## Short note: A first step towards a P wave field modeling plan

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**Abstract**

In 2D and 3D heterogeneous elastic media, the P- and S-wave equations are coupled. In this case, the equation result of $\phi_p$ will take all history and intermediate episodes of P- and S-wave propagation into account. Migration and inversion work, which require accurate P-P events, give us the motivation to model and predict only P-wave events from an elastic world. In this short note, we revisit a wave-theory method that is based on Weglein (2012) that can model both the phase and the amplitude of waves that spend all their history as P-waves. We also provide a basic implementation for obtaining operators that are related to the uncoupled P- and S- components.

# 1 Introduction

As we all know, wave-theory modeling is widely used because of its accuracy and its inclusion of propagation phenomena. However, the wave-theory method has a limitation in selecting a path or wave type of interest from all the events. Conventional finite-difference (wave-theory) modeling methods coded in Cartesian coordinates face the issue that P- and S-wave events come out simultaneously in the final record because all displacements are projected in the $(x, y, z)$ domain. In the 2012 M-OSRP annual report(Weglein, 2012), the formalism of the uncoupled signal-channel P-wave equation was proposed . As that report pointed out, the wave-theory method that Weglein proposed can model and predict P-waves (and P-wave events in recorded data) without using the S-wave field. The P-wave modeling series allows for the selectivity of events that spend all their history as P-waves in a heterogeneous medium, so that we can select the path of interest on the basis of wave-type. In this note, we review the formalism of modeling and selecting P-wave events, and we introduce a basic method for transferring the displacement domain to the P-S domain for implementation.

# 2 Isotropic heterogeneous elastic media

## 2.1 2D media

We are familiar with the acoustic wave equation, as shown here,

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\phi = \rho$$

, where $c$ is the wave velocity, $\omega$ is the angular frequency and $\rho$ is a source term. In an elastic medium, the $\rho$ part on the right will turn out to be very complicated. Here, with the help of the scattering theory, we choose to describe a medium as an isotropic homogeneous whole-space background plus a perturbation in the properties.The whole-space background and the perturbation combine to result in the properties in an actual medium. If we can express the perturbation operator as $(V_{pp}, V_{ps}, V_{sp}, V_{ss})$, then a coupled equation for a P- and S- wave pressure field ($\phi_p$ and $\phi_s$) can be written as

$$\left[\nabla^2 + \frac{\omega^2}{\alpha_0^2}\right]\phi_p = V_{pp}\phi_p + V_{ps}\phi_s + f_p \tag{2.1}$$

$$\left[\nabla^2 + \frac{\omega^2}{\beta_0^2}\right]\phi_s = V_{ss}\phi_s + V_{sp}\phi_p + f_s \tag{2.2}$$

, where $\alpha_0$ is the P-wave velocity and $\beta_0$ is the S-wave velocity. Let us introduce a Green's function as $G_s$, which satisfies

$$\left[\nabla^2 + \frac{\omega^2}{\beta_0^2} - V_{ss}\right]G_s = \delta. \tag{2.3}$$

Notice that the $G_s^0$ is different from $G_s$ and is defined as

$$\left[\nabla^2 + \frac{\omega^2}{\beta_0^2}\right]G_s^0 = \delta. \tag{2.4}$$

Using the Lippmann-Schwinger equation, $G_s$ can be expressed as a Born series with a shear-wave Green's function in the reference medium $G_s^0$, and a shear perturbation operator $V_{ss}$. In this case, $G_s = \sum_{k=0}^{\infty} G_s^0\left(V_{ss}G_s^0\right)$.

Similarly, according to equations (2.2) and (2.3), the shear-wave field can be expressed with the compressional wave field $\phi_p$, the source term $f_s$ and the perturbation operator $V_{sp}$, by using the Lippmann-Schwinger equation, as

$$\phi_s = \int G_s(V_{sp}\phi_p + f_s) \tag{2.5}$$

where $G_s$ is chosen as the causal solution. The final modeling formalism can be expressed as (Weglein, 2012),

$$\mathsf{V} = V_{pp} + V_{ps}\int G_s V_{sp} \tag{2.6}$$

$$\mathsf{f} = V_{ps}\int G_s f_s + f_p \tag{2.7}$$

$$\phi_p^0 = G_p^0 \mathsf{f} \tag{2.8}$$

$$\phi_p = \phi_p^0 + G_p^0 \mathsf{V} \phi_p^0 + G_p^0 \mathsf{V} G_p^0 \mathsf{V} \phi_p^0 + \cdots \tag{2.9}$$

where $\mathsf{V}$ and $\mathsf{f}$ are notations representing a complicated perturbation and a source term, respectively. In this equation, the source $(f_p, f_s)$ is taken into account. If we assume that the source generates only P-waves, namely, that $f_s = 0$, then $\mathsf{f} = f_p$ in equation (2.7). In addition, and equation (2.9) is the modeling equation for P-waves in a 2D heterogeneous elastic medium.

## 2.2 3D media

In 3D isotropic heterogeneous media, the perturbation of three components $(P, S_H, S_V)$ consists of a matrix. The index of $S_H$ represents a shear-horizontal channel, and the index of $S_V$ represents a shear-vertical channel. Similarly as in the 2D case, the source here only generates a P-wave, which is

$$\overrightarrow{f} = \begin{pmatrix} f_p \\ 0 \\ 0 \end{pmatrix}. \tag{2.10}$$

Three coupled equations for a three-component wave field are

$$\begin{aligned}
\left[\nabla^2 + \frac{\omega^2}{\alpha_0^2} - V_{pp}\right]\phi_p &= V_{PS_H}\phi_{S_H} + V_{PS_V}\phi_{S_V} + f_p \\
\left[\nabla^2 + \frac{\omega^2}{\beta_0^2} - V_{S_H S_H}\right]\phi_{S_H} &= V_{S_H P}\phi_P + V_{S_H S_V}\phi_{S_V} \\
\left[\nabla^2 + \frac{\omega^2}{\beta_0^2} - V_{S_V S_V}\right]\phi_{S_V} &= V_{S_V P}\phi_P + V_{S_V S_H}\phi_{S_H}.
\end{aligned} \tag{2.11}$$

After introducing three Green's functions, $G_p^0, G_{S_H}$, and $G_{S_V}$, which are causal solutions of

$$\begin{aligned}
\left[\nabla^2 + \frac{\omega^2}{\alpha_0^2}\right] G_p^0 &= \delta \\
\left[\nabla^2 + \frac{\omega^2}{\beta_0^2} - V_{S_H S_H}\right] G_{S_H} &= \delta \\
\left[\nabla^2 + \frac{\omega^2}{\beta_0^2} - V_{S_V S_V}\right] G_{S_V} &= \delta
\end{aligned} \tag{2.12}$$

, respectively, we can derive the P-wave field by using a scalar equation,

$$\phi_p = \phi_p^0 + G_p^0 \mathsf{V} \phi_p \tag{2.13}$$

$$\mathsf{V} = V_{pp} + V_{PS_H}\left(\sum_{k=0}^{\infty}(G_{S_H} V_{S_H S_V} G_{S_V} V_{S_V S_V})^k\right) G_{S_H}(V_{S_H P} + V_{S_H S_V} G_{S_V} V_{S_V P}) + V_{PS_V} G_{S_V}$$

$$\times \left( V_{S_V P} + V_{S_V S_H} \left( \sum_{k=0}^{\infty} (G_{S_H} V_{S_H S_V} G_{S_V} V_{S_V S_V})^k \right) G_{S_H} (V_{S_H P} + V_{S_H S_V} G_{S_V} V_{S_V P}) \right). \quad (2.14)$$

A Born series provides a modeling formalism for P-wave events(Weglein, 2012),

$$\phi_p = \phi_p^0 + G_p^0 \mathsf{V} \phi_p^0 + G_p^0 \mathsf{V} G_p^0 \mathsf{V} \phi_p^0 + \cdots . \quad (2.15)$$

Here, we make an assumption that the subsurface is isotropic and has horizontal reflectors without anisotropic fracturing or a special structure that could convert a P-wave into a $S_H$-wave. Given the fact that the polarizations of the two shear waves are both perpendicular to the polarization of the P-wave and that the $S_H$-wave vibration is normal to the incidence plane, the P-wave displacement cannot project onto the $S_H$ vibration direction. Therefore, we only consider P- and $S_V$-wave conversions. In this situation, the perturbation part can be simplified as,

$$\mathsf{V} = V_{pp} + V_{PS_V} G_{S_V} V_{S_V P}, \quad (2.16)$$

where $G_{S_V} = \sum_{k=0}^{\infty} G_{S_V}^0 \left( V_{S_V S_V} G_{S_V}^0 \right)$. The case of the 3D isotropic heterogeneous medium P-event modeling degenerates to a 2D case when the assumption of only P- and $S_V$- conversion is made. On the other hand, if the complicated term $\mathsf{V}$ is replaced by $V_{pp}$, the P-events in predicting data will only have intermediate P-wave episodes in their history; i.e.

$$\phi_p = \phi_p^0 + G_p^0 V_{pp} \phi_p^0 + G_p^0 V_{pp} G_p^0 V_{pp} \phi_p^0 + \cdots \quad (2.17)$$

# 3 Basic multi-component elastic-medium method

The operators under the displacement domain are denoted by calligraphic type, such as $\mathcal{L}$, $\mathcal{V}$, that satisfy

$$\mathcal{L}\mathbf{u} = \mathbf{f}$$

$$\rho\omega^2 u_{im} \left(\overrightarrow{r_g}, \overrightarrow{r_s}, \omega\right) + \left(C_{ijkl} u_{km,l} \left(\overrightarrow{r_g}, \overrightarrow{r_s}, \omega\right)\right)_{,j} = -A(\omega)\delta_{im}\delta\left(\overrightarrow{r_g} - \overrightarrow{r_s}\right). \quad (3.1)$$

We have the perturbation under the displacement domain in an isotropic medium, which can be expressed as (Weglein and Stolt, 1992)

$$\mathcal{V} = -\rho_0 \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2 a_\gamma \partial_x^2 + \beta_0^2 \partial_z a_\mu \partial_z & \left(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu\right)\partial_z\partial_x + \beta_0^2 \partial_z a_\mu \partial_x \\ \partial_z \left(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu\right)\partial_x + \beta_0^2 a_\mu \partial_z \partial_x & a_\rho\omega^2 + \alpha_0^2 \partial_z a_\gamma \partial_z + \beta_0^2 a_\mu \partial_x^2 \end{bmatrix} \quad (3.2)$$

where $a_\rho = \rho/\rho_0 - 1$, $a_\gamma = \gamma/\gamma_0 - 1$, $a_\mu = \mu/\mu_0 - 1$ and $\alpha_0$, $\beta_0$ are P- and S-wave velocity in reference medium respectively. Here, for convenience in calculation, the perturbation operator can be transformed to the P-S domain by (Matson, 1997; Clayton and Brown, 1979)

$$V = \Pi \mathcal{V} \Pi^{-1} \Gamma^{-1} \quad (3.3)$$

where

$$\Pi = \begin{bmatrix} \partial_x & \partial_z \\ -\partial_z & \partial_x \end{bmatrix} \quad \Pi^{-1} = \begin{bmatrix} \partial_x & -\partial_z \\ \partial_z & \partial_x \end{bmatrix} \nabla^{-2} \quad \Gamma_0^{-1} = \begin{bmatrix} \frac{1}{\gamma_0} & 0 \\ 0 & \frac{1}{\mu_0} \end{bmatrix}$$

In these matrices, $\gamma_0$ is the P-wave modulus or longitudinal modulus ($\alpha_0 = \sqrt{\frac{\gamma_0}{\rho}}$), and $\mu_0$ is the shear modulus ($\beta_0 = \sqrt{\frac{\mu_0}{\rho}}$) in reference medium. In a perturbation term under the P-S domain, there is an integral operator, $\nabla^{-2}$, which will be discussed in the next section.
At the beginning, the role of $\mathcal{V}$ was to scatter the wave displacements as horizontal and vertical components. After this kind of transformation, the wave can be scattered as P- and S-wave pressure by the new perturbation operator $V$. So that the $G_p^0$ can be used to propagate the wave in reference medium as P-wave pressure and then wave can be scattered by $V$ in consistence as shown in Figure 1.

## 4 Explanation of the $\nabla^{-2}$ operator acting on $G_p^0$

The value of the integral operator $\frac{1}{\nabla^2}$ can be determined by the term that it acts on (Zhang, 2006). For example, we can consider a simple term that is the first term of 2D P-wave-only modeling as

$$G_p^0 \frac{1}{\nabla^2} G_p^0 \tag{4.1}$$

, where $G_p^0$ satisfies

$$\left(\nabla'^2 + \frac{\omega^2}{\alpha_0^2}\right) G_p^0\left(x', z', x'', z'', \omega\right) = \delta\left(x' - x''\right)\delta\left(z' - z''\right). \tag{4.2}$$

Next, we Fourier transform over $x'$ and $z'$ to solve for $G_p^0$. After we transform back to the spatial domain, we can obtain the bilinear form of the Green's function,

$$G_p^0\left(x', z', x'', z'', \omega\right) = (\frac{1}{2\pi})^2 \int\int \frac{e^{ik_x^I(x'-x'')} e^{ik_z^I(z'-z'')}}{k^2 - k_x^{I2} - k_z^{I2}} dk_x^I dk_z^I. \tag{4.3}$$

The term can be written as

$$\begin{aligned} & G_p^0 \frac{1}{\nabla^2} G_p^0 \\ = \; & (\frac{1}{2\pi})^4 \int\int dx'' dz'' \int\int \frac{e^{ik'_x(x_g - x'')} e^{ik'_z(z_g - z'')}}{k^2 - k'^2_x - k'^2_z} dk'_x dk'_z \\ & \times \frac{1}{\nabla''^2} \int\int \frac{e^{ik''_x(x''-x_s)} e^{ik''_z(z''-z_s)}}{k^2 - k''^2_x - k''^2_z} dk''_x dk''_z. \end{aligned} \tag{4.4}$$

The outside term of the Green's function with the integral operator can be expressed as,

$$\frac{1}{\nabla''^2}(-i\pi) e^{ik_s x''} \frac{e^{iq_s(z''-z_s)}}{q_s} = \frac{1}{-k_s^2 - q_s^2}(-i\pi) e^{ik_s x''} \frac{e^{iq_s(z''-z_s)}}{q_s} \tag{4.5}$$

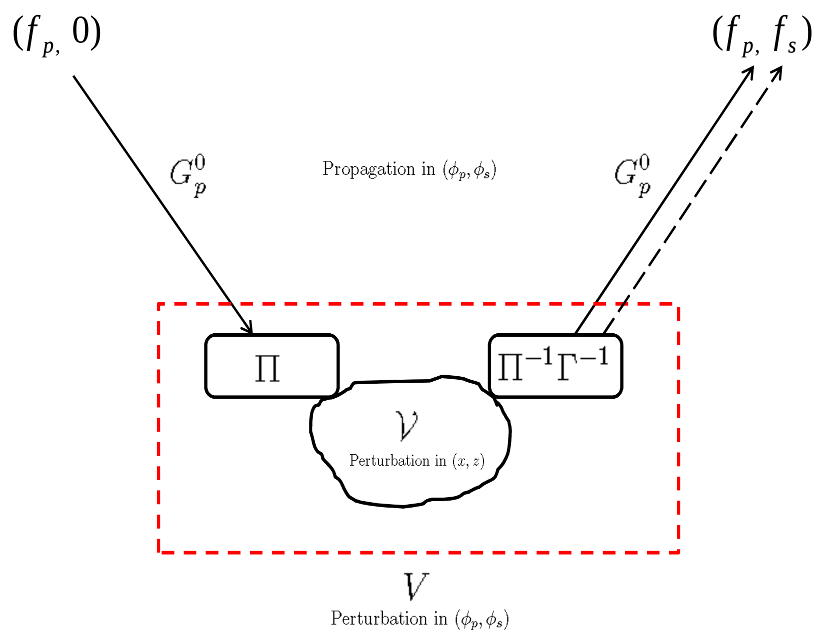


Figure 1: Scattering in P- and S-wave pressure domain.

where $q_s^2 = k^2 - k_s^2$. In this case, the assumption is that the source location must be shallower than the scattering point.

# 5 Conclusions and future plan

In this short note, we conclude that
(a) The formalism for modeling the phase and amplitude of a P-event has been established and is understood, and by using Born series modeling we can select the events that have only intermediate P-wave episodes in their histories. Equations (2.9) and (2.15) are forward series on which the

modeling of all possible P-event histories is based, as shown in Weglein (2012).
(b) The issue of perturbation under the P-S domain can be solved by the transform operator. However, the differential operators are very complicated even when we only look at $V_{pp}$ in equation (2.17). For example, the perturbation term for a 1D earth (i.e., in which properties only vary in z) can be written as

$$V_{pp} = -\nabla^2 a_\gamma - \left[k_0^2(a_\rho \partial_x^2 + \partial_z a_\rho \partial_z) + 4\partial_z a_\mu \partial_z \partial_x^2 - 2\partial_z^2 a_\mu \partial_x^2 - 2a_\mu \partial_z^2 \partial_x^2\right] \frac{1}{\nabla^2}$$

, where $k_0 = \frac{\omega}{\alpha_0}$, $a_\mu = \frac{\beta_0^2}{\alpha_0^2}(\frac{\mu}{\mu_0} - 1)$. In forward modeling, the Born series form can be implemented because the Green's function in the reference medium, the perturbation and properties of the source are known. It is appropriate to examine a single-reflector 1.5D, where the source is a 2D line source (an oblique incident wave) and the properties vary in 1D at the beginning. The algorithm guarantee that the incidence will be pure P-wave. This test will allow us to understand how different types of waves, such as converted and unconverted waves, are constructed by a forward P-wave-only series.
(c) This formalism could be tested further by a modeling project (for example, SEAM) with a smoothed background associated with a small perturbation, in which the converted-wave always is treated as noise (Weglein, 2012).

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