indirect methods like AVO and FWI, have a plethora of approaches and paths, and when less than satisfactory results occur we don't know whether the issue is the chosen problem of interest or the choice among innumerable indirect solutions, or both.

All scientific methods make assumptions — and seismic processing and interpretation methods are no exception. When the assumptions behind seismic methods are satisfied the methods are useful and effective and can support successful drill decisions. When the assumptions are not satisfied the methods can have difficulty or can fail. The latter breakdown can contribute to unsuccessful ill-informed drill decisions, dry hole drilling or suboptimal appraisal and development wells.

The objective of seismic research is to provide new and effective tool box capability for processing and interpretation that will improve the drill success rate and reduce dry hole and suboptimal drilling decisions. Towards that end the starting point in seismic research is to identify the outstanding prioritized problems and challenges that need to be addressed and solved.

The ability to clearly and unambiguously define the origin and root cause behind seismic issues, problems, breakdown and challenges is an essential and critically important step in designing and executing a strategy to provide new and more capable methods to the seismic processing and interpretation toolbox.

Direct inversion methods can provide that problem definition and clarity. They are also unique in providing the confidence that the problem of interest is actually being addressed. For ISS parameter estimation, while the recorded data is of course band limited, the band-limited data is never used to compute the updated inverse operator for the next iterated linear step, since the inverse operator is fixed and analytic for every term in the inverse scattering series. That's one of several important and substantive differences pointed out in this paper between the direct inverse ISS parameter estimation method and all indirect inversion methods, e.g., AVO and FWI. I provide an explicit analytic example and comparison between direct ISS parameter estimation and the indirect linear updating model matching concepts behind AVO and FWI.

All seismic processing methods depend on the amplitude and phase of seismic data. Different processing methods that seek to achieve a certain specific processing goal can have different relative sensitivities to noise and bandwidth. Amplitude analysis for determining earth mechanical property changes is one of the most sensitive. Methods that achieve seismic goals as a sequence of separate intermediate steps have a natural advantage over methods that seek to combine goals. Achieving an intermediate easier goal that's less demanding can significantly enhance the ability to achieve the subsequent more demanding seismic processing objectives. The indirect methods that seek to locate structure and identify changes in earth mechanical properties at once have a terrible dependence on missing low frequency data. However if I first locate structure by wave equation migration (a process that is insensitive to missing low frequency data) then in principle I can determine earth mechanical property changes with a single frequency within the bandwidth. The ISS direct amplitude analysis method described and exemplified and tested and compared in this paper, assumes that a set of less daunting seismic processing tasks, using ISS task specific subseries, that have achieved (e.g., multiple removal, depth imaging) before this task is undertaken. To have a fair comparison, the indirect model matching method is tested with a data with a well located single reflector, and hence there are no imaging issues or multiples in the problem. That allows a pristine, clear and definitive comparison of the amplitude analysis — parameter estimation function of the prototype direct ISS method and the corresponding indirect model-matching iterative updating approach. There are important issues of resolution and illumination, that will impact the results of this paper, with advances in migration theory and algorithms that avoid all high frequency approximations in the imaging principles and wave propagation models that can improve resolution and illumination (e.g., Weglein et al., 2016).

Direct and indirect methods both can play an important role and function in seismic processing: where the former accommodates and addresses the assumed physics and the latter provides a channel for real world phenomena beyond the assumed physics. Both are called for within a comprehensive and effective seismic processing and interpretation strategy.

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REFERENCES

Anderson, J. E., L. Tan, and D. Wang, 2012, Time-reversal checkpointing methods for RTM and FWI: Geophysics, **77**, S93–S103.

Araújo, F. V., A. B. Weglein, P. M. Carvalho, and R. H. Stolt, 1994, Inverse scattering series for multiple attenuation: An example with surface and internal multiples: 64th Annual International Meeting, SEG, Expanded Abstracts, 1039–1041.

Baumstein, A., J. E. Anderson, D. L. Hinkley, and J. R. Krebs, 2009, Scaling of the objective function gradient for full-wavefield inversion: 79th Annual International Meeting, SEG, Expanded Abstracts, 2243–2247.

Ben-Hadj-ali, H., S. Operto, and J. Vireux, 2009, Efficient 3D frequency-domain full-waveform inversion (FWI) with phase encoding: 71st Conference & Exhibition, EAGE, Extended Abstracts, P004.

Ben-Hadj-ali, H., S. Operto, and J. Virieux, 2008, Velocity model building by 3D frequency-domain, full-waveform inversion of wide-aperture seismic data: Geophysics, **73**, VE101–VE117.

Beylkin, G., and R. Burridge, 1990, Linearized inverse scattering problem of acoustics and elasticity: Wave Motion, **12**, 15–22.

Biondi, B., and P. Sava, 1999, Wave-equation migration velocity analysis: 69th Annual International Meeting, SEG, Expanded Abstracts, 1723–1726.

Biondi, B., and W. Symes, 2004, Angle-domain common-image gathers for migration velocity analysis by wavefield-continuation imaging: Geophysics, **69**, 1283–1298.

Blum, E. K., 1972, Numerical analysis and computation: Theory and practice: Addison-Wesley Publishing Company.

Boyse, W. E., and J. B. Keller, 1986, Inverse elastic scattering in three dimensions: The

Journal of the Acoustical Society of America, **79**, 215–218.

Brandsberg-Dahl, S., M. de Hoop, and B. Ursin, 1999, Velocity analysis in the common scattering-angle/azimuth domain: 69th Annual International Meeting, SEG, Expanded Abstracts, 1715–1718.

Brossier, R., S. Operto, and J. Virieux, 2009, Robust elastic frequency-domain full-waveform inversion using the L_1 norm: Geophysical Research Letters, **36**, L20310.

Burridge, R., M. de Hoop, D. Miller, and C. Spencer, 1998, Multiparameter inversion in anisotropic elastic media: Geophysical Journal International, **134**, 757–777.

Carvalho, P. M., 1992, Free-surface multiple reflection elimination method based on nonlinear inversion of seismic data: PhD thesis, Universidade Federal da Bahia. (In Portuguese).

Carvalho, P. M., A. B. Weglein, and R. H. Stolt, 1992, Nonlinear inverse scattering for multiple suppression: Application to real data. Part I: 62nd Annual International Meeting, SEG, Expanded Abstracts, 1093–1095.

Castagna, J. P., and S. W. Smith, 1994, Comparison of AVO indicators: A modeling study: Geophysics, **59**, 1849–1855.

Chavent, G., and C. Jacewitz, 1995, Determination of background velocities by multiple migration fitting: Geophysics, **60**, 476–490.

Claerbout, J. F., 1971, Toward a unified theory of reflector mapping: Geophysics, **36**, 467–481.

Clayton, R. W., and R. H. Stolt, 1981, A Born-WKBJ inversion method for acoustic reflection data: Geophysics, **46**, 1559–1567.

Crase, E., A. Pica, M. Noble, J. McDonald, and A. Tarantola, 1990, Robust elastic nonlinear waveform inversion: Application to real data: Geophysics, **55**, 527–538.

Ferreira, A., 2011, Internal multiple removal in offshore Brazil seismic data using the Inverse

Scattering Series: Master's thesis, University of Houston.

Fichtner, A., 2011, Full seismic waveform modeling and inversion: Springer-Verlag.

Foster, D. J., R. G. Keys, and F. D. Lane, 2010, Interpretation of AVO anomalies: Geophysics, **75**, 75A3–75A13.

Gauthier, O., J. Virieux, and A. Tarantola, 1986, Two dimensional nonlinear inversion of seismic waveforms: Geophysics, **51**, 1387–1403.

Goodway, B., 2010, The magic of Lamé: The Leading Edge, **29**, 1432.

Goodway, B., T. Chen, and J. Downton, 1997, Improved AVO fluid detection and lithology discrimination using Lamé petrophysical parameters; "$\lambda\rho$", "$\mu\rho$", and "λ/μ fluid stack", from P and S inversions: 67th Annual International Meeting, SEG, Expanded Abstracts, 183–186.

Guasch, L., M. Warner, T. Nangoo, J. Morgan, A. Umpleby, I. Stekl, and N. Shah, 2012, Elastic 3D full-waveform inversion: 82nd Annual International Meeting, SEG, Expanded Abstracts, 1–5.

Hawthorn, A., 2009, Real Time Sesimic Measurements Whilst Drilling — A Drilling Optimization Measurement for Subsalt Wells: Presented at the EAGE Subsalt Imaging Workshop, Cairo, Egypt.

Iledare, O. O., and M. J. Kaiser, 2007, Competition and performance in oil and gas lease sales and development in the U.S. Gulf of Mexico OCS Region, 1983-1999: Technical report, U.S. Dept. of the Interior, Minerals Management Service, Gulf of Mexico OCS Region. (OCS Study MMS 2007-034. 106pp.).

Innanen, K. A., and J. E. Lira, 2010, Direct nonlinear q-compensation of seismic primaries reflecting from a stratified, two-parameter absorptive medium: Geophysics, **75**, V13–V23.

Innanen, K. A., and A. B. Weglein, 2007, On the construction of an absorptive-dispersive medium model via direct linear inversion of reflected seismic primaries: Inverse Problems, **23**.

Kapoor, S., D. Vigh, H. Li, and D. Derharoutian, 2012, Full-waveform inversion for detailed velocity model building: 74th Annual Conference and Exhibition, EAGE, Extended Abstracts, W011.

Keys, R. G., and A. B. Weglein, 1983, Generalized linear inversion and the first Born theory for acoustic media: Journal of Mathematical Physics, **24**, 1444–1449.

Li, X., 2011, I.- Multi-parameter depth imaging using the inverse scattering series; II.- Multi-component direct non-linear inversion for elastic earth properties using the inverse scattering series: PhD thesis, University of Houston.

Li, X., F. Liu, and A. B. Weglein, 2011, Dealing with the wavelet aspect of the low frequency issue: A synthetic example: M-OSRP 2010-2011 Annual Report, 82–89.

Liang, H., 2013, Addressing several key outstanding issues and extending the capability of the inverse scattering subseries for internal multiple attenuation, depth imaging, and parameter estimation: PhD thesis, University of Houston.

Liang, H., A. B. Weglein, and X. Li, 2010, Initial tests for the impact of matching and mismatching between the Earth model and the processing model for the ISS imaging and parameter estimation: M-OSRP 2009-2010 Annual Report, 165–180.

Lira, J. E. M., 2009, Compensating reflected primary seismic amplitudes for elastic and absorptive transmission losses when the physical properties of the overburden are unknown: PhD thesis, University of Houston.

Liu, F., 2006, Multi-dimensional depth imaging without an adequate velocity model: PhD thesis, University of Houston.

Luo, Y., P. G. Kelamis, Q. Fu, S. Huo, G. Sindi, S.-Y. Hsu, and A. B. Weglein, 2011, Elimination of land internal multiples based on the inverse scattering series: The Leading Edge, **30**, 884–889.

Matson, K. H., 1997, An inverse-scattering series method for attenuating elastic multiples from multicomponent land and ocean bottom seismic data: PhD thesis, University of British Columbia.

Nolan, C., and W. Symes, 1997, Global solution of a linearized inverse problem for the wave equation: Communications on Partial Differential Equations, **22**, 919–952.

Pratt, R. G., 1999, Seismic waveform inversion in the frequency domain, Part 1: Theory and verification in a physical scale model: Geophysics, **64**, 888–901.

Pratt, R. G., and R. M. Shipp, 1999, Seismic waveform inversion in the frequency domain, Part 2: Fault delineation in sediments using crosshole data: Geophysics, **64**, 902–914.

Rickett, J., and P. Sava, 2002, Offset and angle-domain common image-point gathers for shot-profile migration: Geophysics, **67**, 883–889.

Sava, P., B. Biondi, and J. Etgen, 2005, Wave-equation migration velocity analysis by focusing diffractions and reflections: Geophysics, **70**, U19–U27.

Sava, P., and S. Fomel, 2003, Angle-domain common-image gathers by wavefield continuation methods: Geophysics, **68**, 1065–1074.

Shaw, S. A., 2005, An inverse scattering series algorithm for depth imaging of reflection data from a layered acoustic medium with an unknown velocity model: PhD thesis, University of Houston.

Shaw, S. A., and A. B. Weglein, 2004, A leading order imaging series for prestack data acquired over a laterally invariant acoustic medium. Part II: Analysis for data missing low frequencies: M-OSRP 2003-2004 Annual Report, 140–167.

Shuey, R. T., 1985, A simplification of the zoeppritz equations: Geophysics, **50**, 609–614.

Sirgue, L., O. I. Barkved, J. Dellinger, J. Etgen, U. Albertin, and J. H. Kommedal, 2010, Full-waveform inversion: the next leap forward in imaging at Valhall: First Break, **28**, 65–70.

Sirgue, L., O. I. Barkved, J. P. Van Gestel, O. J. Askim, and J. H. Kommedal, 2009, 3D waveform inversion on Valhall wide-azimuth OBC: 71st Annual Conference and Exhibition, EAGE, Extended Abstracts, U038.

Sirgue, L., B. Denel, and F. Gao, 2012, Challenges and value of applying FWI to depth imaging projects: Presented at the 74th Conference and Exhibition, EAGE, Extended Abstracts.

Smith, G. C., and M. Gidlow, 2000, A comparison of the fluid factor with λ and μ in AVO analysis: 70th Annual International Meeting, SEG, Expanded Abstracts, 122–125.

Stolt, R. H., 1978, Migration by Fourier transform: Geophysics, **43**, 23–48.

———, 1989, Seismic inversion revisited: Geophysical Inversion, Society for Industrial and Applied Mathematics, 3–19. (Proceedings of the Geophysical Inversion Workshop, September 27-29, 1989, Houston, Texas).

Stolt, R. H., and A. B. Weglein, 1985, Migration and inversion of seismic data: Geophysics, **50**, 2458–2472.

———, 2012, Seismic imaging and inversion: Application of linear inverse theory: Cambridge University Press.

Symes, W. W., 2008, Migration velocity analysis and waveform inversion: Geophysical Prospecting, **56**, 765–790.

Symes, W. W., and J. J. Carazzone, 1991, Velocity inversion by differential semblance optimization: Geophysics, **56**, 654–663.

Tarantola, A., 1986, A strategy for nonlinear elastic inversion of seismic reflection data:

Geophysics, **51**, 1893–1903.

———, 1987, Inverse problem theory: Method for data fitting and model parameter estimation: Elsevier.

Taylor, J. R., 1972, Scattering theory: the quantum theory of nonrelativistic collisions: John Wiley & Sons, Inc.

Valenciano, A., B. Biondi, and A. Guitton, 2006, Target-oriented wave-equation inversion: Geophysics, **71**, A35–A38.

Vigh, D., and E. W. Starr, 2008, 3D prestack plane-wave, full-waveform inversion: Geophysics, **73**, VE135–VE144.

Ware, J. A., and K. Aki, 1969, Continuous and discrete inverse-scattering problems in a stratified elastic medium. i. plane waves at normal incidence: The Journal of the Acoustical Society of America, **45**, 911–921.

Weglein, A., J. Mayhan, Y. Zou, Q. Fu, F. Liu, J. Wu, C. Ma, X. Lin, and R. Stolt, 2016, The first migration method that is equally effective for all acquired frequencies for imaging and inverting at the target and reservoir: 86th Annual International Meeting, SEG, Expanded Abstracts, 4266–4272.

Weglein, A. B., 2012a, Short note: A formalism for (1) modeling the amplitude and phase of pressure waves from a heterogeneous elastic medium and (2) selecting and predicting P-wave events that have only experienced pressure-wave episodes in their history: M-OSRP 2011-2012 Annual Report, 364–370.

———, 2012b, Short note: An alternative adaptive subtraction criteria (to energy minimization) for free surface multiple removal: M-OSRP 2011-2012 Annual Report, 375.

———, 2013, A timely and necessary antidote to indirect methods and so-called P-wave FWI: The Leading Edge, **32**, 1192–1204.

———, 2015a, A direct inverse solution for AVO/FWI parameter estimation objectives: 85th Annual Internat. Mtg., SEG, Expanded Abstracts, 3367–3370.

———, 2015b, Direct Inversion and FWI. Invited keynote address given at the SEG Workshop Full-waveform Inversion: Filling the Gaps, Abu Dhabi, UAE, available at `http://mosrp.uh.edu/events/event-news/arthur-b-weglein-will-present-an-invited-key-note-address-on-direct-inversion-at-the-seg-workshop-on-fwi-30-march-1-april-2015-in-abu-dhabi-uae`.

———, 2016, Multiples: Signal or noise?: Geophysics, **81**, V283–V302.

Weglein, A. B., F. V. Araújo, P. M. Carvalho, R. H. Stolt, K. H. Matson, R. T. Coates, D. Corrigan, D. J. Foster, S. A. Shaw, and H. Zhang, 2003, Inverse scattering series and seismic exploration: Inverse Problems, **19**, R27–R83.

Weglein, A. B., D. J. Foster, K. H. Matson, S. A. Shaw, P. M. Carvalho, and D. Corrigan, 2002, Predicting the correct spatial location of reflectors without knowing or determining the precise medium and wave velocity: initial concept, algorithm and analytic and numerical example: Journal of Seismic Exploration, **10**, 367–382.

Weglein, A. B., F. A. Gasparotto, P. M. Carvalho, and R. H. Stolt, 1997, An inverse-scattering series method for attenuating multiples in seismic reflection data: Geophysics, **62**, 1975–1989.

Weglein, A. B., S.-Y. Hsu, P. Terenghi, X. Li, and R. H. Stolt, 2011, Multiple attenuation: Recent advances and the road ahead (2011): The Leading Edge, **30**, 864–875.

Weglein, A. B., F. Liu, X. Li, P. Terenghi, E. Kragh, J. D. Mayhan, Z. Wang, J. Mispel, L. Amundsen, H. Liang, L. Tang, and S.-Y. Hsu, 2012, Inverse scattering series direct depth imaging without the velocity model: first field data examples: Journal of Seismic Exploration, **21**, 1–28.

Weglein, A. B., and K. Matson, 1998, Inverse-scattering interval multiple attenuation: an analytic example and subevent interpretation, *in* Mathematical methods in geophysical imaging: SPIE, 1008–1017.

Weglein, A. B., H. Zhang, A. C. Ramírez, F. Liu, and J. E. M. Lira, 2009, Clarifying the underlying and fundamental meaning of the approximate linear inversion of seismic data: Geophysics, **74**, WCD1–WCD13.

Wu, J., and A. B. Weglein, 2014, Elastic Green's theorem preprocessing for on-shore internal multiple attenuation: Theory and initial synthetic data tests: 84th Annual International Meeting, SEG, Expanded Abstracts, 4299–4304.

Yang, J., 2014, Extending the Inverse Scattering Series free-surface multiple elimination and internal multiple attenuation algorithms by incorporating the source wavelet and radiation pattern: Examining and evaluating the benefit and added-value: PhD thesis, University of Houston.

Yang, J., and A. B. Weglein, 2015, A first comparison of the inverse scattering series non-linear inversion and the iterative linear inversion for parameter estimation: 85th Annual International Meeting, SEG, Expanded Abstracts, 1263–1267.

Zhang, H., 2006, Direct non-linear acoustic and elastic inversion: towards fundamentally new comprehensive and realistic target identification: PhD thesis, University of Houston.

Zhang, H., and A. B. Weglein, 2006, Direct non-linear inversion of multi-parameter 1d elastic media using the inverse scattering series: 76th International Annual Meeting, SEG, Expanded Abstracts, 284–311.

Zhang, H., A. B. Weglein, R. G. Keys, D. J. Foster, and S. A. Shaw, 2006, Discrimination between pressure and fluid saturation using direct non-linear inversion method: an application to time-lapse seismic data: M-OSRP 2005-2006 Annual Report, 312–327.

Zhang, Y., and B. Biondi, 2013, Moveout-based wave-equation migration velocity analysis: Geophysics, **78**, U31–U39.

Zhou, H., L. Amundsen, and G. Zhang, 2012, Fundamental issues in full-waveform inversion: Presented at the 82nd Annual International Meeting, SEG, Expanded Abstracts.

APPENDIX A: THE OPERATOR IDENTITY AND DIRECT INVERSE SOLUTION FOR A 2D HETEROGENEOUS ISOTROPIC ELASTIC MEDIUM

I describe the forward and direct inverse method for a 2D elastic heterogeneous earth (see Zhang, 2006).

The 2D elastic wave equation for a heterogeneous isotropic medium (Zhang, 2006) is

$$L\vec{u} = \begin{pmatrix} f_x \\ f_z \end{pmatrix} \quad \text{and} \quad \hat{L}\begin{pmatrix} \phi^P \\ \phi^S \end{pmatrix} = \begin{pmatrix} F^P \\ F^S \end{pmatrix}, \tag{A-1}$$

where $\vec{u}$, f_x, and f_z are the displacement and forces in displacement coordinates and ϕ_P, ϕ_S and F^P, F^S are the P and S waves and the force components in P and S coordinates, respectively. The operators L and L_0 in the actual and reference elastic media are

$$L = \left[\rho\omega^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x\gamma\partial_x + \partial_z\mu\partial_z & \partial_x(\gamma - 2\mu)\partial_z + \partial_z\mu\partial_x \\ \partial_z(\gamma - 2\mu)\partial_x + \partial_x\mu\partial_z & \partial_z\gamma\partial_z + \partial_x\mu\partial_x \end{pmatrix}\right],$$

$$L_0 = \left[\rho\omega^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0\partial_x^2 + \mu_0\partial_z^2 & (\gamma_0 - \mu_0)\partial_x\partial_z \\ (\gamma_0 - \mu_0)\partial_x\partial_z & \mu_0\partial_x^2 + \gamma_0\partial_z^2 \end{pmatrix}\right],$$

and the perturbation V is

$$V \equiv L_0 - L = \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2\partial_x a_\gamma\partial_x + \beta_0^2\partial_z a_\mu\partial_z & \partial_x(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_z + \beta_0^2\partial_z a_\mu\partial_x \\ \partial_z(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_x + \beta_0^2\partial_x a_\mu\partial_z & a_\rho\omega^2 + \alpha_0^2\partial_z a_\gamma\partial_z + \beta_0^2\partial_x a_\mu\partial_x \end{bmatrix},$$

where the quantities $a_\rho \equiv \rho/\rho_0 - 1$, $a_\gamma \equiv \gamma/\gamma_0 - 1$, and $a_\mu \equiv \mu/\mu_0 - 1$ are defined in terms of the bulk modulus, shear modulus and density (γ_0, μ_0, ρ_0, γ, μ, ρ) in the reference and actual media, respectively.

The forward problem is found from the identity equation 6 and the elastic wave equation A-1 in PS coordinates as

$$\begin{aligned}
&\hat{G} - \hat{G}_0 = \hat{G}_0\hat{V}\hat{G}_0 + \hat{G}_0\hat{V}\hat{G}_0\hat{V}\hat{G}_0 + \cdots, \\
&\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\
&+ \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} + \cdots,
\end{aligned} \tag{A-2}$$

and the inverse solution, equations 15-17, for the elastic equation A-1 is

$$\begin{aligned}
&\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}, \\
&\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\
&= - \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix},
\end{aligned} \tag{A-3}$$

$$\vdots$$

where $\hat{V}^{PP} = \hat{V}_1^{PP} + \hat{V}_2^{PP} + \hat{V}_3^{PP} + \cdots$ and any one of the four matrix elements of V

requires the four components of the data

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}.$$

The 3D heterogeneous isotropic elastic generalization of the above 2D forward and direct inverse elastic isotropic method begins with the linear 3D form found in Stolt and Weglein (2012) page 159.

In summary, from equation A-2, $\hat{D}^{PP}$ can be determined in terms of the four elements of V. The four components $\hat{V}^{PP}$, $\hat{V}^{PS}$, $\hat{V}^{SP}$, and $\hat{V}^{SS}$ require the four components of D. That's what the general relationship $G = G_0 + G_0VG$ requires, i.e., a direct non-linear inverse solution is a solution order-by-order in the four matrix elements of D (in 2D). The generalization of the forward series equation A-2 and the inverse series equation A-3 for a direct inversion of an elastic isotropic heterogeneous medium in 3D involves the 3×3 data, D, and V matrices in terms of P, SH and SV data and start with the linear $G_0V_1G_0 = D$ on page 179 of Stolt and Weglein (2012).

APPENDIX B: NUMERICAL EXAMPLES FOR A 1D NORMAL INCIDENT WAVE ON AN ACOUSTIC MEDIUM

Numerical examples for a 1D normal incident wave on a acoustic medium are shown in this section. First, I examine and compare the convergence of the ISS direct inversion and iterative inversion. Second, the rate of convergence of the ISS inversion subseries is examined and studied using an analytic example, where the ISS method converges and the iterative linear method doesn't and where both methods converge.

The operator identity for a 1D acoustic medium

For a normal incidence plane wave on a 1D acoustic medium (where only the velocity is assumed to vary), the model I consider here consists of two half-spaces with acoustic velocities c_0 and c_1 and an interface located at $z = a$ as shown in Figure 1. If I put the source and receiver on the surface, $z = 0$, the pressure wave

$$D(t) = R\delta(t - 2a/c_0) \tag{A-4}$$

will be recorded, where the reflection coefficient $R = \frac{c_1 - c_0}{c_1 + c_0}$. For this example, $D(t)$ is the only input to the direct ISS inverse and the iterative inversion methods. Since I will assume knowledge of the velocity in the upper half space, c_0, the location of the reflector at $z = a$ is not an issue. I will focus on only determining the change of velocity across the reflector at $z = a$. The operators L_0 and L in the reference and actual acoustic media are

$$L_0 = \frac{d^2}{dz^2} + \frac{\omega^2}{c_0^2} \quad \text{and} \quad L = \frac{d^2}{dz^2} + \frac{\omega^2}{c^2(z)}, \tag{A-5}$$

and I characterize the velocity perturbation as,

$$\alpha(z) \equiv 1 - \frac{c_0^2}{c^2(z)}. \tag{A-6}$$

The perturbation V (Weglein et al., 2003) can be expressed as

$$V(z) = L_0 - L = \frac{\omega^2}{c_0^2} - \frac{\omega^2}{c^2(z)} = k_0^2\alpha(z), \tag{A-7}$$

where ω is the angular frequency and $k_0 = \omega/c_0$. c_0 and $c(z)$ are the reference and local acoustic velocity. Therefore, the inverse series of V [equation 11] becomes

$$\alpha(z) = \alpha_1(z) + \alpha_2(z) + \alpha_3(z) + \cdots . \tag{A-8}$$

That is

$$V_1 = k_0^2\alpha_1, \quad V_2 = k_0^2\alpha_2, \quad \cdots . \tag{A-9}$$

From the inverse scattering series [Equations 15-17], Shaw and Weglein (2004) isolated the leading order imaging subseries and the direct non-linear inversion subseries.

In this section, I will focus on studying the convergence properties of the ISS inversion subseries. The inversion only terms isolated from the inverse scattering series (Zhang, 2006; Li, 2011) are

$$\alpha(z) = \alpha_1(z) - \frac{1}{2}\alpha_1^2(z) + \frac{3}{16}\alpha_1^3(z) + \cdots . \tag{A-10}$$

For a 1D normal incidence case, the linear equation 15 solves for α_1 in terms of the single trace data $D(t)$ (Shaw and Weglein, 2004) as

$$\alpha_1(z) = 4\int_{-\infty}^{z} D(z')dz', \tag{A-11}$$

where $z' = c_0 t/2$. For a single reflector, inserting data D [equation A-4] gives

$$\alpha_1 = 4RH(z-a), \tag{A-12}$$

where R is the reflection coefficient $R = \frac{c_1 - c_0}{c_1 + c_0}$ and H is the Heaviside function. When $z > a$, substituting α_1 into equation A-10, the ISS direct non-linear inversion subseries in terms of R can be written as (where α is the magnitude of $\alpha(z)$ for $z > a$)

$$\alpha = 4R - 8R^2 + 12R^3 + \cdots = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n. \tag{A-13}$$

After solving for α, the inverted velocity $c(z)$ can be obtained through $c_1 = c_0(1-\alpha)^{-1/2}$ [equation A-7].

Considering the convergence property of the series for α or the inversion subseries, I can calculate the ratio test,

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \left|\frac{(n+2)(-R)^{n+1}}{(n+1)(-R)^n}\right| = \left|\frac{n+2}{n+1}R\right|. \tag{A-14}$$

If $\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < 1$, this subseries converges absolutely. That is

$$|R| < \lim_{n\to\infty}\frac{n+1}{n+2} = 1. \tag{A-15}$$

Therefore, the ISS direct non-linear inversion subseries converges when the reflection coefficient $|R|$ is less than 1, which is always true. Hence, for this example, the ISS inversion subseries will converge under any velocity contrasts between the two media.

For the iterative linear inversion, I use the first linear estimate of $\alpha = \alpha_1^1$ to compute the first estimate of $c_1 = c_1^1$. Then I choose the first estimate of $c_1 = c_0(1-\alpha_1^1)^{-1/2} \equiv c_1^1$ as the new reference velocity, $c_0^1 = c_0(1-\alpha_1^1)^{-1/2}$, where $\alpha_1^1 = 4R_1$ and $R_1 = \frac{c_1-c_0}{c_1+c_0}$. Repeating the linear process with a new reflection coefficient R_2 (again exploiting the analytic inverse generously provided by ISS to benefit the iterative linear inverse approach) gives

$$R_2 = \frac{c_1 - c_0^1}{c_1 + c_0^1},\quad \alpha_1^2 = 4R_2 \text{ and } c_1^2 = c_0^1(1-\alpha_1^2)^{-1/2} = c_0^2, \tag{A-16}$$

$$\vdots$$

$$R_{n+1} = \frac{c_1 - c_0^n}{c_1 + c_0^n},\quad \alpha_1^{n+1} = 4R_{n+1} \text{ and } c_1^n = c_0^{n-1}(1-\alpha_1^n)^{-1/2} = c_0^n, \tag{A-17}$$

where α_1^n = nth estimate of α_1 and c_1^n = nth estimate of c_1. The questions are (1) under what conditions does c_1^n approach c_1, and (2) when it converges, what is its rate of convergence.

From the above analysis, I can see that the ISS method for α always converges and the resulting α can be used to find c_1. For the iterative linear inverse, there are values of α_1 such that you cannot compute a real c_1^1. When $\alpha_1^1 > 1$ and $4R > 1$, $R > 1/4$ and you cannot compute an updated reference velocity and the method simply shuts down and fails. The inverse scattering series never computes a new reference and doesn't suffer that problem, with the series for α always converging and then outputting c_1, the correct unknown velocity below the reflector.

The convergence of the ISS direct inversion and iterative inversion

In this section, I will examine and compare the convergence property of the ISS inversion [equation A-13] and the iterative linear inversion for different velocity contrasts in the 1D acoustic case. In the 1D normal incident acoustic model (Figure 1), only one parameter (velocity) varies and a plane wave propagates into the medium. There is only a single reflector and I assume the velocity is known above the reflector and unknown below the reflector. I will compare the convergence of the perturbation α and the inversion results by using the ISS direct non-linear method and the iterative linear method.

With the reference velocity $c_0 = 1500m/s$, two analytic examples with different velocity contrasts for $c_1 = 2000m/s$ and $c_1 = 3000m/s$ are examined. Figure 2 shows the estimated α by the ISS method (green line) for $c_1 = 2000m/s$. The red line represents the actual α that is calculated from the model. The horizontal axis represents the order of the ISS inversion subseries. The vertical axis shows the value of α. The updated estimation of α using the iterative inversion method (blue line) is shown in Figure 3. The horizontal axis represents the iteration numbers in the iterative inversion method. From Figures 2 and 3, I can see that at the small velocity contrast, the estimated α by ISS method becomes the actual α after about five orders calculation and the updated estimation of α by the iterative inversion method goes to zero as expected, because after several iterations, the updated model is close to and approaching to the actual model. Figure 4 represents the velocity estimation. The green blue lines represent the estimated velocity by using the ISS inversion method and the iterative inversion method, respectively. We can see that at the small velocity contrast, both methods converge and produce correct velocity after five orders or iterations and the ISS inversion method converges faster than the iterative inversion method.

Figure 5 shows the estimated α by the ISS method (green line) for $c_1 = 3000m/s$. When the velocity contrast is larger, i.e., $R > 0.25$, the iterative inversion method can not be computable, but the ISS inversion method always converges (see green line in Figure 5) after the summation of more orders in computing α.

As we know, the reflection coefficient R is almost always less than 0.2 in practice, so that both the ISS method and the iterative method converge, but the ISS method converges faster than the iterative method. Moreover, for more complicated circumstances (e.g., the elastic non-normal incidence case), the difference between the ISS method and the iterative method is much greater, not just on the algorithms, but also on data requirements and on how the band-limited noisy nature of the seismic data impact the inverse operators in the iterative method but not in the ISS method.

The rate of convergence of the ISS inversion subseries

The rate of convergence of the estimated α or the ISS inversion subseries [equation A-13] is analytically examined and studied. Since α is always convergent when $R < 1$, the summation of this subseries (Zhang, 2006) is

$$\alpha = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n = 4R\frac{1}{(1+R)^2}. \tag{A-18}$$

If the error between the estimated and the actual α is monotonically decreasing, it means the subseries is a term-by-term added value improvement towards determining the actual medium properties. If this error is increasing before decreasing, it means that the estimate of α becomes worse before it gets better. The error for the first order and the error for the second order have the relation,

$$|\alpha - \alpha_1 - \alpha_2| > |\alpha - \alpha_1|, \tag{A-19}$$

i.e.,

$$|4R\frac{3R^2 + 2R^3}{(1+R)^2}| > |4R\frac{-R^2 - 2R}{(1+R)^2}|. \tag{A-20}$$

After simplification, it gives

$$R^2 + R - 1 > 0. \tag{A-21}$$

I can solve it and obtain the reflection coefficient $R < \frac{-1-\sqrt{5}}{2} = -1.618$ or $R > \frac{-1+\sqrt{5}}{2} = 0.618$. Therefore, when $R > 0.618$, the error increases first. Similarly, if the error for the third order is greater than that for the second order, I get $R > 0.667$. If the error for the fourth order is greater than that for the third order, I obtain $R > 0.721$. In summary, when $R > 0.618$ the error increases and the estimated α gets worse before getting better. The sum of terms in the direct inverse ISS solution (for very large contrasts) requires certain partial sums to be temporally worse in order for the entire series to produce the correct velocity. The dashed green line in Figure 6 shows that when the reflection coefficient R is equal to 0.618, the error for the first order is equal to the error for the second order.

As the analytic calculation, when the reflection coefficient R is smaller than 0.618, this inversion subseries gives a monotonically term-by-term added value improvement towards determining c_1. When the reflection coefficient is larger than 0.618, the ISS inversion series still converges, but the estimation of α will become worse before it gets better. Each term in the series works towards the final goal. Sometimes when more terms in the series are included, the estimation looks temporally worse, but once it starts to improve the estimation at a specific order, the approximations never become worse again, every single term after that order will produce an improved estimation. The locally worse partial sum behavior is, in fact, purposeful and essential for convergence to and for computing the exact velocity. The direct inverse solution fulfills its commitment to always predict c_1, and not necessarily to having order-by-order improvement. The ISS direct inversion always converges in contrast to the iterative linear inverse method. This property has also been indicated by Carvalho (1992) in the free-surface multiple elimination subseries, e.g., what appears to make a second-order free-surface multiple larger with a first-order free-surface algorithm is actually helpful and necessary for preparing the second-order multiple to be removed by the higher-order terms.

FIGURE CAPTIONS

Figure 1: 1D acoustic model with velocities c_0 over c_1.

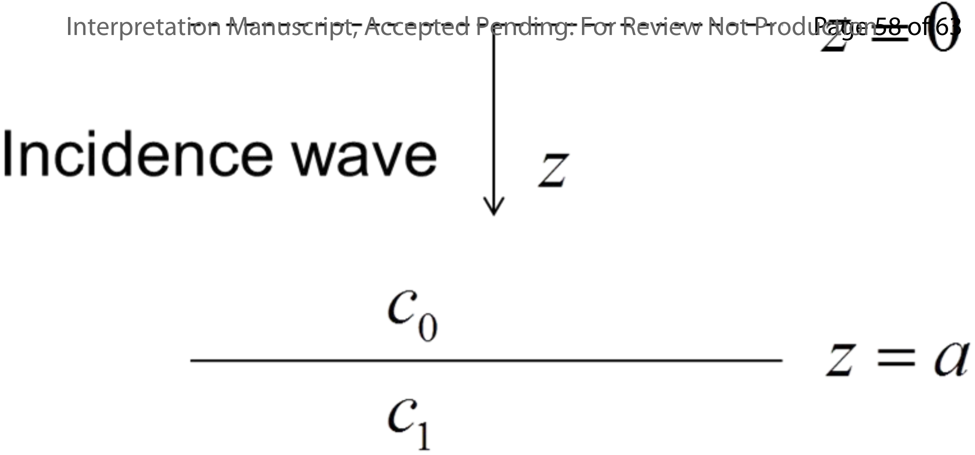
Figure 2: The estimated α at $R = 0.1429$: The horizontal axis is the order of the ISS subseries and the vertical axis shows the value of α. The red line shows the actual value of $\alpha = 0.4375$. The green line shows the estimation of α using the ISS inversion method order-by-order.

Figure 3: The updated α at $R = 0.1429$: The horizontal axis is the iteration numbers and the vertical axis shows the updated value of α. The blue line represents the updated estimation of α using the iterative inversion method.

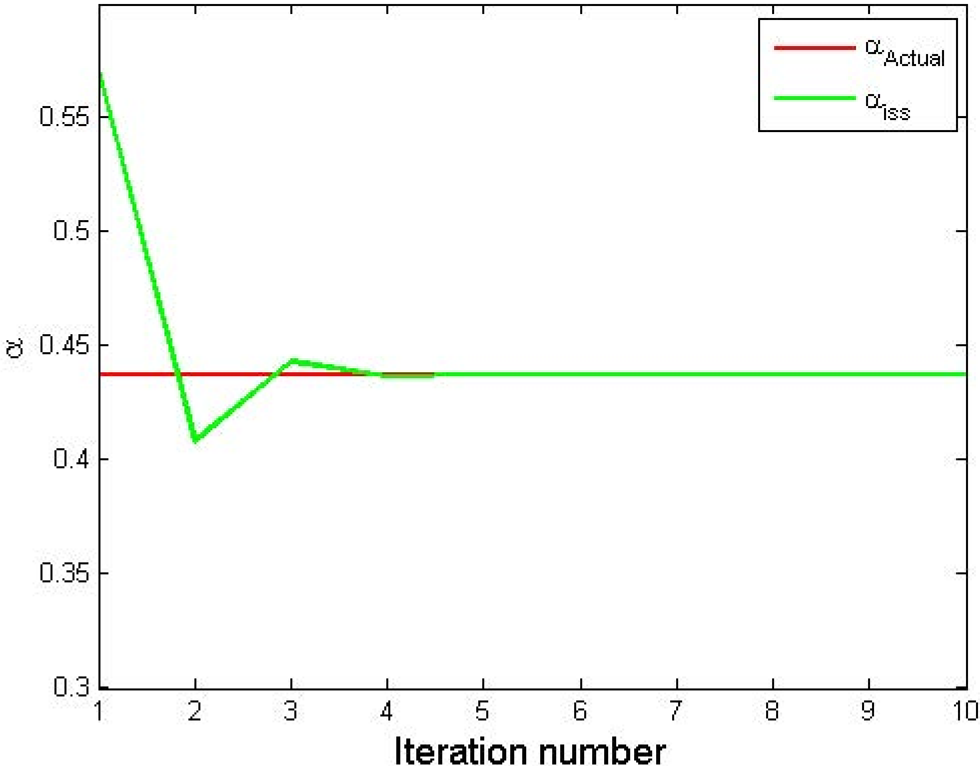
Figure 4: The estimated velocity by using the ISS inversion method (green line) and the iterative inversion method (blue line).

Figure 5: The estimated α at $R = 0.3333$: The horizontal axis is the order of the ISS subseries and the vertical axis represents the value of α. The red line shows the actual value of $\alpha = 0.7500$. The green line shows the estimation of α using the ISS inversion method order-by-order.

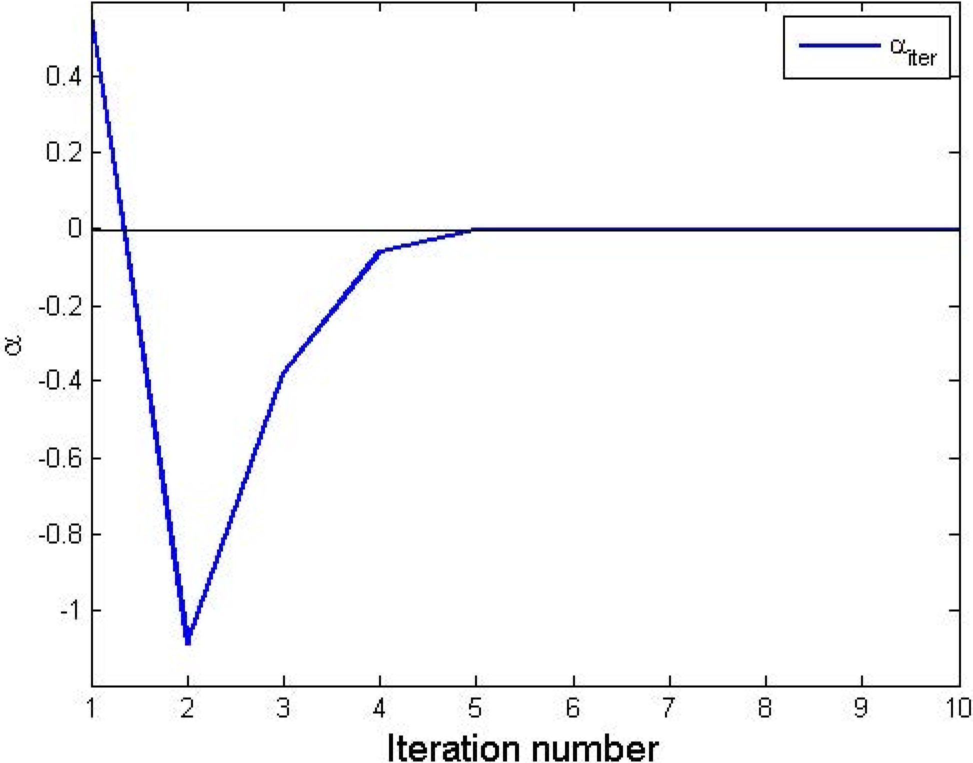
Figure 6: The error (dashed green line) of estimated α at $R = 0.6180$ and $\alpha = 0.9443$.



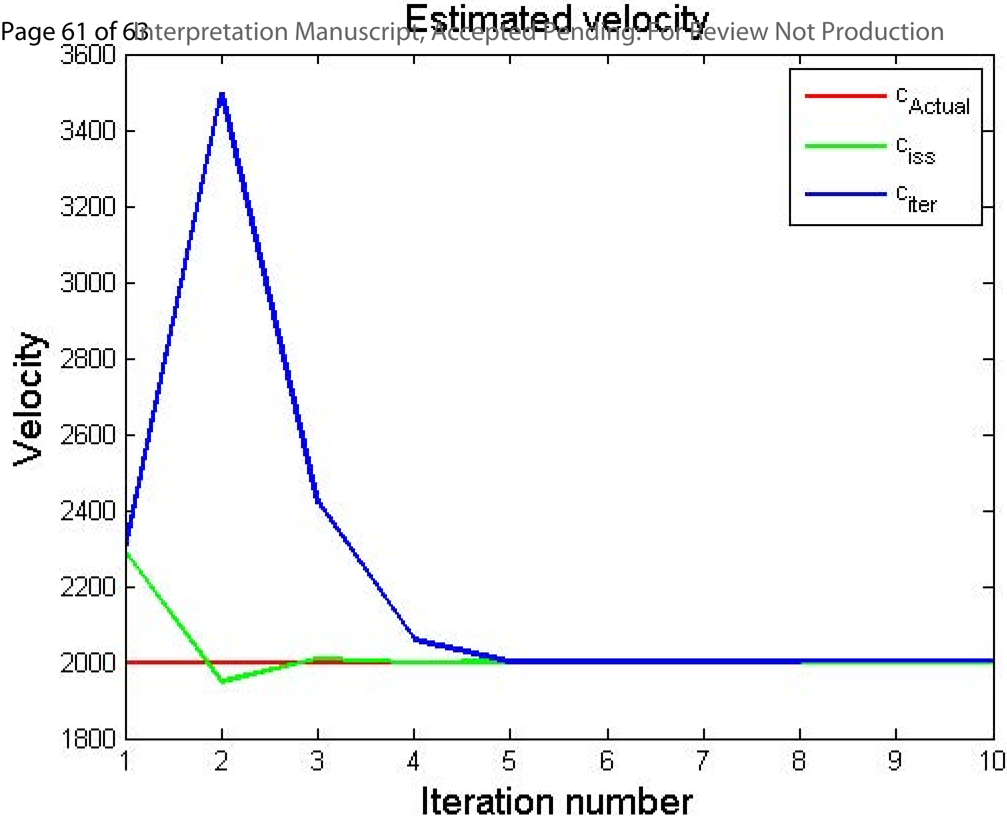
$z = 0$
Incidence wave
z
c_0
$z = a$
c_1

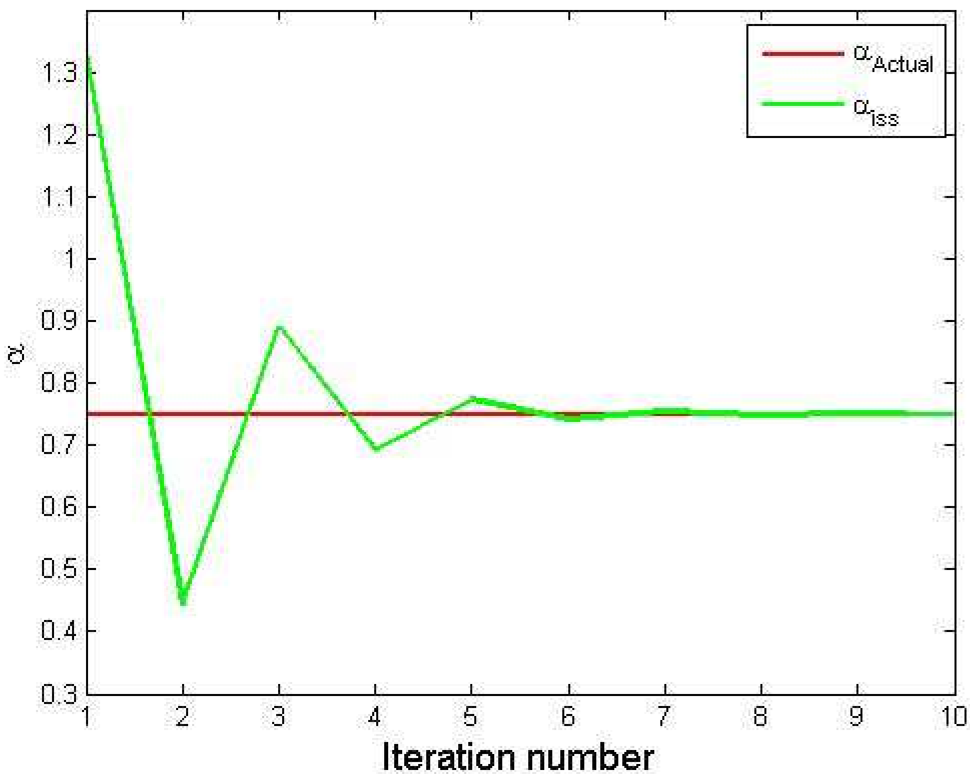


α_{Actual}
α_{Iss}
α
0.55
0.5
0.45
0.4
0.35
0.3
1
2
3
4
5
6
7
8
9
10
Iteration number

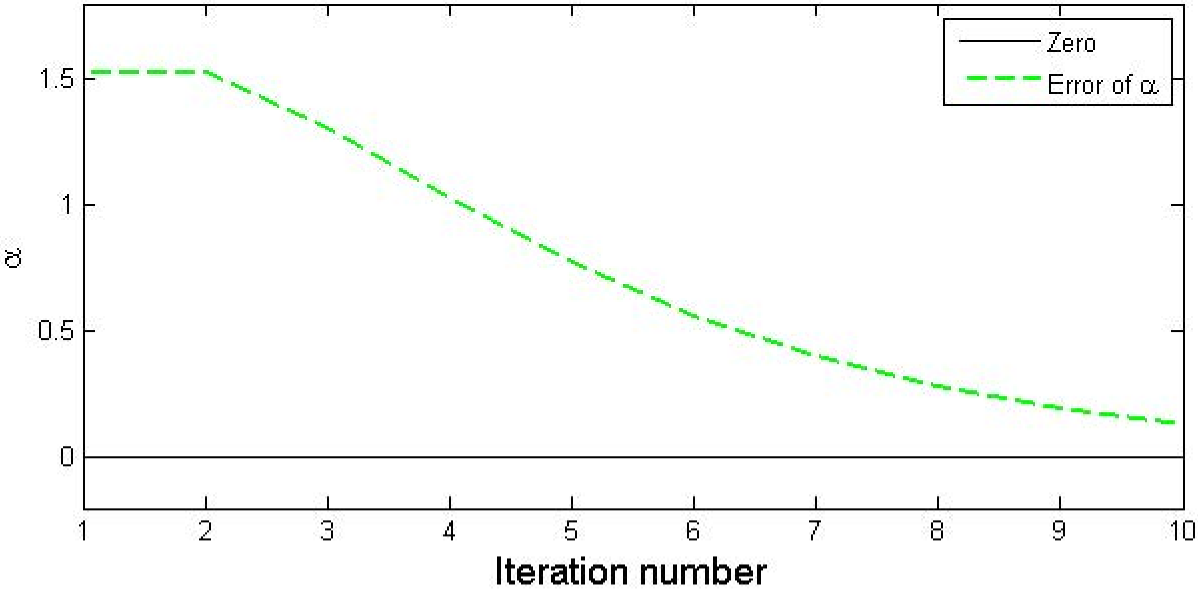


α_{iter}
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
α
1
2
3
4
5
6
7
8
9
10
Iteration number


Estimated velocity
3600
3400
3200
3000
2800
2600
2400
2200
2000
1800
Velocity
1
2
3
4
5
6
7
8
9
10
Iteration number
c_{Actual}
c_{iss}
c_{iter}



α_{Actual}
α_{iss}
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
α
1
2
3
4
5
6
7
8
9
10
Iteration number


Zero
Error of α
1.5
1
0.5
0
α
1
2
3
4
5
6
7
8
9
10
Iteration number