GEOPHYSICS, VOL. 75, NO. 2 (MARCH-APRIL 2010); P. V13–V23, 3 FIGS.
10.1190/1.3337695

Direct nonlinear Q-compensation of seismic primaries reflecting from a stratified, two-parameter absorptive medium

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ABSTRACT

Q-compensation of seismic primaries that have reflected from a stratified, absorptive-dispersive medium may be posed as a direct, nonlinear inverse scattering problem. If the reference medium is chosen to be nonattenuating and homogeneous, an inverse-scattering Q-compensation procedure may be derived that is highly nonlinear in the data, but which operates in the absence of prior knowledge of the properties of the subsurface, including its Q structure. It is arrived at by (1) performing an order-by-order inversion of a subset of the Born series, (2) isolating and extracting a component of the resulting nonlinear inversion equations argued to enact Q-compensation, and (3) mapping the result back to data space. Once derived, the procedure can be understood in terms of nonlinear interaction of the input primary reflection data: the attenuation of deeper primaries is corrected by an operator built (automatically) using the angle- and frequency variations of all shallower primaries. A simple synthetic example demonstrates the viability of the procedure in the presence of densely sampled, broadband reflection data.

INTRODUCTION

Certain problems of reflection seismic data processing, which when solved using linear algorithms require an accurate input model of subsurface medium properties, have in recent years proven tractable without that information. The cost of this improved capability is that the resulting algorithms are nonlinear in the input data. Surface-related multiple elimination-type methods are well-known example, for which the cost often has proven very worthwhile. (For detailed examples and discussion, see Carvalho, 1992; Verschuur et al., 1992; Verschuur and Berkhout, 1997; Weglein et al., 1997; Weglein et al., 2003; and Weglein and Dragoset, 2005.) The purpose of this paper is to examine and assess the extent to which this is also true for the problem of Q-compensation of primary reflections.

We begin with a particular definition of the problem, a necessary step, because the state-of-the-art and precise goals of deterministic Q-compensation are difficult to neatly pin down. Recovery from the resolution-compromising effects of absorption can occur within many otherwise distinct procedures, such as inversion (e.g., Dahl and Ursin, 1992; Ribodetti and Virieux, 1998; Causse et al., 1999; Hicks and Pratt, 2001; Dasios et al., 2004), downward continuation/imaging (e.g., Mittet et al., 1995; Song and Innanen, 2002; Wang, 2003; Mittet, 2007), and deterministic deconvolution (e.g., Bickel and Natarajan, 1985; Hargreaves and Calvert, 1991; Wang, 2006; Zhang and Ulrych, 2007). We define *Q-compensation* as the estimation of the primary reflection data set that would have been measured in the absence of the absorptive component of wave propagation; i.e., we will pose it such that it is maximally isolated from the other tasks of seismic inversion, as recommended by Hargreaves and Calvert (1991).

Building on earlier studies (Innanen and Weglein, 2003, 2005; Innanen and Lira, 2008), we seek such an algorithm by making use of inverse scattering, a framework capable of providing procedures that trade nonlinearity for subsurface information, for processing both multiples (as mentioned above) and primaries (Weglein et al., 2001, 2003; Amundsen et al., 2005; Shaw, 2005; Liu, 2006; Zhang, 2006). The end goal is to define a processing procedure, which (absent an input Q estimate) returns a Q-compensated, prestack primary data set due to waves reflecting from an anelastic medium with arbitrary variability in three dimensions. Here we derive and numerically test a candidate algorithm which is appropriate for primaries reflecting from a simpler medium, a two-parameter absorptive medium with arbitrary variability in depth. Because we do not take any steps during the derivation that could not — at least in principle — also be taken under conditions of more complex heterogeneity or model type, this result may be regarded as a potentially useful waypoint towards the end goal.

Manuscript received by the Editor 16 June 2009; published online 5 April 2010.
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Inverse scattering, absent manipulation, takes as its input measurements of a scattered field, and creates as its output the perturbation that gave rise to the field. This runs counter to the goals of Q-compensation as we have defined the procedure, because

1) The scattered field in general contains all reflected events, including primaries and multiples, whereas (because there exist reliable methods for multiple removal that are not sensitive to the elasticity/anelasticity of the subsurface) we will perform inverse operations on primaries only.
2) Of all the processing steps enacted upon primaries within the full inverse problem, we wish only one, the correction for absorptive propagation, to be actually carried out.
3) We wish to estimate not the perturbation, a model-like quantity, but rather a data-like quantity, a set of reflected primaries that have been Q-compensated.

After posing the scattering problem to accommodate absorptive media, most of the strategy in the algorithm development we present is geared towards managing these three issues. Our route is as follows.

We begin by creating a forward-modeling procedure for absorptive-dispersive primaries based on the Born series. The result is a nonlinear scattering-based series calculation of *primaries only* in a layered absorptive-dispersive medium, which is accurate for large, extended perturbations. This is useful for our current purposes, because such partial series may be inverted, order-by-order, in exactly the same fashion as the full inverse scattering series, to generate nonlinear direct inversion procedures that take as their input data reflected primaries. We continue by carrying out this inversion upon the absorptive-dispersive primary series above. The resulting nonlinear inverse scattering equations, which construct approximations of the actual wavespeed and Q perturbations in the medium, are therefore of a form that addresses item one above.

Next we observe that, because of the direct, analytic nature of these inverse equations, it is possible to make informed conjectures regarding where and how in the mathematics the correction for Q takes place, and by extension, how to suppress all of the other nonlinear operations. Doing so, we argue, amounts to an extraction and separate execution of the Q-compensation part of the full inversion of primary data; this addresses item two above.

Finally, we point out that given a homogeneous reference medium, the relationship between the linear components of the parameter perturbations and the data is very simple — essentially a Fourier transform. In the second step above, all nonlinear aspects of the processing (apart from those that we argue are concerned with Q-compensation) have been suppressed. It follows that in all respects apart from absorption, the output maintains a simple, linear relationship with the data. We map the output trivially back to data space using this relationship, which amounts to a change of variables and an inverse Fourier transform. The final result, which has addressed item three above, is deemed to be the Q-compensated data set.

DIRECT NONLINEAR Q-COMPENSATION

We define the reference medium such that the Green's functions for a source at $\mathbf{x_s}$ and a receiver at $\mathbf{x}$ at angular frequency ω obey the scalar (nonabsorptive) equation

$$\left[\nabla^2 + \frac{\omega^2}{c_0^2}\right] G_0(\mathbf{x}|\mathbf{x_s};\omega) = \delta(\mathbf{x} - \mathbf{x_s}). \quad (1)$$

The solution to equation 1 in 2D for a line source at (x_s,z_s) and a line receiver at (x_g,z_g), is expressible analytically as, e.g.,

$$G_0(x_g,z_g,x_s,z_s,\omega) = \frac{1}{2\pi}\int dk_x' e^{ik_x'(x_g - x_s)} \frac{e^{iq'|z_g - z_s|}}{i2q'}, \quad (2)$$

where $q' = \omega/c_0(1 - c_0^2 k_x'^2/\omega^2)^{1/2}$, and we take $\int$ to indicate $\int_{-\infty}^{\infty}$. We define the wavefield in the actual medium as satisfying a two-parameter (nearly constant Q) absorptive wave equation:

$$[\nabla^2 + K^2] G(\mathbf{x}|\mathbf{x_s};\omega) = \delta(\mathbf{x} - \mathbf{x_s}), \quad (3)$$

where, following, e.g., Aki and Richards (2002),

$$K \equiv \frac{\omega}{c(\mathbf{x})}\left[1 + \frac{F(\omega)}{Q(\mathbf{x})}\right], \quad (4)$$

and

$$F(\omega) = \frac{i}{2} - \frac{1}{\pi}\ln\left(\frac{\omega}{\omega_r}\right). \quad (5)$$

In the inverse developments to follow, F is assumed known, and Q as assumed unknown. Treating the quantities in square brackets in equations 1 and 3 as the operators $\mathbf{L}_0$ and $\mathbf{L}$, respectively, (see, e.g., Weglein et al., 2003), and defining two dimensionless perturbation quantities:

$$\alpha(\mathbf{x}) = 1 - \frac{c_0^2}{c^2(\mathbf{x})}, \quad \beta(\mathbf{x}) = \frac{1}{Q(\mathbf{x})}, \quad (6)$$

we arrive at a perturbation operator (defined as the difference between $\mathbf{L}_0$ and $\mathbf{L}$) appropriate for this Q problem (Innanen and Weglein, 2007; Innanen et al., 2008):

$$\mathbf{L}_0 - \mathbf{L} \approx \frac{\omega^2}{c_0^2}[\alpha(\mathbf{x}) - 2F(\omega)\beta(\mathbf{x})]. \quad (7)$$

We next restrict the medium such that α and β vary in depth only, and use the above quantities to form a partial Born series:

$$\psi_P = \psi_1 + \psi_2 + \psi_3 \ldots, \quad (8)$$

whose terms are judged, via arguments based on relative scattering geometry, to construct reflected primaries that have been distorted by Q. This is a two-parameter extension of the scalar acoustic construction discussed by Innanen (2008). At first order, for instance, after Fourier transforming over x_s, we have

$$\psi_1(k_s,\omega) = \int dx' \int dz' G_0(x_g,z_g,x',z',\omega) \times \frac{\omega^2}{c_0^2}\gamma(z') G_0(x',z',k_s,z_s,\omega)$$
$$= -\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}\gamma(z'),$$

where $\gamma(z) = \alpha(z) - 2F(\omega)\beta(z)$, k_s is the Fourier conjugate of x_s, $q_s = \omega/c_0(1 - c_0^2 k_s^2/\omega^2)^{1/2}$, and $\theta = \sin^{-1}\frac{k_s c_0}{\omega}$, and for convenience, we have set $x_g = z_g = z_s = 0$. At second order, we have

$$\psi_2(k_s,\omega) = -\frac{(-i2q_s)}{8\cos^4\theta}\int dz' e^{i2q_s z'}\gamma(z')\left(\int_0^{z'} dz''\gamma(z'')\right).$$

Continuing with this program of retention and rejection at all orders, the details of which are included in Appendix A, we produce the series ψ_P. This forward-modeling expression can be evaluated with any source- and receiver depth by reinstating $z_g, z_s \neq 0$. With $z_g = z_s = 0$, we next identify ψ_P with the primary data D arising from a reflection experiment:

$$D(k_s,\omega) = \frac{-1}{4\cos^2\theta}\int dz' e^{i2q_s z'}\gamma(z') \times \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-iq_s}{\cos^2\theta}\int_0^{z'} dz''\gamma(z'')\right)^n. \quad (9)$$

Computing and summing a large number of terms in equation 9 generates an approximation of attenuated primary data appropriate for large and extended perturbations γ.

We next form an inverse series for the perturbations α and β, in which the nth term is defined to be nth order in the *primary* data modeled above. Let this series be $\gamma = \gamma_1 + \gamma_2 + \ldots$, or, explicitly,

$$[\alpha(z) - 2F(\omega)\beta(z)] = [\alpha_1(z) - 2F(\omega)\beta_1(z)] + [\alpha_2(z) - 2F(\omega)\beta_2(z)] + \cdots.$$

This is substituted into equation 9, and like orders are equated (Innanen, 2008), similarly to Carvalho's derivation of the full inverse scattering series (Carvalho, 1992; Weglein et al., 1997). The inverse solution is generated by sequentially solving for and summing contributions to the perturbation over many orders. At first order, we have

$$D(k_s,\omega) = \frac{-1}{4\cos^2\theta}\int dz' e^{i2q_s z'}[\alpha_1(z') - 2F(\omega)\beta_1(z')]. \quad (10)$$

Innanen and Weglein (2007) describe in detail how this equation can be used to separately determine α_1 and β_1 as functions of pseudodepth. The process of linear separation requires the data to be combined across sets of incidence angles $\vartheta = \{\theta_1, \theta_2, \ldots\}$, and can also involve a weighting scheme W, hence in general, the outputs must be considered functions of these variables also:

$$\alpha_1 = \alpha_1(z|\vartheta, W),$$
$$\beta_1 = \beta_1(z|\vartheta, W). \quad (11)$$

Quantities α_1 and β_1 have qualitative interpretations that depend on the size and extent of the actual perturbations α and β. If the actual perturbations are small and transient, and some scheme of averaging (e.g., Clayton and Stolt, 1981) is invoked to manage the overdeterminedness of the problem, these quantities can be considered model-like, and if interpreted as inverse Born-approximate model-parameter estimates, represent an endpoint of the procedure. Alternatively, if the perturbation is large and extended, which we assume is the case in this paper, quantities α_1 and β_1 bear scant resemblance to the actual perturbations α and β. In fact, they are data-like: they depend on experimental variables, and they have amplitudes and discontinuities that are only distantly and nonlinearly related to those of α and β, while being closely and linearly related to those of the reflected primary events. For this reason, in this paper we refer to the α_1, β_1 quantities as being essentially linearly transformed and weighted versions of the input primary data.

Continuing next with the nonlinear components of the inversion procedure, at second order, we find a relationship between α_1, β_1 and α_2, β_2:

$$\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}[\alpha_2(z'|\vartheta,W) - 2F(\omega)\beta_2(z'|\vartheta,W)]$$
$$= -\frac{(-i2q_s)}{8\cos^4\theta}\int dz' e^{i2q_s z'}[\alpha_1(z'|\vartheta,W) - 2F(\omega)\beta_1(z'|\vartheta,W)] \times \left(\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(\omega)\beta_1(z''|\vartheta,W)]\right).$$

This continues at third order, wherein a relationship between α_1, β_1 and α_3, β_3 is determined. By this time, a pattern is discernable in the mathematics. Assuming the continuation of this pattern, we sum the equations over all orders. Defining

$$\alpha_P(z|\vartheta,W) \equiv \sum_{n=0}^{\infty}\alpha_{n+1}(z|\vartheta,W)$$

and

$$\beta_P(z|\vartheta,W) \equiv \sum_{n=0}^{\infty}\beta_{n+1}(z|\vartheta,W),$$

there results a closed-form set of nonlinear equations

$$\alpha_P(k_z,\theta|\vartheta,W) - 2F(k_z,\theta)\beta_P(k_z,\theta|\vartheta,W)$$
$$= \int dz' e^{-ik_z\left[z' + \frac{1}{2\cos^2\theta}\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(k_z,\theta)\beta_1(z''|\vartheta,W)]\right]} \times [\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)], \quad (12)$$

where $k_z \equiv -2q_s$ is the Fourier conjugate of depth z, and F has been written as a function of the reference plane-wave variables θ and k_z rather than ω. These equations constitute a direct inversion of the primary data, exact to within the accuracy of the primary approximation series in equation 9, and appropriate for a layered, two-parameter, absorptive-dispersive medium. Further details of the derivation of equation 12 are included in Appendix B. The quantities α_P and β_P are the nonlinearly determined profiles associated with $c(z)$ and $Q(z)$; each can, in principle, be individually determined via equation 12 if desired, which itself might be of independent interest. This is shown in Appendix C.

However, our current goal is to carry out a single inverse task, that of compensating for Q, and to ultimately recover a data set, not a set of parameter perturbations. To accomplish this, we examine equation 12 more closely. We note that the outputs α_P and β_P would be related linearly to the inputs α_1 and β_1, except that α_1 and β_1 also appear in the argument of the exponential function in the integrand. All of the nonlinearity of the inversion resides here. Then, we make the following statements. The principal role of α_1 in the argument of the exponential is to nonlinearly accomplish aspects of the inversion associated with wavespeed deviations between the reference and actual media (e.g., to correctly locate linearly misplaced reflectors at depth). And, the principal role of β_1 in the argument is to accomplish aspects of the inversion associated with deviations between reference and actual Q values — meaning, predominantly, compensation. The arguments for these statements are twofold. First, equation 12 is a two-parameter version of a scheme derived elsewhere for a one-parameter acoustic medium, i.e., involving α only (Innanen, 2008). Those one-parameter equations include an exponential function with an argument identical to the first (α_1) term in the exponential of equation 12. Because the β_1 component of the exponential function appears only when absorptive inverse issues appear, we ascribe to it the role of managing these issues. Second, this component of the exponential function (by virtue of the complex nature of the coefficient F) is the only part of the function that grows exponentially, and therefore alone has the numerical capability to perform the (ill-conditioned) boosting of high frequencies characteristic of Q-compensation. We will now permit these two arguments to lead us to a proposed form of a Q-compensation algorithm, and discuss the possibility that they are only approximately true (and the consequences of this) in the discussion section of this paper.

We set α_1 in the argument of the exponential to zero, and suggest that as a consequence, (1) the (now-altered) outputs α_P and β_P undergo nonlinear correction for the attenuation and dispersion associated with propagation in an absorptive medium, but (2) they undergo linear treatment in all other respects.

Calling the partially treated outputs α_Q and β_Q, we have instead

$$\alpha_Q(k_z,\theta|\vartheta,W) - 2F(k_z,\theta)\beta_Q(k_z,\theta|\vartheta,W) = \int dz' e^{-ik_z\left[z' - \frac{F(k_z,\theta)}{\cos^2\theta}\int_0^{z'} dz''\beta_1(z''|\vartheta,W)\right]} \times [\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)]. \quad (13)$$

By assumption, the form of these equations ensures that only Q-compensation (and not, say, any repositioning of reflectors away from their linearly-determined depths) takes place as the data (through α_1 and β_1) are processed.

Finally, we will argue for an approach to make the output of this processing a data-like quantity. Equation 13 follows the basic template

$$A = \int e^B C. \quad (14)$$

We proceed by comparing this template, and the associated elements of equation 13, to equation 10. Evidently if β_1 in the exponential function were set to zero, we would exactly recover the data by carrying out the integral in equation 13. If our previous suggestion is correct, i.e., that $\beta_1 \neq 0$ in the exponential is responsible solely for Q-compensation, then it would appear that the left-hand side is already a data-like quantity, different only from the input data set in that its amplitudes have been corrected for absorption and dispersion.

Hence, after constructing the input C and the operator B by linearly transforming and weighting the data, and computing the right-hand side of equation 13, we use the linear relationship defined in equation 10 to map not α_1 and β_1, but rather the left-hand side of equation 13, back to the (k_s,ω) domain, through, in essence, a change of variables. Our suggestion is that this mapped quantity is a Q-compensated data set in the Fourier domain. That is, we define

$$D_{\mathrm{comp}}(k_z,\theta|\vartheta,W) \equiv -\frac{1}{4\cos^2\theta}[\alpha_Q(k_z,\theta|\vartheta,W) - 2F(k_z,\theta)\beta_Q(k_z,\theta|\vartheta,W)], \quad (15)$$

where $\vartheta = \vartheta(\theta,k_z)$ and $W = W(\theta,k_z)$. Changing variables back to k_s, ω and inverse Fourier transforming, the Q-compensated data set is estimated as

$$D_{\mathrm{comp}}(x_s,t) = \left(\frac{1}{2\pi}\right)^2 \int\int dk_s d\omega e^{ik_s x_s} e^{i\omega t} D_{\mathrm{comp}}(k_s,\omega|\vartheta,W). \quad (16)$$

SYNTHETIC EXAMPLE

To exemplify this procedure, we construct a simple synthetic primary data set corresponding to a suite of plane sources and line receivers over the two-interface absorptive-dispersive model in Figure 1. The resulting primary data (Figure 2a), generated analytically in the frequency/wavenumber domain and numerically inverse Fourier transformed to produce the plots, are used as input to the linear inverse scattering equations, which involves a transformation and weighting thereof. Multiples are not modeled; we assume multiples have been removed as a preprocessing step. Then these data-like quantities are used to construct both the operator e^B and the operand C as in equation 14. The Q-compensated data set (Figure 2b) is formed by transforming the result, A in equation 14, to the (k_s,ω) domain, and then performing straightforward inverse Fourier transforms. The Q-compensated results are compared in detail with the input in Figure 3 for three offsets, (a) 0 m, (b) 170 m, and (c) 335 m. For illustration purposes, *after* all processing is complete, we convolve the input and output with a Ricker wavelet, which is a cosmetic step; the procedure assumes the source wavelet has been decon-

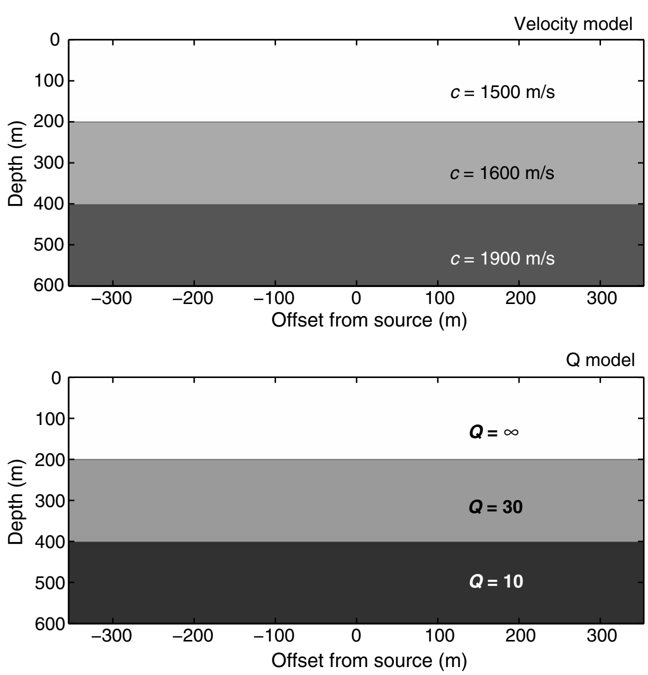

Figure 1. Two-interface absorptive-dispersive model.

volved from the input data. The dispersion correction, and the similarity of the output to the idealized, nonattenuated test trace, indicate that the algorithm is largely achieving its stated goal. We note a slight undercorrection at large angle.

In equations 13 and 15, there are two sets of angles: an input set (ϑ) used to separate α_1 and β_1, and to construct the correction operator, and an output angle θ, which is varied to recover the full, corrected prestack data set. Neither inverse scattering theory nor our manipulations of it specifically impose any relationship between the two. In the full inverse-scattering parameter-estimation problem (Appendix C), this freedom might be exploited for purposes of regularization, or to incorporate prior information. However, for the problem at hand, we have used input angles ϑ which "cluster" around the output angle θ. That is, we have decided to correct particular angle and wavenumber components of the data using the data at those same components and their immediate neighbors. In the example above, the input set $\vartheta = \{\theta, \theta + \Delta\theta\}$ (where $\Delta\theta$ is the smallest provided by the synthetic data after the change of variables) was used for each θ of corrected data. The angle pairs were weighted equally ($W = 1$); little of the additional freedom W provides to precondition the data has been explored as yet.

Noisy examples have not been included at this proof-of-concept stage; we have found our approach to share the basic response to noise of all standard Q-compensation schemes. We point out that (as is often done in standard Q-compensation) with a straightforward alteration of the function F as it appears in the argument of the exponential in equation 12, this algorithm may be transformed into a "dispersion compensation" procedure, which is well conditioned and largely unaffected by noise.

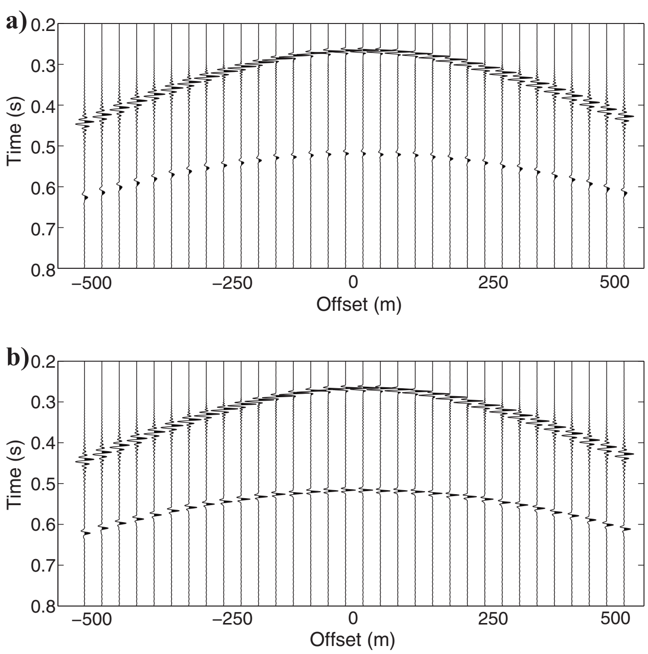

Figure 2. (a) Synthetic prestack input primary data from the model in Figure 1 (decimated for purposes of display), and (b) Q- compensated output data (likewise decimated). The amplitude and phase signature of the shallower primary is caused by the strong absorptive reflection coefficient associated with the top interface.

DISCUSSION

We choose as the definition of Q-compensation, *the estimation of an output data set that is identical to the input, except that all absorptive propagation effects are absent*. We present a candidate scheme, based on nonlinear inverse scattering, whose output, we argue, fits this definition. In applying it, a correction operator is automatically constructed from the data themselves, with no requirement for an input estimate of subsurface medium parameters, including its Q structure. Synthetic examples illustrate the scheme in action, and provide proof-of-concept-level evidence of the validity of the approach.

The behavior of this algorithm can be interpreted in terms of data events interacting nonlinearly. Consider again the schematic form

$$A = \int e^B C, \qquad (17)$$

where

$$A = \alpha_Q(k_z, \theta|\vartheta, W) - 2F(k_z,\theta)\beta_Q(k_z,\theta|\vartheta, W),$$
$$B = -ik_z\left[z' - \frac{F(k_z,\theta)}{\cos^2\theta}\int_0^{z'} dz''\beta_1(z''|\vartheta, W)\right], \qquad (18)$$

and

$$C = \alpha_1(z'|\vartheta, W) - 2F(k_z,\theta)\beta_1(z'|\vartheta, W),$$

and recall that α_1 and β_1 are effectively linearly transformed, weighted forms of the data in the pseudodepth domain (i.e., vertical two-way traveltime scaled with the constant reference wavespeed c_0). The quantity C is being operated on by $\int e^B$ to produce A. One fre-

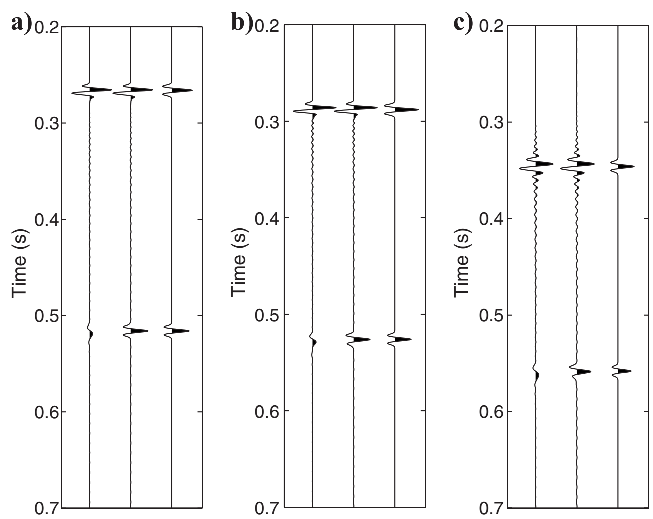

Figure 3. Detail of Q-compensation for three offsets of prestack data: (a) 0-m offset, (b) 170-m offset, and (c) 335-m offset. The left-hand trace in each panel is the input, the middle trace is the output, and the right-hand trace, for benchmarking, is an idealized trace constructed without Q, and normalized to the maximum value of the output traces. Because the benchmark traces involve no absorption, their shallow primaries differ from their counterparts in that they do not have the amplitude and phase signature associated with the absorptive reflection coefficient. The relevant events for comparison are the deeper primaries.

quency-wavenumber component of the output data (ω and k_s, via k_z and θ) A, comes from contributions from the input data C at all pseudodepths. For each contributing depth, the C component is boosted by e^B, which from equation 18 can be seen to be the cumulative influence of the values of β_1 that are shallower than the contributing depth. The quantities β_1 and B are constructed from the angle and frequency dependence of the primaries in the input data. Therefore, this Q-compensation operator, acting on a given primary, can be understood to have been constructed from the cumulative angle and frequency variations (i.e., the generalized AVO/AVA behavior) of all shallower primaries.

We have made two major assumptions in deriving the scheme:

1) By suppressing certain components of the full nonlinear inversion equations derived from inverse scattering, we isolate the Q-compensation activity inherent to the inversion.
2) With trivial linear transformation and changes of variable, the output of this isolated inverse step can be treated as an equivalent data set, different only from the input in the lack of absorption in the primary events.

The soundness of these assumptions is likely best argued for with success in testing, some of which we have provided with our proof-of-concept example. But, we might already anticipate that slightly more sophisticated choices ultimately could lead to more accurate results, in particular with respect to the first assumption above. Part of the argument for isolating and extracting the Q-compensation component of the inverse equations lay in comparing the two-parameter absorptive system of inverse equations with its one-parameter acoustic counterpart. But the two-parameter linear inverse problem, by which we determine the correction operator, is subject to phenomena not shared by one-parameter problems, for instance, leakage, or the tendency for one parameter's *actual* variations to be accounted for with variations in multiple *linearly estimated* parameters (discussed for the absorptive problem by Innanen and Weglein, 2007). Continued study of these issues may lead to a more sophisticated program for isolation of the absorption compensation component of the nonlinear equations.

CONCLUSIONS

Direct, nonlinear methods bring a greatly reduced requirement for prior information as compared to their linear counterparts. But they demand broadband, densely sampled, wide-aperture, deghosted, deconvolved (of the source wavelet), and demultipled data in return. Data fidelity, bandwidth, and coverage are the first requirements in considering methods such as this one. The data set used in the synthetic example is broadband and includes low (<1 Hz) frequencies (although not close to zero frequency — the nearly constant Q model we are using in fact diverges at and near that limit). The requirement for this kind of data is typical of nonlinear, wave-theoretic inverse methods. The best outcome will result from actual acquisition of maximally low-frequency data, of course; however, various assumptions (for instance that of a piecewise-constant overburden) additionally can be made, removing the sensitivity to cutoff of low frequencies.

Two other issues are at the forefront when it comes to contemplating field data application of an algorithm of this kind. The first has to do with the way in which the data are interrogated for information in constructing the operator, which is closely related to linear inversion. Briefly, it is the frequency- and angle dependence of the transmission-altered reflection coefficients of the primaries (as we have stated above, loosely a brand of AVO/AVA behavior specific to absorptive-dispersive media) that drives the construction of the operator. That this behavior exists is a straightforward prediction of wave theory. However, it may appear as subtle variations in field data. Detecting it is critical to the procedure.

The second is a consequence of the algorithm's interest in amplitude variations in the data. As it stands, the algorithm considers data to be due to a layered, two-parameter (anacoustic), absorptive-dispersive medium. When that is true, as in our synthetic examples, the results are of high quality. When that is (at best) only approximately true, as in a seismic field data application, the results will presumably suffer. The basic framework and arguments underlying this candidate direct nonlinear Q-compensation procedure have been purposefully chosen never to *fundamentally* restrict the results to either anacoustic or layered (1D) media. However, some specific aspects of the procedure, for instance the availability of closed forms, are a consequence of these simplifications. Two clear next steps are to alter the construction of the corrective operator to be in accordance with a more suitable anelastic and heterogeneous medium model, and to reformulate the algorithm to allow attenuating reference media, although we anticipate a greater degree of algorithm complexity as a result.

ACKNOWLEDGMENTS

The sponsors and personnel of M-OSRP and CREWES are thanked for their support. K. Innanen was also supported by U. S. D.O.E. Grant No. DOE-De-FG02-05ER15697.

APPENDIX A

PRIMARIES IN A LAYERED ABSORPTIVE MEDIUM

In this appendix, we take the scattering quantities defined in the body of the paper, and use them to construct the first three terms in the absorptive-dispersive Born series. Arguments from relative scattering geometry are used to extract a subset of terms from this series, which are judged to construct only the absorptive-dispersive primaries. Patterns in these terms are used to deduce a full nonlinear absorptive-dispersive primary approximation.

We proceed assuming a nonabsorptive reference medium. For waves at oblique incidence (i.e., a nonzero angle θ) above a layered absorptive medium, with reflected waves detected at a lateral receiver location x_g, the first-order term of the Born series is

$$\begin{aligned}\psi_1(x_g,z_g,k_s,z_s,\omega) &= \int dx' \int dz' G_0(x_g,z_g,x',z',\omega) \\ &\quad \times k^2 \gamma(z') G_0(x',z',k_s,z_s,\omega) \\ &= -\frac{1}{4}\frac{k^2}{q_s^2} e^{-iq_s(z_g+z_s)} \\ &\quad \times e^{ik_s x_g} \int dz' e^{i2q_s z'} \gamma(z'), \qquad (\text{A-1})\end{aligned}$$

where $\gamma(z) = \alpha(z) - 2F(\omega)\beta(z)$, and $k = \omega/c_o$. This term constructs only primaries, and as such is kept in full as the first-order term in the primary series also. For convenience, we set $x_g = z_g = z_s = 0$ and rename the linear term ψ_{1P}:

$$\psi_{1P}(k_s,\omega) = -\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}\gamma(z'). \quad \text{(A-2)}$$

The second-order term of the Born series also is needed in its entirety in the primary approximation. We have, again with $x_g = z_g = z_s = 0$,

$$\psi_{2P}(k_s,\omega) = -\frac{(-i2q_s)}{16\cos^4\theta}\int dz' e^{iq_s z'}\gamma(z') \times \int dz'' e^{iq_s|z'-z''|}\gamma(z'')e^{iq_s z''} = -\frac{1}{8\cos^4\theta}(-i2q_s)\int dz' e^{i2q_s z'}\gamma(z') \times \left(\int_0^{z'} dz''\gamma(z'')\right). \quad \text{(A-3)}$$

At third order, we begin with the full Born series term

$$\psi_3(k_s,\omega) = -\frac{(-i2q_s)^2}{64\cos^6\theta}\int dz' e^{iq_s z'}\gamma(z') \times \int dz'' e^{iq_s|z'-z''|}\gamma(z'') \times \int dz''' e^{iq_s|z''-z'''|}\gamma(z''')e^{iq_s z'''}, \quad \text{(A-4)}$$

but reject the component for which the "middle" scattering location z'' is shallower than both z' and z''', which begins the construction of multiples (Weglein et al., 1997). This means rejecting one of the four components of equation A-4 that arise when the absolute value bars are evaluated case-wise. Retaining the other three components, we have

$$\psi_{3P}(k_s,\omega) = -\frac{(-i2q_s)^2}{64\cos^6\theta}\int dz' e^{i2q_s z'}\gamma(z') \times \int_0^{z'} dz''\gamma(z'')\int_0^{z''} dz'''\gamma(z''') = -\frac{(-i2q_s)^2}{32\cos^6\theta}\int dz' e^{i2q_s z'}\gamma(z')\left(\int_0^{z'} dz''\gamma(z'')\right)^2, \quad \text{(A-5)}$$

where again for convenience, $x_g = z_g = z_s = 0$ (for cases involving nonzero source and receiver depths, or several x_g values, the simple exponential factors outside the integrals may be easily reinstated). The pattern visible from orders one to three persists at higher order. Collecting all terms that fit the same pattern creates an approximation of primaries appropriate for large, extended absorptive-dispersive perturbations. The approximation is a straightforward extension of the scalar (acoustic) approximation discussed by Innanen (2008). Calling the approximation ψ_P, we have

$$\psi_P(k_s,\omega) = \sum_{n=0}^{\infty}\psi_{nP}(k_s,\omega) = -\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}\gamma(z') \times \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{iq_s}{\cos^2\theta}\int_0^{z'} dz''\gamma(z'')\right)^n. \quad \text{(A-6)}$$

This may be summed to closed form, as was done in direct nonlinear imaging by Shaw et al. (2004):

$$\psi_P(k_s,\omega) = -\frac{1}{4\cos^2\theta}\int dz' \times e^{i2q_s\left[z' - (1/2)\cos^{-2}\theta\int_0^{z'} dz''\gamma(z'')\right]}\gamma(z'). \quad \text{(A-7)}$$

In this paper, the summed form is of less significance, because our aim will be to perform an order-by-order inversion. As the key result of this appendix, then, we have the series in equation A-6, expressed explicitly in terms of the wavespeed and Q perturbations α and β:

$$\psi_P(k_s,\omega) = -\int dz' \frac{e^{i2q_s z'}}{4\cos^2\theta}[\alpha(z') - 2F(\omega)\beta(z')] \times \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-iq_s}{\cos^2\theta}\int_0^{z'} dz''[\alpha(z'') - 2F(\omega)\beta(z'')]\right)^n. \quad \text{(A-8)}$$

APPENDIX B

DIRECT NONLINEAR ABSORPTIVE INVERSION

In this appendix, we perform a direct, order-by-order inversion of the absorptive-dispersive primary approximation derived in Appendix A. We assume that the data (1) contain only primaries, (2) have been deconvolved of the source wavelet, and (3) have been deghosted. These assumptions are typical for direct nonlinear primary algorithms based on the inverse scattering series (Weglein et al., 2003). If this is the case, and if the perturbations α and β are of such a size and extent that equation A-8 is accurate, we may write

$$D(k_s,\omega) = -\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}[\alpha(z') - 2F(\omega)\beta(z')] - \frac{(-i2q_s)}{8\cos^4\theta}\int dz' e^{i2q_s z'}[\alpha(z') - 2F(\omega)\beta(z')] \times \left(\int_0^{z'} dz''[\alpha(z'') - 2F(\omega)\beta(z'')]\right)$$

$$- \frac{(-i2q_s)^2}{32\cos^6\theta}\int dz' e^{i2q_s z'}[\alpha(z') - 2F(\omega)\beta(z')] \times\left(\int_0^{z'} dz''[\alpha(z'') - 2F(\omega)\beta(z'')]\right)^2 + \cdots, \tag{B-1}$$

where θ and q_s are particular arrangements of experimental variables k_s and ω:

$$\theta = \sin^{-1}\frac{k_s c_0}{\omega}, \quad q_s = \frac{\omega}{c_0}\sqrt{1 - \frac{k_s^2 c_0^2}{\omega^2}}. \tag{B-2}$$

The aim in the remainder of this appendix will be to directly invert equation B-1.

An inverse series for absorptive primaries

We form an inverse series for the perturbations α and β, in which the nth term is defined to be nth order in the *primary* data modeled in equation A-8. Let this series be

$$[\alpha(z) - 2F(\omega)\beta(z)] = [\alpha_1(z) - 2F(\omega)\beta_1(z)] + [\alpha_2(z) - 2F(\omega)\beta_2(z)] + \cdots. \tag{B-3}$$

This is substituted into equation B-1, and like orders are equated (Innanen, 2008) in a manner similar to Carvalho's derivation of the full inverse scattering series (Carvalho, 1992). At first order, we have

$$D(k_s,\omega) = -\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}[\alpha_1(z') - 2F(\omega)\beta_1(z')]. \tag{B-4}$$

At second order, we have

$$\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}[\alpha_2(z') - 2F(\omega)\beta_2(z')] = -\frac{(-i2q_s)}{8\cos^4\theta}\int dz' e^{i2q_s z'}[\alpha_1(z') - 2F(\omega)\beta_1(z')] \times\left(\int_0^{z'} dz''[\alpha_1(z'') - 2F(\omega)\beta_1(z'')]\right). \tag{B-5}$$

At third order, we have

$$\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s z'}[\alpha_3(z') - 2F(\omega)\beta_3(z')] = -\frac{(-i2q_s)}{8\cos^4\theta}\int dz' e^{i2q_s z'}[\alpha_1(z') - 2F(\omega)\beta_1(z')] \times\left(\int_0^{z'} dz''[\alpha_2(z'') - 2F(\omega)\beta_2(z'')]\right) - \frac{(-i2q_s)}{8\cos^4\theta}\int dz' e^{i2q_s z'}[\alpha_2(z') - 2F(\omega)\beta_2(z')] \times\left(\int_0^{z'} dz''[\alpha_1(z'') - 2F(\omega)\beta_1(z'')]\right) - \frac{(-i2q_s)^2}{32\cos^6\theta}\int dz' e^{i2q_s z'}[\alpha_1(z') - 2F(\omega)\beta_1(z')] \times\left(\int_0^{z'} dz''[\alpha_1(z'') - 2F(\omega)\beta_1(z'')]\right)^2. \tag{B-6}$$

This continues. Just as in the full inverse scattering series, the sequential direct solution for perturbation components at each order, followed by their summation, produces the desired solution. Our approach will be to carry out the inversion explicitly on the first three orders only, thereafter deducing a pattern that holds over all orders.

First order

The construction of the first-order components of the absorptive-dispersive perturbations α_1 and β_1 from the data (i.e., the solution of equation B-4), and the resulting issues of conditioning, detectability, and relationships with the actual medium perturbations, have been described in detail by Innanen and Weglein (2007), and will not be reviewed extensively here. Briefly put, two profiles $\alpha_1(z|\vartheta,W)$ and $\beta_1(z|\vartheta,W)$, over layered absorptive media, may be constructed given a single shot record or receiver record of reflected primary data and the acoustic reference wavespeed c_0, which is assumed to agree with the actual medium at and above the sources and receivers. Because two or more plane-wave incidence angles are required to separately construct the profiles, but many varied sets of these angles may do so, we define the quantity $\vartheta = \{\theta_1,\theta_2,\ldots\}$ to represent the particular set of angles used. In addition, because the freedom also exists to weight the data at each angle, we define W to represent the particular weighting scheme (if any) chosen. The profiles are then functions of these quantities also. Summarizing, we establish a mapping between $D(k_s,\omega)$ and $\alpha_1(z|\vartheta,W)$, $\beta_1(z|\vartheta,W)$. The mapping is simple, generally a linear combination of Fourier components of the data.

Second order

The second-order term in equation B-5 is close to a form suitable for the direct nonlinear inverse equations. Because the relationships in equations B-4–B-6 hold for all k_s and ω, by comparing integrands in equation B-5, we see that instances of $\alpha_2 - 2F\beta_2$ occurring under Fourier integrals may be replaced by

$$\alpha_2(z) - 2F(\omega)\beta_2(z) = -\frac{(-i2q_s)}{2\cos^2\theta}[\alpha_1(z) - 2F(\omega)\beta_1(z)] \times \left(\int_0^z dz'[\alpha_1(z') - 2F(\omega)\beta_1(z')] \right). \quad \text{(B-7)}$$

This will be useful for manipulations at third order. We further change variables to θ and $k_z = -2q_s$:

$$\int dz' e^{-ik_z z'}[\alpha_2(z'|\vartheta,W) - 2F(k_z,\theta)\beta_2(z'|\vartheta,W)] = \frac{-ik_z}{2\cos^2\theta}\int dz' e^{-ik_z z'}[\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)] \times \left(\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(k_z,\theta)\beta_1(z''|\vartheta,W)] \right), \quad \text{(B-8)}$$

where we have employed the specific forms for α_1 and β_1 derived above, including the set of angles ϑ and weights W. Because the first-order input to the second-order term has these dependences, so also must the second-order perturbations $\alpha_2 = \alpha_2(z|\vartheta,W)$ and $\beta_2 = \beta_2(z|\vartheta,W)$.

Third order

The third-order problem requires a greater level of manipulation. Using equation B-7 and the relationship

$$\int_{-\infty}^{z} f(z') \int_{-\infty}^{z'} f(z'')dz''dz' = \frac{1}{2}\left(\int_{-\infty}^{z} f(z')dz' \right)^2, \quad \text{(B-9)}$$

the first two terms on the right-hand side of equation B-6 are seen to be of the same form as the third, albeit with different constant factors. After adding these three terms together, equation B-6 becomes

$$\frac{1}{4\cos^2\theta}\int dz' e^{i2q_s(k_s,\omega)z'}[\alpha_3(z') - 2F(\omega)\beta_3(z')] = \frac{(-i2q_s)^2}{16\cos^6\theta}\int dz' e^{i2q_s z'}[\alpha_1(z') - 2F(\omega)\beta_1(z')] \times \left(\int_0^{z'} dz''[\alpha_1(z'') - 2F(\omega)\beta_1(z'')] \right)^2. \quad \text{(B-10)}$$

Simplifying, and changing variables to k_z and θ, we have

$$\int dz' e^{-ik_z z'}[\alpha_3(z'|\vartheta,W) - 2F(k_z,\theta)\beta_3(z'|\vartheta,W)] = \frac{(-ik_z)^2}{4\cos^4\theta}\int dz' e^{-ik_z z'}[\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)] \left(\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(k_z,\theta)\beta_1(z''|\vartheta,W)] \right)^2. \quad \text{(B-11)}$$

Again, because at first and second orders the outputs are functions of the set of angles and weights used in the first-order procedure, so must the third-order terms, i.e., $\alpha_3 = \alpha_3(z'|\vartheta,W)$ and $\beta_3 = \beta_3(z'|\vartheta,W)$.

Direct nonlinear absorptive inversion equations

A pattern is discernible in equations B-8 and B-11, whose form, like in the forward case, persists at higher order. In fact, α_{n+1} and β_{n+1} are related to α_1 and β_1 via

$$\int dz' e^{-ik_z z'}[\alpha_{n+1}(z'|\vartheta,W) - 2F(k_z,\theta)\beta_{n+1}(z'|\vartheta,W)] = \frac{1}{n!}\left(\frac{-ik_z}{2\cos^2\theta}\right)^n \int dz' e^{-ik_z z'}[\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)] \left(\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(k_z,\theta)\beta_1(z''|\vartheta,W)] \right)^n. \quad \text{(B-12)}$$

Defining

$$\alpha_P(z|\vartheta,W) \equiv \sum_{n=0}^{\infty} \alpha_{n+1}(z'|\vartheta,W),$$

$$\beta_P(z|\vartheta,W) \equiv \sum_{n=0}^{\infty} \beta_{n+1}(z'|\vartheta,W), \quad \text{(B-13)}$$

creating an instance of equation B-12 for every value of $n \geq 0$, and summing, we obtain

$$\alpha_P(k_z,\theta|\vartheta,W) - 2F(k_z,\theta)\beta_P(k_z,\theta|\vartheta,W) = \int dz' e^{-ik_z\left[z' + \frac{1}{2\cos^2\theta}\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(k_z,\theta)\beta_1(z''|\vartheta,W)]\right]} \times [\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)], \quad \text{(B-14)}$$

having recognized the integral on the left-hand side as a Fourier transform.

APPENDIX C

ABSORPTIVE MODEL CONSTRUCTION VIA NONLINEAR DIRECT INVERSION

The aims of inverse scattering procedures vary from the construction of spatial distributions of perturbation quantities to the construction of processed data sets. The direct nonlinear primary inversion quantities derived in Appendix B lend themselves to either goal. In this appendix, we will address the former of these aims for layered, two-parameter absorptive-dispersive media.

We begin by rewriting equation B-14

$$\alpha_P(k_z|\vartheta,W) - 2F(k_z,\theta)\beta_P(k_z|\vartheta,W) = \Delta(k_z,\theta|\vartheta,W), \tag{C-1}$$

where we define

$$\Delta(k_z,\theta|\vartheta,W) \equiv \int dz' e^{-ik_z\left[z' + \frac{1}{2\cos^2\theta}\int_0^{z'} dz''[\alpha_1(z''|\vartheta,W) - 2F(k_z,\theta)\beta_1(z''|\vartheta,W)]\right]} \times [\alpha_1(z'|\vartheta,W) - 2F(k_z,\theta)\beta_1(z'|\vartheta,W)]. \tag{C-2}$$

We wish to separately calculate α_P and β_P at each relevant depth wavenumber k_z; given at least two angles per depth wavenumber k_z, and if desired, an additional weighting scheme, this is an overdetermined problem. Some notational care will be required, because as we see in equations C-1 and C-2 earlier sets of angles and weights, $\vartheta = \{\theta_1, \theta_2, \ldots\}$ and W are already in play.

We proceed by defining a new set of angles $\tilde{\vartheta} = \{\tilde{\theta}_1, \tilde{\theta}_2, \ldots\} \neq \vartheta$ and weights $\tilde{W} \neq W$. There is no formal requirement that $\tilde{\vartheta}$ and $\tilde{W}$ be related to ϑ and W, although intuition and common sense might lead us in one direction or another. During the numerical application of the direct Q-compensation algorithms in this paper, we have argued towards a relationship for that specific situation, but here we will leave them distinct and unrelated. The solutions in this appendix therefore are considered to be functions of both.

Given the $N > 2$ angles $\tilde{\vartheta}$, the N resulting instances of equation C-1 can be written in matrix form:

$$\mathbf{F}(k_z,\tilde{\vartheta})\begin{bmatrix}\alpha_P(k_z|\vartheta,W)\\ \beta_P(k_z|\vartheta,W)\end{bmatrix} = \mathbf{\Delta}(k_z,\tilde{\vartheta}|\vartheta,W), \tag{C-3}$$

where

$$\mathbf{F}(k_z,\tilde{\vartheta}) = \begin{bmatrix} 1 & -2F(k_z,\tilde{\theta}_1)\\ 1 & -2F(k_z,\tilde{\theta}_2)\\ \vdots & \vdots \\ 1 & -2F(k_z,\tilde{\theta}_N)\end{bmatrix} \tag{C-4}$$

and

$$\mathbf{\Delta}(k_z,\tilde{\vartheta}|\vartheta,W) = \begin{bmatrix}\Delta(k_z,\tilde{\theta}_1|\vartheta,W)\\ \Delta(k_z,\tilde{\theta}_2|\vartheta,W)\\ \vdots \\ \Delta(k_z,\tilde{\theta}_N|\vartheta,W)\end{bmatrix}. \tag{C-5}$$

A new weighting scheme $\tilde{W}$ can be brought in via whatever choices are made in inverting $\mathbf{F}(k_z,\tilde{\vartheta})$. That is, $\mathbf{F}^{-1} = \mathbf{F}^{-1}(k_z,\tilde{\vartheta},\tilde{W})$. Now this means that the outputs α_P and β_P are dependent on k_z, but also on (1) the weights and angles, ϑ and W, used to create the linear output and on (2) the weights and angles, $\tilde{\vartheta}$ and $\tilde{W}$, used above to create the nonlinear output. That is,

$$\begin{bmatrix}\alpha_P(k_z|\tilde{\vartheta},\tilde{W},\vartheta,W)\\ \beta_P(k_z|\tilde{\vartheta},\tilde{W},\vartheta,W)\end{bmatrix} = \mathbf{F}^{-1}(k_z,\tilde{\vartheta},\tilde{W})\mathbf{\Delta}(k_z,\tilde{\vartheta}|\vartheta,W). \tag{C-6}$$

Finally, profiles may be generated through inverse Fourier transforms:

$$\alpha_P(z|\tilde{\vartheta},\tilde{W},\vartheta,W) = \frac{1}{2\pi}\int dk_z e^{ik_z z}\alpha_P(k_z|\tilde{\vartheta},\tilde{W},\vartheta,W),$$
$$\beta_P(z|\tilde{\vartheta},\tilde{W},\vartheta,W) = \frac{1}{2\pi}\int dk_z e^{ik_z z}\beta_P(k_z|\tilde{\vartheta},\tilde{W},\vartheta,W). \tag{C-7}$$

The freedom to *twice* choose both the subsets of the data we use and their weights, during the calculation of the profiles in equation C-7, suggests a large range of types of inverse result is possible.

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