Progressing amplitude issues for testing 1D analytic data in leading order internal multiple algorithms

Wilberth Herrera*, Chao Ma*, Hong Liang*, Paolo Terenghi*, and Arthur B. Weglein*
M-OSRP/Physics Dept./UH*

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Abstract

In Ramírez (2007), a subseries of the Inverse Scattering Subseries (ISS) was isolated, whose specific task is to eliminate internal multiples of first order. This subseries naturally splits into two subseries: the Leading-Order Internal-Multiple-Eliminator Subseries (LOIMES) and the Higher-Order Internal-Multiple-Eliminator Subseries (HOIMES).

The purpose of this report is to propose a modification of the LOIMES. The motivation for such a modification is twofold. First, the original formulation carries a limitation for correctly accommodating spike-like data, due to the presence of powers of the data higher than one, which are not well defined mathematically when the data are spike-like. Second, we wish to apply the LOIMES to the Internal-Multiple-Attenuation Subseries (IMAS) obtained in Ma and Weglein (2012). The proposal splits into two cases: spike-like data and non-spike (but continuous) data. For the spike-like case, the proposal correctly overcomes the limitation of the original approach by explicitly avoiding the higher powers of the data, and it also allows for the elimination of the effect described in Ma and Weglein (2012), and originated by the presence of a specific 1st. order internal multiple in the input data of the leading order contribution of the original IMAS. For continuous data the proposal fixes a mathematical issue that is present in the original approach, regarding the behavior of the subseries when ϵ (the parameter introduced to avoid self interactions) goes to zero. At the same time, however, the proposal brings new questions to the subject because it is not general enough to deal with all types of continuous data; it only works for a very restricted class.

1 Introduction

One of the main challenges of exploration seismology is to locate hydrocarbon targets beneath the earth's surface. To achieve this goal, there is a sequence of steps to be performed in the data resulting from seismic experiments: random-noise attenuation, deghosting, source wavelet deconvolution, removal of free-surface multiples, removal of internal multiples, imaging, and inversion. All these steps must be done in the same order in which they are listed. In particular, all current imaging algorithms assume that the data consist exclusively of primaries, which means that any other type of event (i.e., ghosts and multiples) is considered to be noise by the imaging process and therefore needs to be removed from the data before the application of any imaging algorithm.

Today, there are a number of methodologies in the oil industry that are designed to handle the different steps just mentioned. In particular, for the removal of internal multiples, one of the standard methods is the energy-minimization adaptive substraction. This method works by using the internal multiples predicted by a given model, and then systematically substracting this prediction from the actual data, using the minimum energy criteria: the energy after the removal should be minimal. However, this method fails, among other situations, when an internal multiple is interfering with a primary. The reason is that in this situation the minimum energy criterium is no longer valid, and the adaptive substraction affects also the amplitude of the primary.

A new criteria for adaptive subtraction in then necessary, but is not yet available. As such criteria must deal with factors from the system (wavelet, ghosts internal multiples, etc.) and outside the system (such as the irregular shape the free surface), what we can do in the meantime is to lower the burden for the adaptive substraction. This can be done by applying to the system all the preprocessing tools we have at hand. In this way, we help the adaptive subtraction to take care mostly of the factors outside the system, and hence to improve its effectiveness.

Using the ISS and the concept of specific-task subseries, a multidimensional algorithm to remove free-surface multiples was derived in Carvalho (1992), using no information about the earth's subsurface. Later on, this work was extended in Araújo (1994), where a multidimensional algorithm was derived to attenuate internal multiples present in the data. However, to reach the goal of lowering the burden of the adaptive subtraction as much as possible, it is important to move the atenuation of internal multiples to a total elimination.

In response to this necessity, a further subseries was isolated in Ramírez (2007). The specific task of this subseries is to remove, rather than attenuate, internal multiples of first order. However, this subseries is unable to deal with spike-like data, as in this case there is a mathematical inconsistency: the subseries contains powers of the data higher than one and these powers are not well defined when the data are spike-like (analytic). The subseries splits into two subseries: the Leading-Order Internal-Multiple-Eliminator Subseries (LOIMES) and the Higher-Order Internal-Multiple-Eliminator Subseries (HOIMES). The LOIMES eliminates internal multiples that are of first-order and whose downward reflection takes place at the shallowest reflector, while the HOIMES eliminates the first-order internal multiples generated at any reflector other than the shallowest.

Although in practice the field data is never a spike, it is very important to test any new algorithm with analytic data, as in this case the data is error-free and we have total control on them. This means that if we test the algorithm with analytic data, any error in the output is an error in the algorithm. In other words, with analytic data we can isolate and test the concept behind the algorithm. Once the algorithm is successful with analytic data, we can go ahead and test it with synthetic data, and eventually with field data.

In this report, we modify the original derivation of the LOIMES to allow for spike-like data. We focus on the LOIMES because of its immediate application to recent developments in Ma and Weglein (2012) and Liang and Weglein (2012), where the original Internal-Multiple-Attenuator Subseries (IMAS) of Araújo and Weglein is extended to allow the input data to include first-order internal multiples. In particular we show, by specific example, how this modified algorithm for the LOIMES can be used to move the work in Ma and Weglein (2012), from an attenuator to an eliminator of the effect of a particular internal multiple in the input data. Along the way, we also find the need to rederive the LOIMES (and in general the first-order IMES) when the data are not spike-like (but are continuous). In this case, we find a derivation that is valid only for highly constrained data. This last fact, together with the analysis of Ramírez and Weglein in the original derivation of the first-order IMES, strongly suggests that more research in the subject is necessary, as there is no physical reason for the elimination to take place only within some subset of data.

The organization of this report is as follows: in section 2, we review the original derivation of the IMES proposed by Ramírez and Weglein. In section 3 we point out the limitation of the LOIMES in dealing with spike-like data and explain how to overcome this limitation, by a modification of the algorithm specific for such data. We also apply this modified algorithm to promote (for a specific earth model) the IMAS in Ma and Weglein (2012) to being an eliminator for the effect of the inclusion of a specific internal multiple in the input data. In section 4, we propose a slightly different way to derive the LOIMES for continuous data, and we also make a few comments about the HOIMES. Finally, in section 5 we present final comments and conclusions. There are two appendices, in which we show the details of the calculations needed to follow the main body of this report.

2 Review of the (LO)IMES

In this section, we will provide the line of thought for the original derivation of the LOIMES, and at the same time we will highlight the problem we aim to solve. For simplicity, in this report we will focus on a 1D earth with normal incidence.

The key point in the original approach of Ramírez (2007), in moving from the attenuator to the eliminator, is to take into account certain self interactions of the effective data, denoted $b_1(z)$, that contain the correct amplitude compensation for eliminating the internal multiples rather than just for attenuating them. The resulting Internal-Multiple-Eliminator Subseries (IMES) is

$$b^{IM}(k) = b^{IM}_{LO}(k) + b^{IM}_{HO}(k), \tag{2.1}$$

where

$$b^{IM}_{LO}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \left(\frac{1}{1-b_1(z')^2}\right) b_1(z') \times \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{2.2}$$

and

$$b^{IM}_{HO}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \frac{2G(z') \int_{-\infty}^{z'-\epsilon} dz''' J(z''')}{1-\int_{-\infty}^{z'-\epsilon} dz''' J(z''')} \times \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{2.3}$$

where

$$J(z''') = \frac{b_1(z''')^2}{1 - b_1(z''')^2} \qquad\qquad G(z') = \frac{b_1(z')}{1 - b_1(z')^2}. \tag{2.4}$$

The task of the leading-order eliminator b^{IM}_{LO} is to eliminate, when it is added to the effective data, the internal multiples of first order generated at the shallowest reflector. The higher-order eliminator b^{IM}_{HO} eliminates the first-order internal multiples created at deeper reflectors, and assumes that b^{IM}_{LO} has been applied to data.

For now, we will focus on the leading-order eliminator b^{IM}_{LO}, whose initial terms are as follows:

$$b^{IM}_{LO} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \left(b_1(z') + b_1(z')^3 + b_1(z')^5 + ... \right) \times$$

$$\int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''). \tag{2.5}$$

Expanding (2.5), we notice that the resulting first term is exactly the first term in the IMAS discussed in Araújo (1994), and the following terms contain the self interactions (in the middle integral) mentioned in the second paragraph of the present section. Now, we will briefly describe the origin of these self interactions by analyzing the first self-interacting term-namely, the one containing $b_1(z')^3$:

$$b^{IM}_5(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z')^3 \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''). \tag{2.6}$$

The whole term, being of fifth order in the data, must reside somewhere within the fifth inverse scattering equation. The correct term of this equation turns out to be $V_1 G_0 V_3 G_0 V_1$, from which, after selecting the model-type independent contribution, writing it in terms of effective data b_1, picking up the term with the right nonlinear characteristics to predict the internal multiple's time, and finally selecting the *lower-higher-lower* contribution, we are left with

$$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \hat{b}_3(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{2.7}$$

where $\hat{b}_3(z')$ is the data representation of the model-type independent part of the third equation in the inverse scattering series

$$V_3 = -V_1 G_0 V_1 G_0 V_1 - V_1 G_0 V_2 - V_2 G_0 V_1. \tag{2.8}$$

Finally, we still need to focus on $B_3(k)$, the part of $\hat{b}_3(z')$ coming from $V_1 G_0 V_1 G_0 V_1$:

$$B_3(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z''). \tag{2.9}$$

From (2.6) and (2.7), it is clear that the self interactions must arise from (2.9), so we need to split $B_3(k)$ in a way that makes these self interactions evident. The result proposed in Ramírez (2007) is:

$$
\begin{aligned}
&B_3(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z+\epsilon}^{\infty} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{z'-\epsilon} dz'' e^{ikz''} b_1(z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z+\epsilon}^{\infty} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{z'-\epsilon} dz'' e^{ikz''} b_1(z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') \delta(z-z') \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') \delta(z'-z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') \delta(z-z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') \delta(z-z') \int_{-\infty}^{z'-\epsilon} dz'' e^{ikz''} b_1(z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z+\epsilon}^{\infty} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') \delta(z'-z'') \\
&+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') \delta(z'-z'') \\
&= B_{31}(k) + B_{32}(k) + B_{33}(k) + B_{34}(k) \\
&\qquad + B_{35}(k) + B_{36}(k) + B_{37}(k) + B_{38}(k) + B_{39}(k). \qquad (2.10)
\end{aligned}
$$

In the above expression for $B_3(k)$, we can see that the self-interaction terms come from the Delta functions in the last five terms. Performing the integrals with the Delta functions in $B_{35}(k)$, followed by an inverse Fourier transform, we end up with $B_3(z') = b_1(z')^3$. Inserting this portion of $\hat{b}_3(z')$ into (2.7), we find exactly (2.6), the second term of b_{LO}^{IM}.

The next self-interaction contribution to b_{LO}^{IM}, $b_1(z')^5$, is obtained by analogous arguments applied to $V_1 G_0 V_5 G_0 V_1$, to finally get

$$
\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z')^5 \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''). \qquad (2.11)
$$

The closed form, eq. (2.2), becomes evident by calculating, following the procedure just described, a few terms beyond $b_1(z')^5$.

3 LOIMES and spike-like data

In this section, we will describe a limitation of the formalism described above to eliminate internal multiples, and we will also explain the solution when the data are spike-like. As a result, the original algorithm for the LOIMES will change and the correct prescription will be provided (at least when the data are spike-like). We will also apply this prescription to illustrate how to promote the IMAS discussed in Ma and Weglein (2012) to an eliminator of certain unwanted events predicted by the IMAS under some circumstances. In the next section we will discuss an approach solving the limitation when the data are not spike-like but instead are continuous.

3.1 Statement of the problem

As we mentioned earlier, we consider a 1D earth with normal incidence and two interfaces at pseudodepths $z_1 \equiv c_0 t_1/2$ and $z_2 \equiv c_0 t_2/2$ with respect to a homogeneous reference medium with constant velocity c_0. The terms t_1 and t_2 are the traveltimes associated with the primaries created at the first (shallowest) and second (deepest) reflector, respectively. Consider also spike-like data, assumed to be built up from primaries and the unique first-order internal multiple allowed by this two-layer example (strictly speaking, for this subsection we do not need any internal multiple in the data, but it is included for further convenience):

$$D(t) = R_1\delta(t-t_1) + T_{01}R_2T_{10}\delta(t-t_2) - T_{01}R_2R_1R_2T_{10}\delta(t-(2t_2-t_1)) \tag{3.1}$$

where $2t_2 - t_1$ is the traveltime associated with the first-order internal multiple and T_{ij} denotes the transmission coefficient when the wave travels from the ith medium to the jth medium. R_k is the reflection coefficient at the kth layer for a downward incident wave. Expressed in depth units the data become

$$b_1(z) = R_1\delta(z-z_1) + T_{01}R_2T_{10}\delta(z-z_2) - T_{01}R_2R_1R_2T_{10}\delta(z-(2z_2-z_1)). \tag{3.2}$$

If we try to compute the second term of b_{LO}^{IM}, eq. (2.6), using the data given by (3.2), we immediately run into serious theoretical issues because $b_1(z')^3$ will involve terms with powers of the Delta functions higher than one, i.e., terms like $\delta^3(z-z_1)$, etc. Unfortunately, the powers of the Delta function are not well-defined mathematical objects and hence the spike-like data do not fit in this formalism.

3.2 Fixing the problem

We will now propose a different way to deal with spike-like data to eliminate internal multiples of first order generated at the shallowest reflector; i.e.; we will explain how to deal with the LOIMES when the data are spike-like.

The starting point is eq. (2.10): it turns out that this expression has a subtle inconsistency. To see this, take the limit $\epsilon \to 0$, and use the following relations, involving definite integrals

$$lim_{\epsilon\to 0}\int_{-\infty}^{z-\epsilon} dz' f(z') = \int_{-\infty}^{z} dz' f(z') \qquad lim_{\epsilon\to 0}\int_{z+\epsilon}^{\infty} dz' f(z') = \int_{z}^{\infty} dz' f(z'). \tag{3.3}$$

The resulting expression is

$B_3(k) = B_3(k)$

$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z')\delta(z-z') \int_{-\infty}^{\infty} dz e^{ikz''} b_1(z'')\delta(z'-z'')$

$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z')\delta(z-z') \int_{z'}^{\infty} dz e^{ikz''} b_1(z'')$

$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z')\delta(z-z') \int_{-\infty}^{z'} dz e^{ikz''} b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z}^{\infty} dz'e^{-ikz'}b_1(z') \int_{-\infty}^{\infty} dze^{ikz''}b_1(z'')\delta(z'-z'')$

$$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{-\infty}^{z} dz'e^{-ikz'}b_1(z') \int_{-\infty}^{\infty} dze^{ikz''}b_1(z'')\delta(z'-z''), \tag{3.4}$$

which is obviously inconsistent, because the contribution of the interaction terms is not zero. To fix this problem, let's go back to eq. (2.9) and split the second and third intervals of integration as follows:

$\int_{-\infty}^{\infty} = \int_{-\infty}^{z-\epsilon} + \int_{z-\epsilon}^{z+\epsilon} + \int_{z+\epsilon}^{\infty}$

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{z'-\epsilon} + \int_{z'-\epsilon}^{z'+\epsilon} + \int_{z'+\epsilon}^{\infty}. \tag{3.5}$$

The resulting expression is

$B_3(k) = \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{-\infty}^{z-\epsilon} dz'e^{-ikz'}b_1(z') \int_{z'+\epsilon}^{\infty} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z+\epsilon}^{\infty} dz'e^{-ikz'}b_1(z') \int_{-\infty}^{z'-\epsilon} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z+\epsilon}^{\infty} dz'e^{-ikz'}b_1(z') \int_{z'+\epsilon}^{\infty} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{-\infty}^{z-\epsilon} dz'e^{-ikz'}b_1(z') \int_{-\infty}^{z'-\epsilon} dz''e^{ikz''}b_1(z'')$

$\int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz'e^{-ikz'}b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz'e^{-ikz'}b_1(z') \int_{z'+\epsilon}^{\infty} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz'e^{-ikz'}b_1(z') \int_{-\infty}^{z'-\epsilon} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{z+\epsilon}^{\infty} dz'e^{-ikz'}b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz''e^{ikz''}b_1(z'')$

$+ \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{-\infty}^{z-\epsilon} dz'e^{-ikz'}b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz''e^{ikz''}b_1(z'') =$

$B_{31}(k) + B_{32}(k) + B_{33}(k) + B_{34}(k)$

$$+ B'_{35}(k) + B'_{36}(k) + B'_{37}(k) + B'_{38}(k) + B'_{39}(k). \tag{3.6}$$

In the limit $\epsilon \to 0$, (3.6) reduces trivially to $B_3(k) = B_3(k)$, so we will use this expression, instead of (2.10), as the starting point for the derivation of the LOIMES. In our present approach all the arguments in Ramírez (2007) for the derivation of the LOIMES, prior to eq. (2.10), are unchanged. The difference is that instead of $B_{35}(k)$ in eq. (2.10) we now consider the analogous term $B'_{35}(k)$ in eq. (3.6), as both contain two interactions. Thus, the recipe now is that the second term in b_{LO}^{IM} becomes

$$b_5^{IM}(k) = \int_{-\infty}^{\infty} dze^{ikz}b_1(z) \int_{-\infty}^{z-\epsilon} dz'e^{-ikz'}B'_{35}(z) \int_{z'+\epsilon}^{\infty} dz''e^{ikz''}b_1(z''). \tag{3.7}$$

3.3 Example: The three-layer earth

In appendix A we work out the details for the above expression for the same earth model as that of section 3.1: a 1D and three-layer (or two-interface) earth with normal incidence. The only difference with respect to section 3.1 is that this time we assume spike-like and primary-only data. The result for $B'_{35}(z)$ is eq. (A.7), which upon its insertion into (3.7), results in

$$b_5^{IM} = R_1^3 R_2'^2 e^{ik(2z_2 - z_1)} = T_{01}T_{10} * R_1^2 * (T_{01}R_2R_1R_2T_{10})e^{ik(2z_2-z_1)}$$

whose explicit calculation is also performed in appendix A, and the result is eq. (A.10). In the above expression, the notation is also as in section 3.1: T_{ij} denotes the transmission coefficient when the wave travels from the ith medium to the jth medium, and R_k is the reflection coefficient at the kth layer for a downward incident wave.

Eq. (A.10) is consistent with the one obtained in Ramírez (2007) for the same earth configuration. However, in that reference it is assumed that $\delta^n(z - z_i) = \delta(z - z_i)$, where $\delta^n(z - z_i)$ means the nth power of $\delta(z - z_i)$. This statement is wrong and is only true for the definition of the Delta function used for numerical simulations:

$$\delta(z - z_i) = \begin{cases} 1 & z = z_i \\ 0 & z \neq z_i \end{cases}.$$

Going back to our approach, we can now perform an analysis similar to the one presented in Ramírez (2007): when the above expression is added to both the data b_1^{IM} and the first term of the eliminator series $b_3^{IM} = T_{01}T_{10} * (T_{01}R_2R_1R_2T_{10})$ (which is also the second term in the IMAS), we have

$$b_1^{IM} + b_3^{IM} + b_5^{IM} = \ primaries + [-1 + T_{01}T_{10} * (1 + R_1^2)] * (T_{01}R_2R_1R_2T_{10}). \tag{3.8}$$

In the above expression the (-1) term comes from the original first-order internal multiple in the data, whose amplitude is $-T_{01}R_2R_1R_2T_{10}$, and the $1 + R_1^2$ term contains the first two terms in the geometric series expansion for 1:

$$1 = T_{01}T_{10}\left(\frac{1}{T_{01}T_{10}}\right) = T_{01}T_{10}\frac{1}{(1 - R_1^2)} = T_{01}T_{10}(1 + R_1^2 + R_1^4 + R_1^6 + R_1^8 + \ldots). \tag{3.9}$$

This means that $b_3^{IM} + b_5^{IM}$ is closer to 1 than is the attenuator b_3^{IM}, and therefore the internal multiple's amplitude is better estimated, which means that $[-1 + T_{01}T_{10} * (1 + R_1^2)]$ is closer to zero and hence this is a first step toward the complete removal of the internal multiple.

As we explained before, in Ramírez (2007) the next term in the eliminator series, b_7^{IM}, whose amplitude is $T_{01}T_{10} * R_1^4 * T_{01}R_2R_1R_2T_{10}$, arises when we are selecting the appropriate part of $V_1V_5V_1$ by a procedure similar to the one applied in the same reference to $V_1V_3V_1$ to get b_5^{IM}. This procedure will bring, when the data are spike-like, the same issue that we had with b_5^{IM}, i.e., an interaction of the form $b_1(z)^5$ implying a fifth power of the Delta function.

3.4 A modified closed form of the LOIMES

The issue explained in the last paragraph of the previous subsection is solved in exactly the same way we solved the analogous problem for b_5^{IM} i.e. by selecting the same term proposed in Ramírez (2007) for b_7^{IM} and performing the resulting integrals with finite intervals of integration. This procedure can be applied to each higher-order term in the original IMES. After computing a few higher-order terms, we can write a closed form for b_{LO}^{IM}:

$b_{LO}^{IM} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \times$

$$\mathcal{F}^{-1}\left(\int_{-\infty}^{\infty} dz' e^{ikz'} b_1(z') \frac{1}{1-\int\int b_1(z')}\right) \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') \tag{3.10}$$

where $\mathcal{F}^{-1}$ means the inverse Fourier transform and

$$\frac{1}{1-\int\int b_1(z')} \equiv 1 + \int\int b_1(z') + \left(\int\int b_1(z')\right)^2 + \left(\int\int b_1(z')\right)^3 + \ldots \tag{3.11}$$

and

$\left(\int\int b_1(z')\right)^n \equiv \int_{z'-\epsilon}^{z'+\epsilon} dz_1 e^{-ikz_1} b_1(z_1) \int_{z_1-\epsilon}^{z_1+\epsilon} dz_2 e^{ikz_2} b_1(z_2) \times$

$\int_{z_2-\epsilon}^{z_2+\epsilon} dz_3 e^{-ikz_3} b_1(z_3) \int_{z_3-\epsilon}^{z_3+\epsilon} dz_4 e^{ikz_4} b_1(z_4) \cdots \times$

$$\int_{z_{(2n-2)}-\epsilon}^{z_{(2n-2)}+\epsilon} dz_{(2n-1)} e^{-ikz_{(2n-1)}} b_1(z_{(2n-1)}) \int_{z_{(2n-1)}-\epsilon}^{z_{(2n-1)}+\epsilon} dz_{2n} e^{ikz_{2n}} b_1(z_{2n}), \qquad n > 0. \tag{3.12}$$

$$\left(\int\int b_1(z')\right)^n \equiv 1, \qquad n = 0. \tag{3.13}$$

In this way we have successfully addressed the problem of incorporating the spike-like data in the LOIMES.

3.5 Application to the IMAS: Removal of effects of internal multiples in the input data

We will now explain an application of the modified LOIMES proposed in this report. In particular, we will see how to eliminate the effect, created by the IMAS, when the input data include internal multiples, and we are working with a specific 1D earth model. Speciffically, this effect is a component of the recorded data whose traveltime cannot be related to a set of reflections and transmissions originated in the reflector boundaries at the subsurface. In other words, it is an event that does not exist in the earth.

In the original IMAS algorithm in Araújo (1994), one of the basic assumptions is that the input data were made only of primaries and that the internal multiples' times are constructed via interactions of these primaries. In other words, the internal multiples are constructed using the primaries as subevents. However, the data collected from the seismic experiment obviously contain internal multiples, and a consequence of their inclusion as part of the input data for the IMAS is that, under certain conditions, the effects mentioned in the paragraph above are created. The presence of such events can potentially decrease the effectiveness of the subsequent imaging algorithm applied to the data. For this reason, it is important to find the way in which the ISS deals with the presence of those events.

The answer to this is provided in Ma and Weglein (2012) and Liang and Weglein (2012), each of which propose an extension of the IMAS. This extended IMAS contains some terms attenuating the amplitude of the unusual events, created by the presence of internal multiples in the input data. We will go a step further and explain how this attenuator subseries can be extended to an eliminator (of effects of internal multiples in the input data) subseries by using the modified LOIMES proposed in this report. For this we will focus on the simplest situation in which such a unusual event is created: a 1D earth with three reflectors and with the traveltime t_3 of the primary associated with the third layer satisfying $t_3 > 2t_2 - t_1$, where t_2 and t_1 are the traveltimes of the primaries associated with the second and first layer, respectively, and $t_2 > t_1$, as before. We also assume normal incidence and include in the input data the first-order internal multiple, associated with the first (shallowest) layer, and with traveltime $2t_2 - t_1$.

With the assumptions of the paragraph above, the second term of this IMAS becomes

$$b_3(k) + \int_{-\infty}^{+\infty} dz_1' e^{ikz_1'} b_1(z_1') \int_{-\infty}^{z_1'-\epsilon} dz_2' e^{-ikz_2'} b_3(z_2') \int_{z_2'+\epsilon}^{\infty} dz_3' e^{ikz_3'} b_1(z_3'), \tag{3.14}$$

where

$$b_3(k) = \int_{-\infty}^{+\infty} dz_1' e^{ikz_1'} b_1(z_1') \int_{-\infty}^{z_1'-\epsilon} dz_2' e^{-ikz_2'} b_1(z_2') \int_{z_2'+\epsilon}^{\infty} dz_3' e^{ikz_3'} b_1(z_3') \tag{3.15}$$

is the leading order contribution in the original IMAS.

The reason for the second term in (3.14) is as follows. The inclusion of the first-order internal multiple $IM_1 = -T_{01}R_2R_1R_2T_{10}e^{ik[2z_2-z_1]}$ in the input data (z_m, T_{ij} and R_k are defined as in section 3.1) results in the presence of the term $SE = (T_{01}T_{12}R_3T_{21}T_{10})^2(-T_{01}R_2R_1R_2T_{10})e^{ik[2z_2-z_1]}$ in (3.15). Now, if the ISS is right, this event should be attenuated at least in some way; in other words, it should be possible to find a term from the ISS predicting the same phase of the SE but with an attenuated amplitude and positive sign. This is exactly what the second term in (3.14) does: it creates the term $(T_{01}T_{10})(T_{01}T_{12}R_3T_{21}T_{10})^2(T_{01}R_2R_1R_2T_{10})e^{ik[2z_2-z_1]}$, which when added to SE event, results in

$$(1 - T_{01}T_{10})(T_{01}T_{12}R_3T_{21}T_{10})^2(-T_{01}R_2R_1R_2T_{10})e^{ik[2z_2-z_1]}. \tag{3.16}$$

The above expression is an attenuator of the amplitude of SE, because $T_{01}T_{10} < 1$. Our claim in this report is that the ISS is able to completely remove SE rather than just to attenuate it.

In the particular earth model we are considering, it is easy to prove our claim: notice in (3.16) that if we can add to (3.14) more terms from the ISS, such that the correction to the amplitude of SE is changed from $T_{01}T_{01}$ to 1, then the amplitude of the SE is canceled. It is clear that the contributions of the extra terms must match exactly those of (3.9), when we explained how to promote the IMAS proposed by Araújo and Weglein to the role of an eliminator. But we know that in (3.9) this contribution comes from the LOIMES, when eliminating the first-order internal multiple generated at the shallowest reflector. Hence, if it is possible to somehow include the modified LOIMES described in this report into (3.14), then we will be able to eliminate the SE. It turns out that the right place to plug in the LOIMES is in the second term of (3.14), because this is the term responsible for the factor $T_{10}T_{01}$ in (3.16). Therefore, at least for this configuration, (3.14) can be promoted to being an eliminator of SE. This subseries takes the form:

$$b_3(k) + \int_{-\infty}^{+\infty} dz_1' e^{ikz_1'} b_1(z_1') \int_{-\infty}^{z_1'-\epsilon} dz_2' e^{-ikz_2'} b_{LO}^{IM}(z_2') \int_{z_2'+\epsilon}^{\infty} dz_3' e^{ikz_3'} b_1(z_3'), \tag{3.17}$$

whose first term is exactly (3.14). To see explicitly how this subseries works, we write the second term in (3.17) in expanded form:

$$\int_{-\infty}^{+\infty} dz_1' e^{ikz_1'} b_1(z_1') \int_{-\infty}^{z_1'-\epsilon} dz_2' e^{-ikz_2'} (b_3(z_2') + b_5^{IM}(z_2') + ...) \int_{z_2'+\epsilon}^{\infty} dz_3' e^{ikz_3'} b_1(z_3'). \tag{3.18}$$

On the other hand, for the particular earth configuration studied in this example, we have

$$b_3(z) = -T_{01}T_{10} * IM_1 + SE + ... \tag{3.19}$$

$$b_3(z) + b_5^{IM}(z) + ... = -T_{01}T_{10}(1 + R_1^2 + ...) * IM_1 + ... \tag{3.20}$$

Inserting (3.20) into (3.18):

$\int_{-\infty}^{+\infty} dz_1' e^{ikz_1'} b_1(z_1') \int_{-\infty}^{z_1'-\epsilon} dz_2' e^{-ikz_2'} (-T_{01}T_{10}(1 + R_1^2 + ...) * IM_1 + ...) \int_{z_2'+\epsilon}^{\infty} dz_3' e^{ikz_3'} b_1(z_3') =$

$$- T_{01}T_{10}(1 + R_1^2 + ...) * \int_{-\infty}^{+\infty} dz_1' e^{ikz_1'} b_1(z_1') \int_{-\infty}^{z_1'-\epsilon} dz_2' e^{-ikz_2'} IM_1 \int_{z_2'+\epsilon}^{\infty} dz_3' e^{ikz_3'} b_1(z_3') + ... \tag{3.21}$$

Among other terms, (3.21) produces, when the middle integral is combined with the two outer integrals containing the primary associated with the third layer, the event SE. Hence, using (3.9), (3.21) reproduces $-SE$ plus other contributions. Therefore, the first term of eq. (3.14), becomes

$$SE - SE + ... = ... \tag{3.22}$$

From the above expression, it becomes evident that the amplitude of SE is completely removed, as desired.

It is not coincidence that the terms added are exactly those of the LOIMES, as we are trying to eliminate the contribution of an event created by the first-order internal multiple generated at the shallowest reflector; this internal multiple is exactly the contribution that the LOIMES takes care of. In more general earth models, in which events similar to SE can be generated by first-order internal multiples generated at reflectors other than the shallowest, we would need the HOIMES.

It is worth mentioning that (3.17) will bring more terms than the ones needed for the elimination of the SE. It would be interesting to do further research into the specific tasks of these terms.

4 LOIMES and continuous data

In section 2, we modified the LOIMES to correctly accomodate spike-like data. The key point was writing the correct splitting of $B_3(k)$, eq. (3.6), as opposed to eq. (2.10), and then selecting $B'_{35}(k)$ instead of $B_{35}(k)$. Analogously, in this section we will propose a derivation for the LOIMES, suitable for nonspike-like but continuous data, starting from eq. (3.6) and $B'_{35}(k)$.

4.1 LOIMES and the mean value theorem

Our goal is to make explicit the interactions contained in the finite-interval integrations in eq. (3.6). For that we will use a sort of "complex mean value theorem" (CMVT) :

$$\int_a^b f(z)dz = (b-a)f(\eta) \quad for\ some\ \eta \in (b,a), \tag{4.1}$$

where $f(z)$ is a complex-valued, real function. Also we assume that the real and imaginary parts of $f(z)$ are continuous on (a,b).

For complex-valued functions, eq. (4.1) is not true in general, but it can be for certain cases. Hence, (4.1) is a restriction for the data in which the present approach to the LOIMES can be applied. To determine whether the CMVT applies to a given data, we need to split $f(z)$ into real and imaginary parts and apply the usual mean value theorem (MVT) to each of them. If η in (4.1) is the same for both integrals, then we can proceed with the application of the LOIMES to these specific data.

Let's now apply the CMVT to $B'_{35}(k)$. From (3.6)

$B'_{35}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz e^{ikz''} b_1(z'') =$

$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z')(2\epsilon) e^{ik(z'+\beta)} b_1(z'+\beta) =$

$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) e^{ik\beta} b_1(z+\alpha)(2\epsilon)^2 b_1(z+\alpha+\beta) =$

$$e^{ik\beta} \int_{-\infty}^{\infty} (2\epsilon)^2 dz e^{ikz} b_1(z) b_1(z+\alpha) b_1(z+\alpha+\beta), \tag{4.2}$$

where $z+\alpha$ and $z'+\beta$ are the parameter η introduced by the mean value theorem, for the middle and left integrals in $B'_{35}(k)$, respectively. Note that α and β represent the deviation from the center of the interval of integration of the respective integrals, and hence

$\alpha < \epsilon/2 \qquad \beta < \epsilon/2 \qquad \alpha + \beta < \epsilon.$

Notice that the factor $(2\epsilon)^2$ makes the value of $B_{35}^{'}(k)$ go to zero when $\epsilon \to 0$, as is desired. Using the fact that ϵ is small (and also $\epsilon/2$), and with the continuity of $b_1(z)$, we can make the following approximation:

$b_1(z) \approx b_1(z+\alpha) \approx b_1(z+\beta) \approx b_1(z+\alpha+\beta).$

$$e^{ik\beta} \approx 1, \tag{4.3}$$

Hence we end up with

$$B_{35}'(k) = (2\epsilon)^2 \int_{-\infty}^{\infty} dz e^{ikz} b_1(z)^3, \tag{4.4}$$

which upon a Fourier transform becomes $B_{35}'(z) = (2\epsilon)^2 b_1(z)^3$. We propose eq. (4.4) as the part of $\hat{b}_3$ to be inserted into (3.7) to get b_5^{IM}, the second term in b_{LO}^{IM}. Using the fact that $B_{35}'(z) = (2\epsilon)^2 b_1(z)^3 = (2\epsilon)^2 B_{35}(z)$, the result is the original term (2.6) times a factor $(2\epsilon)^2$:

$$(2\epsilon)^2 \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z')^3 \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''). \tag{4.5}$$

As we anticipated in the introduction, this CMVT scheme has its own issues. For example, by applying the CMVT to obtain subsequent terms in b_{LO}^{IM}, we predict a subseries whose closed form is, when the data are continuous and hence are not spike-like:

$$b_{LO}^{IM} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \left(\frac{1}{1-(2\epsilon\ b_1(z'))^2}\right) b_1(z') \times \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''). \tag{4.6}$$

Notice the 2ϵ factor in the quotient that is present in (4.6) but is not present in the original series (2.2). This means that both subseries agree only if $\epsilon = 1/2$. At first we may think that we have a generalization of (2.2), but this is not true because we should keep in mind that we want a subseries that eliminates multiples and for this we need to predict the right amplitude, which is exactly what (2.2) does. This means that any deviation from the amplitude predicted by (2.2) will result in the failure of the series to eliminate internal multiples. This forces us to interpret the condition $\epsilon = 1/2$ as a restriction to the class of experiments to which the LOIMES can be applied, namely, those for which $\epsilon = 1/2$.

We now discuss some intriguing relations between the approach we have just described for continuous data, and another procedure commonly used in the physics literature to circumvent difficulties similar to the ones encountered in this report.

A common approach used to overcome difficulties such as ill-definiteness of the higher powers of the Delta function is to introduce into $B_{35}(k)$ the two parameters α and β into the arguments of the Delta function, as follows

$B_{35}(\alpha,\beta,k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z')\delta(z-z'+\alpha) \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'')\delta(z'-z''+\beta) =$
$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z')\delta((z+\alpha)-z') e^{ik(z'+\beta)} b_1(z'+\beta)$

$$= e^{ik\beta} \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) b_1(z+\alpha) b_1(z+\alpha+\beta). \tag{4.7}$$

In this way, the self interactions are removed and are, when the data are spike-like, the higher-than-one powers of the Delta function. The next step would be to define

$$B_{35}(k) \equiv lim_{\alpha,\beta\to 0} B_{35}(\alpha,\beta,k), \tag{4.8}$$

where the limit is performed $after$ the integral $B_{35}(\alpha,\beta,k)$ has been calculated. What we have described is analogous to the procedure used in Green's function theory, in which the resulting integral defining the Green's function is not well defined (due to the presence of poles in the path of integration). Hence it is made well defined by deforming the contour of integration, which we accomplished by introducing a small parameter ϵ to avoid the poles of the integrand, followed by the limit $\epsilon \to 0$.

Unfortunately, this solution is not powerful enough for our present issues, and the reason is that if we take (4.8) as the definition for the interactions, then with the spike data $B_{35}(k)$ becomes zero, which is obviously not what we want.

If we assume (1) that the data are continuous and (2) that the operation of taking limits commutes with the integral, then (4.8) reduces to the original integral $B_{35}(k)$ containing interactions. This means that at least for continuous data, we can consider (4.8) to be an equivalent expression for the interaction contribution to the eliminator subseries b_{LO}^{IM}.

At this point it is worth comparing (4.2) with (4.7), the expression obtained using the approach described earlier that was based on the CMVT. We can see that they are similar, with the obvious difference of the factor $(2\epsilon)^2$ in (4.2). In this way, the CMVT approach reproduces the regularization scheme just explained and at the same time fixes the problem with the original splitting of $B_3(k)$, eq. (2.10). It's fair to say that it is not expected for (4.8) to fix the issue related to the limit $\epsilon \to 0$, as it was obtained from the old expression for $B_3(k)$, eq. (2.10), which is not well behaved in this limit. However it is interesting that we partially reproduce the result of the CMVT. This might mean that although not strictly correct, eq. (2.10) may still be useful for studying some properties and obtaining some insight about the LOIMES; after all, in practice ϵ is small but not zero. Another nice feature of the CMVT is that whereas in the regularization scheme both α and β were introduced in a somewhat arbitrary way, here they arise naturally: they are the coordinates of the point whose image is used by the CMVT.

Given the similarities between those two approaches, it would be interesting to perform a more detailed study of the relation between them, in order to better understand the nature of the LOIMES.

As we explained earlier, the LOIMES matches the amplitude of the internal multiple by using interactions in the integrals of certain terms of the ISS. Now, if we just require the filtering (or extraction) of the self interaction contained in the integrals in (3.6) instead of insisting on looking for a convenient expression for the value of the integrals with a finite interval of integration (i.e., looking for an expression in which are the self interactions become evident), we only need to multiply the integral times a Delta function with the correct argument. In this way we arrive, for the non-spike data, at an expression similar to the right-hand side of (2.10), but we must keep in mind that this expression is not equal to $B_3(k)$ anymore. Now we can proceed by selecting the original $B_{35}(k)$, instead of $B'_{35}(k)$, to be the term associated with the LOIMES. By repeating this filtering process in the appropriate higher-order terms, we arrive at the original form for the LOIMES, eq. (2.2). In general this filtering process can be applied also to the HOIMES, obtaining in this way the original expression for the IMES, as stated in eqs. (2.1)-(2.4).

The advantage of this argument is that, although not mathematically rigorous, it is fairly general and can include all continuous data, as opposed to the scheme proposed in this report. Again, the spike-like are not included, as this would bring the original problem with the powers of the Delta function. However we can now argue, on the basis of of the results of Appendix A, that the filtering process is not necessary for the Delta function. That is because of the function's very particular properties under integration; it automatically selects the self-interaction part of the integral, without the need of any further filtering process. This argument is highly plausible, even though ideally it would be desirable to have a filter working with all kinds of data at once including the Delta function.

Although the central subject of this report is the LOIMES, it is worthwhile to say some words about the HOIMES. As can be seen from the general form, eqs. (2.3) and (2.4), this subseries also contains interaction terms, thereby causing the same problem that the leading-order eliminator subseries has with spike-like data. A detailed analysis of such a case is beyond the objective of this report, but it is easy to provide some evidence that the formalism described here can be also applied to the HOIMES. For this, let's focus on continuous data, so we can apply the CMVT approach.

The first term in the HOIMES is

$$b^{IM}_{HO} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} 2(2\epsilon) b_1(z') \int_{-\infty}^{z'-\epsilon} dz''' b_1^2(z''') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{4.9}$$

and it was derived in Ramírez (2007) on the basis of the symmetry $B_{36}(k) = B_{39}(k)$ in (2.10). So, a first hint that the HOIMES can be described by the CMVT approach is that this symmetry is preserved by the corresponding terms in (3.6): $B'_{36}(k) = B'_{39}(k)$. This fact is proved in Appendix B, where we also use this symmetry to show that, in this case, the parameters arising from the application of the CMVT are unambiguously zero. Thus, by using $B'_{36}(k)$ and $B'_{39}(k)$ instead of $B_{36}(k)$ and $B_{39}(k)$, and using the MVT, we get our proposal for the first term of the HOIMES:

$$b^{IM}_{HO} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} 2(2\epsilon) b_1(z') \int_{-\infty}^{z'-\epsilon} dz''' b_1^2(z''') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{4.10}$$

which also contains the factor 2ϵ, characteristic of the CMVT approach. Notice that this modified term is, as in the LOIMES, the old term multiplied by a factor of 2ϵ. The rule is that for each time the CMVT is applied, there is a factor of 2ϵ and also an interaction of the data. More precisely, if the MVT is applied $n$times in a single term of the ISS, we get a factor of $(2\epsilon)^n$ and a factor of $b_1(z)^{n+1}$.

Following these criteria, and provided the corresponding symmetries are preserved, we conjecture that the HOIMES predicted by the MVT is

$$b^{IM}_{HO} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \frac{2G(z') \int_{-\infty}^{z'-\epsilon} dz''' J(z''')}{1-\int_{-\infty}^{z'-\epsilon} dz''' J(z''')} \times \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') \tag{4.11}$$

where

$$J(z''') = \frac{2\epsilon b_1(z''')^2}{1-(2\epsilon b_1(z'''))^2} \qquad G(z') = \frac{b_1(z')}{1-(2\epsilon b_1(z'))^2}. \tag{4.12}$$

By expanding (4.11), we have

$$b^{IM}_{HO} = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (2(2\epsilon) b_1(z') \int_{-\infty}^{z'-\epsilon} b_1^2(z') +$$
$$2(2\epsilon)^3 b_1(z') \int_{-\infty}^{z'-\epsilon} b_1^4(z') + 2(2\epsilon)^3 b_1^3(z') \int_{-\infty}^{z'-\epsilon} b_1^2(z') +$$
$$2(2\epsilon)^2 b_1(z') \int_{-\infty}^{z'-\epsilon} b_1^2(z') \int_{-\infty}^{z'-\epsilon} b_1^2(z')) \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''). \tag{4.13}$$

Notice that this conjecture also works only for $\epsilon = 1/2$, as in this case it coincides with (2.3), the old version of the HOIMES. Notice also that the factors 2ϵ in each term satisfy the general rule just explained in the paragraph above.

5 Discussion and conclusions

As mentioned in the introduction, the present report is oriented to lower the burden of the adaptive subtraction of internal multiples, by promoting to elimination, the attenuation provided by the leading order contribution of the original attenuator subseries. In particular, we have rederived and modified the LOIMES in order to accommodate spike-like data. As a result we find a modified closed form for the Leading-Order Internal-Multiple-Eliminator Subseries (LOIMES) originally proposed in Ramírez (2007). Such a closed form, eqs. (3.10)-(3.12), is only valid for this kind of data.

The relevance of this work is that now we can test the algorithm itself: as the analytic data is perfect, any problem in the output is caused by the algorithm, which means that it must ve revisited. Also, as we did in this work, this allows the elimination subseries to enhance the effectiveness of other algorithms, which are also being tested with analytic data.

Also, we illustrate how to apply the modified closed form of the LOIMES to promote the IMAS of Ma and Weglein (2012), eq. (3.14), to the role of an eliminator of some effects, caused by the inclusion of internal multiples in the input data. We do this for the simplest case, in which the contribution of (3.14) differs from that of the original IMAS of Araujo and Weglein: a four-layer $1D$-earth, with normal incidence. The traveltime t_3 represents the primary associated with the third (deepest) layer satisfying $t_3 > 2t_2 - t_1$, where t_2 and t_1 are the traveltimes of the primaries associated with the second and first (shallowest) layer, respectively. We also include in the input data the first-order internal multiple with traveltime $2t_2 - t_1$, associated with the first layer.

As was explained in section 3.1, both the LOIMES and the HOIMES were first derived from (2.10), which is not strictly correct. Hence the need to rederive both suberies, including for continuous data starting from the correct expression, eq. (3.6). We do this for the LOIMES, and we conjecture the answer for the HOIMES. Unfortunately, the derivation we found is not general enough to include all types of continuous data, but only a very restricted class-i.e., continuous data that satisfy the CMVT, eq. (4.1).

A further research topic in this direction is to write the modified closed form, analogous to (3.10), corresponding to the HOIMES. The potential applications are (1) elimination of effects, created by the original IMAS, when the input data includes first-order internal multiples, whose downward reflection is generated at deeper reflectors, and more important, (2) elimination of first-order internal multiples, created at salt deposits beneath the earth's surface.

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A Calculating the modified LOIMES for spike data.

In this appendix we will show explicitly, for 1D and three-layer earth, how to perform the integral (3.7), when the input data are the two spike-like primaries, with normal incidence, associated with the two interfaces.

$$D(t) = R_1\delta(t - t_1) + \underbrace{T_{01}R_2T_{10}}_{R_2'}\delta(t - t_2). \tag{A.1}$$

The notation is the same as in section 3.1: t_1 and t_2 are the traveltimes associated with the primaries created at the first and second reflector, respectively, and $t_2 > t_1$, T_{ij} denotes the transmission coefficient when the wave travels from the ith medium to the jth medium, and R_k is the reflection coefficient at the kth layer for a downward incident wave. We will also need the pseudodepths $z_1 \equiv c_0t_1/2$ and $z_2 \equiv c_0t_2/2$, of the two interfaces, with respect to a homogeneous reference medium with constant velocity c_0.

Inserting (A.1) into the right integral of $B'_{35}(k)$, we get by following eq. (90) in Weglein et al. (2003)
$\int_{z'-\epsilon}^{z'+\epsilon} dz''e^{ikz''}b_1(z'') = \int_{z'-\epsilon}^{z'+\epsilon} dz''e^{ikz''}[R_1\delta(z'' - z_1) + R_2'\delta(z'' - z_2)] =$

$R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)$.

Substituting the above result into the second integral in $B'_{35}(k)$, we have

$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z')[R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$

$R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)] = \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'}[R_1\delta(z'-z_1)+R'_2\delta(z'-z_2)]\times$

$[R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)] =$

$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1)R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$

$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1)R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)]+$

$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R'_2\delta(z'-z_2)R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$

$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R'_2\delta(z'-z_2)R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2) =$

$$I_1+I_2+I_3+I_4. \tag{A.2}$$

Performing the four integrations, we arrive at

$I_1 = \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1)R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1) =$

$R_1^2 e^{-ikz_1} e^{ikz_1} H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)\underbrace{H(z_1-(z_1-\epsilon))}_{=1}\underbrace{H((z_1+\epsilon)-z_1)}_{=1} =$

$R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)$

$I_2 = \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1)R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2) =$

$R_1 R'_2 e^{-ikz_2} e^{ikz_2} H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)H(z_2-(z_1-\epsilon))\underbrace{H((z_1+\epsilon)-z_2)}_{=0} = 0$

$I_3 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R'_2\delta(z'-z_2)R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1) =$

$R_1 R'_2 e^{-ikz_2} e^{ikz_1} H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)\underbrace{H(z_1-(z_2-\epsilon))}_{=0}H((z_2+\epsilon)-z_1) = 0$

$I_4 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R'_2\delta(z'-z_2)R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2) =$

$(R'_2)^2 e^{-ikz_2} e^{ikz_2} H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)H(z_2-(z_2-\epsilon))H((z_2+\epsilon)-z_2) =$

$$(R'_2)^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2). \tag{A.3}$$

Finally, substituting the value of the integrals in (A.3) into the third integral in $B'_{35}(k)$, we end up with

$B'_{35}(k) = \int_{-\infty}^{\infty} dz e^{ikz}[R_1\delta(z-z_1)+R'_2\delta(z-z_2)]\times$

$[R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)+(R'_2)^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)] =$

$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^2 H(z_1-(z-\epsilon)) H((z+\epsilon)-z_1)+$
$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) (R_2')^2 H(z_2-(z-\epsilon)) H((z+\epsilon)-z_2)+$
$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1^2 H(z_1-(z-\epsilon)) H((z+\epsilon)-z_1)$
$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) (R_2')^2 H(z_2-(z-\epsilon)) H((z+\epsilon)-z_2) =$

$$I_1' + I_2' + I_3' + I_4'. \tag{A.4}$$

Evaluating the integrals above, we have
$I_1' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^2 H(z_1-(z-\epsilon)) H((z+\epsilon)-z_1) =$
$R_1^3 e^{ikz_1} H(z_1-(z_1-\epsilon)) H((z_1+\epsilon)-z_1) = R_1^3 e^{ikz_1}$
$I_2' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) (R_2')^2 H(z_2-(z-\epsilon)) H((z+\epsilon)-z_2) =$
$R_1 (R_2')^2 e^{ikz_1} H(z_2-(z_1-\epsilon)) \underbrace{H((z_1+\epsilon)-z_2)}_{=0} = 0$
$I_3' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1^2 H(z_1-(z-\epsilon)) H((z+\epsilon)-z_1) =$
$R_1^2 R_2' e^{ikz_2} \underbrace{H(z_1-(z_2-\epsilon))}_{=0} H((z_2+\epsilon)-z_1) = 0$
$I_4' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) (R_2')^2 H(z_2-(z-\epsilon)) H((z+\epsilon)-z_2) =$

$$(R_2')^3 e^{ikz_2} H(z_2-(z_2-\epsilon)) H((z_2+\epsilon)-z_2) = (R_2')^3 e^{ikz_2}. \tag{A.5}$$

Adding the integrals above, we finally have

$$B_{35}'(k) = R_1^3 e^{ikz_1} + (R_2')^3 e^{ikz_2}. \tag{A.6}$$

When transformed to the space domain, (A.6) becomes

$$B_{35}'(z) = R_1^3 \delta(z-z_1) + (R_2')^3 \delta(z-z_2). \tag{A.7}$$

Now we will evaluate b_5^{IM}, the second term in b_{LO}^{IM}, using (A.7):

$$b_5^{IM}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} B_{35}'(z) \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z). \tag{A.8}$$

The first integral in the above expression is
$\int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z) = \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} [R_1 \delta(z''-z_1) + (R_2')\delta(z''-z_2)] =$
$R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) + R_2' e^{ikz_2} H(z_2-(z'+\epsilon)).$

Substituting the above result into the second integral of (A.8), we get

$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'}[R_1^3\delta(z'-z_1)+(R_2')^3\delta(z'-z_2)][R_1 e^{ikz_1}H(z_1-(z'+\epsilon))+R_2' e^{ikz_2}H(z_2-(z'+\epsilon))]$

$=\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3\delta(z'-z_1)R_1 e^{ikz_1}H(z_1-(z'+\epsilon))+$

$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3\delta(z'-z_1)R_2' e^{ikz_2}H(z_2-(z'+\epsilon))$

$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3\delta(z'-z_2)R_1 e^{ikz_1}H(z_1-(z'+\epsilon))+$

$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3\delta(z'-z_2)R_2' e^{ikz_2}H(z_2-(z'+\epsilon))=$

$$I_1''+I_2''+I_3''+I_4''. \tag{A.9}$$

Evaluating the above integrals, we have

$I_1''=\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3\delta(z'-z_1)R_1 e^{ikz_1}H(z_1-(z'+\epsilon))=R_1^4\underbrace{H(z_1-(z_1+\epsilon))}_{=0}H((z-\epsilon)-z_1)=0$

$I_2''=\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3\delta(z'-z_1)R_2' e^{ikz_2}H(z_2-(z'+\epsilon))=$

$R_1^3R_2' e^{ik(z_2-z_1)}H(z_2-(z_1+\epsilon))H((z-\epsilon)-z_1)$

$I_3''=\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3\delta(z'-z_2)R_1 e^{ikz_1}H(z_1-(z'+\epsilon))=$

$R_1(R_2')^3 e^{ik(z_1-z_2)}\underbrace{H(z_1-(z_2+\epsilon))}_{=0}H((z-\epsilon)-z_2)=0$

$I_4''=\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3\delta(z'-z_2)R_2' e^{ikz_2}H(z_2-(z'+\epsilon))=$

$(R_2')^4\underbrace{H(z_2-(z_2+\epsilon))}_{=0}H((z-\epsilon)-z_2)=0$

Substituting the only nonzero value, I_2'', in the last integral of (A.8), we finally have

$b_5^{IM}=\int_{-\infty}^{\infty} dz e^{ikz}[R_1\delta(z-z_1)+R_2'\delta(z-z_2)]R_1^3R_2' e^{ik(z_2-z_1)}H(z_2-(z_1+\epsilon))H((z-\epsilon)-z_1)=$

$R_1^4R_2' e^{ikz_2}H(z_2-(z_1+\epsilon))\underbrace{H((z_1-\epsilon)-z_1)}_{=0}+R_1^3R_2'^2 e^{ik(2z_2-z_1)}H(z_2-(z_1+\epsilon))H((z_2-\epsilon)-z_1)=$

$$R_1^3R_2'^2 e^{ik(2z_2-z_1)}. \tag{A.10}$$

B Calculating α and β for the HOIMES

In this appendix we will show that the ISS requires the parameters α and β to be zero. We begin by showing that the integrals $B_{36}'(k)$ and $B_{39}'(k)$ in (3.6) have the same value.

$B_{39}'(k)=\int_{-\infty}^{\infty} dz e^{ikz}b_1(z)\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'}b_1(z')\int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''}b_1(z'')=$

$\int_{-\infty}^{\infty} dz e^{ikz}b_1(z)\int_{-\infty}^{\infty} dz' e^{-ikz'}b_1(z')H((z-\epsilon)-z')\times$

$\int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') H((z'+\epsilon)-z'') H(z''-(z'-\epsilon)) =$

$\int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') H((z'+\epsilon)-z'') H(z''-(z'-\epsilon)) \times$

$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) H((z-\epsilon)-z')$

Making the change of variables $z'' \to z$ and $z \to z''$, we get:

$B'_{39}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') H((z'+\epsilon)-z) H(z-(z'-\epsilon)) \times$

$\int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') H((z''-\epsilon)-z') =$

$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') H(z'-(z-\epsilon)) H((z+\epsilon)-z') \times$

$\int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') H(z''-(z'+\epsilon)) =$

$$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') = B'_{36}(k). \tag{B.1}$$

The following step is to apply the mean-value theorem to both $B'_{36}(k)$ and $B'_{39}(k)$:

$B'_{36}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') =$

$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) 2\epsilon e^{-ik(z+\alpha)} b_1(z+\alpha) \int_{z+\alpha+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'') =$

$$(2\epsilon) e^{-ik\alpha} \int_{-\infty}^{\infty} dz b_1(z) b_1(z+\alpha) \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') H(z''-(z+\alpha+\epsilon)) \tag{B.2}$$

$B'_{39}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} b_1(z'')$

$= \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') H((z-\epsilon)-z') 2\epsilon e^{ik(z'+\beta)} b_1(z'+\beta)$

$= (2\epsilon) e^{ik\beta} \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' b_1(z') b_1(z'+\beta) H((z-\epsilon)-z')$

$= (2\epsilon) e^{ik\beta} \int_{-\infty}^{\infty} dz' b_1(z') b_1(z'+\beta) \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) H((z-\epsilon)-z')$

$$= (2\epsilon) e^{ik\beta} \int_{-\infty}^{\infty} dz b_1(z) b_1(z+\beta) \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') H((z''-(z+\epsilon)). \tag{B.3}$$

We have just proved that $B'_{36}(k) = B'_{39}(k)$. Hence, (B.2) and (B.3) must be equal-i.e.,

$(2\epsilon) e^{-ik\alpha} \int_{-\infty}^{\infty} dz'' b_1(z) b_1(z+\alpha) \int_{-\infty}^{\infty} dz e^{ikz''} b_1(z'') H(z''-(z+\alpha+\epsilon))$

$$= (2\epsilon) e^{ik\beta} \int_{-\infty}^{\infty} dz b_1(z) b_1(z+\beta) \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') H(z''-(z+\epsilon)). \tag{B.4}$$

From the phase outside the integral, the argument of the data and the argument of the step function respectively, in the expression above, we get the following over constrained, but consistent set of equations:

$$\alpha = -\beta, \qquad \alpha = \beta, \qquad \alpha = 0. \tag{B.5}$$

whose only solution is $\alpha = \beta = 0$. This shows that the parameters introduced by the CMVT are zero. Notice that those α and β are not the same as the α and β of section 4.

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