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# Wavelet estimation for a multidimensional acoustic or elastic earth

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## ABSTRACT

A new and general wave theoretical wavelet estimation method is derived. Knowing the seismic wavelet is important both for processing seismic data and for modeling the seismic response. To obtain the wavelet, both statistical (e.g., Wiener-Levinson) and deterministic (matching surface seismic to well-log data) methods are generally used. In the marine case, a far-field signature is often obtained with a deep-towed hydrophone. The statistical methods do not allow obtaining the phase of the wavelet, whereas the deterministic method obviously requires data from a well. The deep-towed hydrophone requires that the water be deep enough for the hydrophone to be in the far field and in addition that the reflections from the water bottom and structure do not corrupt the measured wavelet. None of the methods address the source array pattern, which is important for amplitude-versus-offset (AVO) studies.

This paper presents a method of calculating the total wavelet, including the phase and source-array pattern. When the source locations are specified, the method predicts the source spectrum. When the source is completely unknown (discrete and/or continuously distributed) the method predicts the wavefield due to this source. The method is in principle exact and yet *no* information about the properties of the earth is required. In addition, the theory allows either an acoustic wavelet (marine) or an elastic wavelet (land), so the wavelet is consistent with the earth model to be used in processing the data. To accomplish this, the method requires a new data collection procedure. It requires that the field and its normal derivative be measured on a surface. The procedure allows the multidimensional earth properties to be arbitrary and acts like a filter to eliminate the scattered energy from the wavelet calculation. The elastic wavelet estimation theory applied in this method may allow a true land wavelet to be obtained. Along with the derivation of the procedure, we present analytic and synthetic examples.


## INTRODUCTION

In seismic exploration a man-made source of energy produces a wave which propagates into the subsurface. The reflection data recorded on the surface depend on (1) the properties of the earth's structure (2) the energy source and recording system.

The purpose of seismic exploration is to extract information about the subsurface from these data. Consequently, it is important to attempt to identify and remove the effects of the source characteristics from this reflected energy. We suggest that a wavelet estimation method be applied which is theoretically consistent with the process to be applied to the reflection data. That is, for acoustic processing (e.g., acoustic migration) an acoustic wavelet would be appropriate, whereas an elastic wavelet would be appropriate for elastic wave-equation data processing.

It has been shown [Loveridge et al. (1984)] that for amplitude-versus-offset (AVO) studies, the source array pattern can be important. We show a method of obtaining this array pattern.

The recently published one-dimensional (1-D) acoustic wavelet estimation method of Loewenthal et al. (1985) assumes (1) that the medium above the receivers is known and (2) that both the field and the normal derivative are measured. Our method makes analogous assumptions for multidimensional acoustic and elastic media. In a related

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paper, Sonneland et al. (1986) use the 1-D acoustic wave equation with two independent vertical measurements of the pressure to perform a combined designature-dereverberation. Hargreaves (1984) has presented a wave-field extrapolation method for source signature identification. Although his method produced favorable results in some cases, e.g., deep water, he states a problem resulting from corruption due to the scattered field. The method in this paper directly addresses this problem, automatically filtering the scattered field from the wavelet calculation.

Our purpose is to present a new general method of source signature identification which also requires two separate field measurements. It is applicable for an arbitrary inhomogeneous multidimensional acoustic or elastic earth. Furthermore, no information about the properties of the subsurface is required. Each shot record produces an effective acoustic or elastic wavelet for that particular shot record. The effective wavelet can vary from one shot record to the next. The wavelet obtained is the source wavelet or driving function being imparted to the medium. In the second section, we present the general wavelet estimation procedure and show that the method relies only on the incident wave, effectively filtering the scattered energy from the integral. We illustrate this procedure with an analytic example. The marine exploration environment is treated next. We show an application to a distributed source and the source array pattern, followed by an application of the method for a multidimensional elastic problem. Finally, we give numerical examples which address the issues of time and spatial sampling, finite aperture, approximations to the field derivative, and finding the source array pattern.

The procedure (for either the acoustic or elastic model) derives and from a comparison of the Lippmann-Schwinger equation and Green's theorem. The former originates in scattering theory and the latter in boundary-value problems. These two equations were compared, for a different purpose, in Weglein and Silvia (1981) and Silvia and Weglein (1981).

## METHOD FOR ACOUSTIC WAVELET

Consider a point source at the spatial position $\mathbf{r}_s$. The constant-density acoustic wave equation for the pressure field $P$ due to a source $A(t)$ at $\mathbf{r}_s$ is

$$\left(\nabla^2 - \frac{1}{c^2(\mathbf{r})}\frac{\partial^2}{\partial t^2}\right) P(\mathbf{r}, \mathbf{r}_s, t) = A(t)\delta(\mathbf{r} - \mathbf{r}_s).$$

Taking the Fourier transform in time gives

$$\left(\nabla^2 + \frac{\omega^2}{c^2(\mathbf{r})}\right)\tilde{P}(\mathbf{r}, \mathbf{r}_s, \omega) = \tilde{A}(\omega)\delta(\mathbf{r} - \mathbf{r}_s). \tag{1}$$

In this context the wavelet estimation problem is to determine $\tilde{A}(\omega)$. Characterize the velocity configuration $c(\mathbf{r})$ in terms of a reference value $c_0$ and a variation in the index of refraction $\alpha$:

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2}[1 - \alpha(\mathbf{r})].$$

Equation (1) can then be rewritten as

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right)\tilde{P} = \frac{\omega^2}{c_0^2}\alpha\tilde{P} + \tilde{A}(\omega)\delta(\mathbf{r} - \mathbf{r}_s). \tag{2}$$

We now proceed to derive two integral equations for $\tilde{P}$. One is the Lippmann-Schwinger integral equation and the other is derived from Green's second identity. These two equations will lead to the desired relationship between the sought after $\tilde{A}(\omega)$ and the measurements of $\tilde{P}$ on the surface $z = 0$.

Formally inverting the operator $\nabla^2 + \omega^2/c_0^2 = L_0 = G_0^{-1}$ in equation (2), we have the operator relationship

$$\tilde{P} = G_0 \frac{\omega^2}{c_0^2}\alpha\tilde{P} + \tilde{A}G_0$$

or, equivalently,

$$\tilde{P}(\mathbf{r}, \mathbf{r}_s, \omega) = \tilde{A}(\omega)G_0(\mathbf{r}, \mathbf{r}_s, \omega) + \int_\infty G_0(\mathbf{r}, \mathbf{r}', \omega)\frac{\omega^2}{c_0^2}\alpha(\mathbf{r}')\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)\, dr', \tag{3}$$

where

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right)G_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}')$$

with outgoing wave-boundary conditions.

Equation (3) is the Lippmann-Schwinger integral equation and is valid for all $\mathbf{r}$.

A second integral equation for $\tilde{P}$ is derived from Green's second identity:

$$\int_V (A\nabla^2 B - B\nabla^2 A)\, d^3\mathbf{r} = \int_S (A\nabla B - B\nabla A) \cdot \mathbf{n}\, ds,$$

where $S$ is the surface which encloses the volume $V$. Let $A = \tilde{P}$ and $B = G_0$ in Green's second identity:

$$\begin{aligned}\int_V [\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)\nabla'^2 G_0(\mathbf{r}', \mathbf{r}, \omega) - G_0(\mathbf{r}', \mathbf{r}, \omega)\nabla'^2\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)]\, dr' \\ = \int_S [\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)\nabla' G_0(\mathbf{r}', \mathbf{r}, \omega) - G_0(\mathbf{r}', \mathbf{r}, \omega)\nabla'\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)] \cdot \mathbf{n}\, ds,\end{aligned} \tag{4}$$

where $S$ is the surface illustrated in Figure 1. $\mathbf{r}_s$ is located on or above the $z = 0$ boundary of the surface $S$. We choose the convention for the delta function to be

$$\int_V \delta(\mathbf{r} - \mathbf{a})f(\mathbf{r})\, d\mathbf{r} = f(\mathbf{a}),$$

where $\mathbf{a}$ is strictly within the volume. For $\mathbf{a}$ outside the volume, or on the boundary, this integral vanishes.

Substituting the equations

$$\nabla'^2 \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) = -\frac{\omega^2}{c_0^2} \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) + \frac{\omega^2}{c_0^2} \alpha(\mathbf{r}') \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) + \tilde{A}(\omega)\delta(\mathbf{r}' - \mathbf{r}_s)$$

and

$$\nabla'^2 G_0(\mathbf{r}', \mathbf{r}, \omega) = -\frac{\omega^2}{c_0^2} G_0(\mathbf{r}', \mathbf{r}, \omega) + \delta(\mathbf{r}' - \mathbf{r})$$

into equation (4) and taking $\mathbf{r}$ within the volume, we obtain

$$\tilde{P}(\mathbf{r}, \mathbf{r}_s, \omega) = \int_V G_0(\mathbf{r}, \mathbf{r}', \omega) \frac{\omega^2}{c_0^2} \alpha(\mathbf{r}') \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)\, dr' + \int_S [\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) \nabla' G_0(\mathbf{r}', \mathbf{r}, \omega) - G_0(\mathbf{r}', \mathbf{r}, \omega) \nabla' \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)] \cdot \mathbf{n}\, ds. \quad (5)$$

Thus, for $\mathbf{r}$ within the volume we have two expressions for $\tilde{P}$. These two integral equations [equations (3) and (5)] were studied in by Weglein and Silvia (1981).

If the support of $\alpha$ is within the volume $V$, then

$$\int_V G_0(\mathbf{r}, \mathbf{r}', \omega) \frac{\omega^2}{c_0^2} \alpha(\mathbf{r}') \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)\, dr' = \int_\infty G_0(\mathbf{r}, \mathbf{r}', \omega) \frac{\omega^2}{c_0^2} \alpha(\mathbf{r}') \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)\, dr'.$$

For points $\mathbf{r}$ within the volume and the support of $\alpha$ within the volume $V$, equations (3) and (5) give two expressions for the field in the reference medium. From these we obtain

$$\tilde{A}(\omega) G_0(\mathbf{r}, \mathbf{r}_s, \omega) = \int_S [\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) \nabla' G_0(\mathbf{r}', \mathbf{r}, \omega) - G_0(\mathbf{r}', \mathbf{r}, \omega) \nabla' \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)] \cdot \mathbf{n}\, ds$$

or

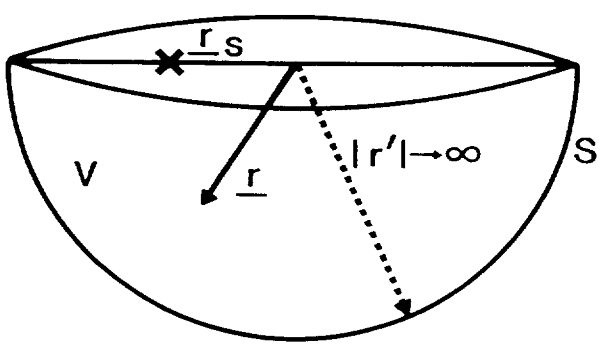


FIG. 1. Subsurface volume $V$.

$$\tilde{A}(\omega) = \frac{1}{G_0(\mathbf{r}, \mathbf{r}_s, \omega)} \cdot \int_S [\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) \nabla' G_0(\mathbf{r}', \mathbf{r}, \omega) - G_0(\mathbf{r}', \mathbf{r}, \omega) \nabla' \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)] \cdot \mathbf{n}\, ds. \quad (6)$$

The upper boundary of the volume $V$ is defined by the measurement surface $z = 0$. Therefore the support of $\alpha$ within $V$ means that the reference medium must only agree with the actual medium above the receiver plane. The numerator and denominator can be evaluated at any point inside the volume. In principle, one evaluation point will suffice to determine $\tilde{A}(\omega)$. However, in practice $N$ such evaluations of the numerator and denominator will give $N$ independent estimates of $\tilde{A}(\omega)$ for each source location. A statistical weighting of these estimates could then be used to evaluate the wavelet optimally. Consequently, in practice the method would involve a combination of deterministic and statistical procedures. We show later that this redundancy can be used to solve for the source array pattern.

The total field $\tilde{P}$ and its normal derivative $\nabla' \tilde{P} \cdot \mathbf{n}$ depend on both the wavelet and the subsurface properties. However, we have shown that

$$\int_S [\tilde{P} \nabla' G_0 - G_0 \nabla' \tilde{P}] \cdot \mathbf{n}\, ds$$

depends only on the wavelet. The major conclusion is that measurement of $\tilde{P}$ and $\nabla \tilde{P} \cdot \mathbf{n}$ on the surface of the earth is sufficient to determine the source wavelet without requiring any information about the subsurface properties. We now show that the above integral depends only on the incident field by showing that the integral for the scattered wave is zero. Thus, since

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right) G_0(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}')$$

and

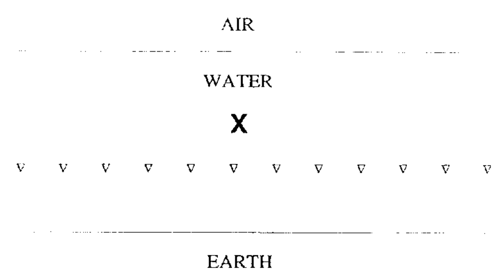


FIG. 2. The marine exploration geometry.

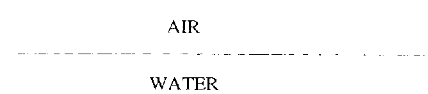


FIG. 3. Background medium for marine case.

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right)\tilde{P}_s = \frac{\omega^2}{c_0^2}\,\alpha\tilde{P},$$

we have, by substituting into Green's second identity,

$$\int_S [\tilde{P}_s \nabla' G_0 - G_0 \nabla' \tilde{P}_s] \cdot \mathbf{n}\, ds$$

$$= \tilde{P}_s - \int_V G_0 \frac{\omega^2}{c_0^2}\alpha(\mathbf{r}')\tilde{P}\, dr' = \tilde{P}_s - \tilde{P}_s = 0,$$

and thus the only contribution to the integral in equation (6) is the incident wave.

We illustrate this method with 1-D and multidimensional examples.

## WAVELET ESTIMATION

We now demonstrate the method with the analytic example of a point scatterer. Numerical examples of the method will be given later.

### One-dimensional example—Localized inhomogeneity

We illustrate this method for the 1-D model

$$\frac{1}{c^2(\mathbf{r})} = \frac{1}{c_0^2}[1 - \lambda\delta(x - x_0)],$$

i.e., $\alpha = \lambda\delta(x - x_0)$.

We evaluate equation (6) as

$$\frac{1}{G_0}\int_S [\tilde{P}\nabla' G_0 - G_0 \nabla' \tilde{P}] \cdot \mathbf{n}\, ds.$$

The Green's function $G_0$ is given by

$$G_0 = \frac{1}{2ik} e^{ik|x - x'|} \tag{7}$$

and satisfies

$$\frac{d^2 G_0}{dx^2} + k^2 G_0 = \delta(x - x'), \tag{8}$$

where $k = \omega/c_0$ and $c_0$ is the reference velocity.

For an impulsive source at $x_s$, the incident field is

$$\frac{\tilde{A}(\omega)e^{ik|x - x_s|}}{2ik}.$$

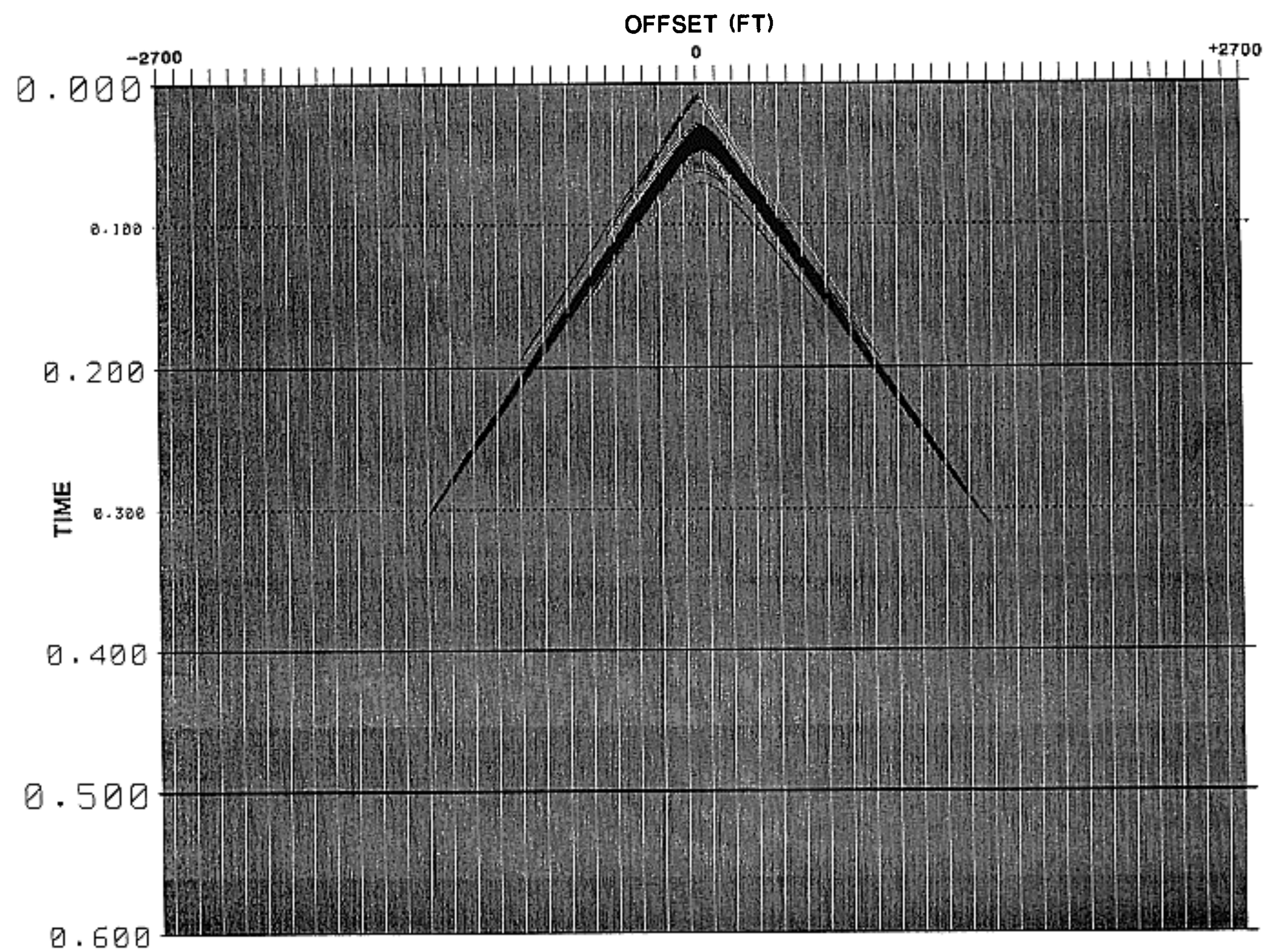


FIG. 4. Shot gather: point source, incident wave.

The total wave field $\tilde{P}$ satisfies

$$\frac{d^2\tilde{P}}{dx^2} + k^2(1-\alpha)\tilde{P} = \tilde{A}(\omega)\delta(x - x_s). \tag{9}$$

The problem is to determine the wavelet $\tilde{A}(\omega)$ from the boundary measurements of $\tilde{P}$. The Lippmann-Schwinger equation, valid for all $x$, is

$$\tilde{P} = \tilde{P}_0 + G_0 k^2 \alpha \tilde{P},$$

or (10)

$$\tilde{P}(x, x_s, \omega) = \frac{\tilde{A}(\omega)}{2ik} e^{ik|x-x_s|} + \frac{1}{2ik}\int_{-\infty}^{\infty} e^{ik|x-x'|}k^2\alpha(x')\tilde{P}(x', x_s, \omega)\, dx'.$$

For the localized scatterer, $\alpha = \lambda\delta(x - x_0)$, the total wave field is

$$\tilde{P}(x, x_s, \omega) = \frac{\tilde{A}(\omega)}{2ik} e^{ik|x-x_s|} + \frac{1}{2ik} e^{ik|x-x_0|}k^2\lambda\tilde{P}(x_0, x_s, \omega). \tag{11a}$$

To find $\tilde{P}(x_0, x_s, \omega)$, set $x = x_0$ in the last equation:

$$\tilde{P}(x_0, x_s, \omega) = \frac{\tilde{A}(\omega)}{2ik} e^{ik|x_0-x_s|} + \frac{1}{2ik} k^2\lambda\tilde{P}(x_0, x_s, \omega)$$

or

$$\tilde{P}(x_0, x_s, \omega) = \frac{\dfrac{\tilde{A}(\omega)}{2ik} e^{ik|x_0-x_s|}}{1 + \dfrac{ik\lambda}{2}},$$

and $\tilde{P}(x, x_s, \omega)$ from equation (11a) becomes

$$\tilde{P}(x, x_S, \omega) = \frac{\tilde{A}(\omega)}{2ik} e^{ik|x-x_s|} - \frac{\lambda}{4}\frac{\tilde{A}(\omega)e^{ik|x-x_0|}e^{ik|x_0-x_s|}}{1 + \dfrac{ik\lambda}{2}}. \tag{11b}$$

The total wave field $\tilde{P}(x, x_s, \omega)$ is a function of the wavelet $\tilde{A}(\omega)$ and the medium.

The second integral representation, derived from Green's second identity, is

$$\int_a^b \left(\tilde{P}\frac{d^2G_0}{dx^2} - G_0\frac{d^2\tilde{P}}{dx^2}\right)dx = \left|_a^b \left(\tilde{P}\frac{dG_0}{dx} - G_0\frac{d\tilde{P}}{dx}\right)\right., \tag{12}$$

where $a < x_0 < b$. Using the differential equations (8) and (9) in the left-hand side of equation (12) leads to

$$\int_a^b [\tilde{P}(x', x_s, \omega)\delta(x - x') - k^2\alpha(x')\tilde{P}(x', x_s, \omega)G_0(x, x', \omega) - G_0(x, x', \omega)\tilde{A}(\omega)\delta(x' - x_s)]\, dx' = \left|_a^b \left(\tilde{P}\frac{dG_0}{dx'} - G_0\frac{d\tilde{P}}{dx'}\right)\right.$$

For $a < x < b$ and $x_s < a$, this last expression is

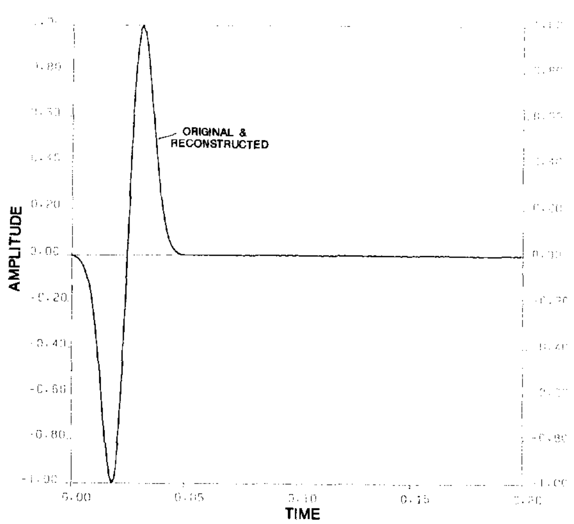


FIG. 5. Wavelet reconstruction for shot gather in Figure 4 with analytic derivative.

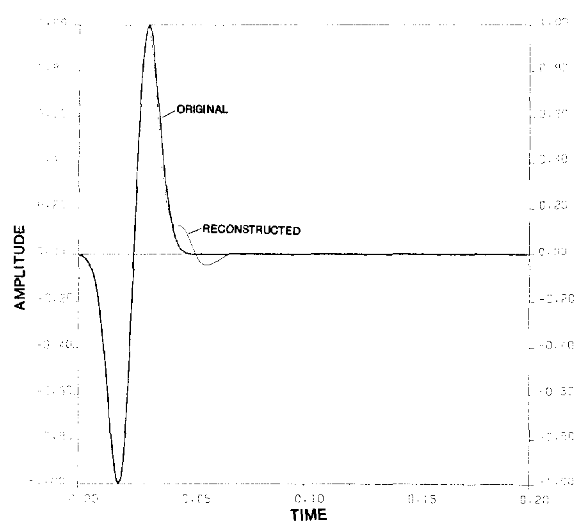


FIG. 6. Wavelet reconstruction for shot gather in Figure 4 with Eulerian approximation to derivative.

$$- k^2 \int_a^b G_0(x, x', \omega)\alpha(x')\tilde{P}(x', x_s, \omega)\, dx'$$

$$+ \tilde{P}(x, x_s, \omega)$$

$$= \Big|_a^b \left\{ \tilde{P}(x', x_s, \omega) \frac{dG_0(x, x', \omega)}{dx'} - G_0(x, x', \omega) \frac{d\tilde{P}(x', x_s, \omega)}{dx'} \right\}$$

and, solving for $\tilde{P}(x, x_s, \omega)$,

$$\tilde{P}(x, x_s, \omega) = k^2 \int_a^b G_0(x, x', \omega)\alpha(x')\tilde{P}(x', x_s, \omega)\, dx' + \Big|_a^b \left\{ \tilde{P}(x', x_s, \omega) \frac{dG_0(x, x', \omega)}{dx'} - G_0(x, x', \omega) \frac{d\tilde{P}(x', x_s, \omega)}{dx'} \right\}. \tag{13}$$

For this example, the support of $\alpha(x)$ is at $x_0$, thus

$$\int_a^b G_0(x, x', \omega)k^2\alpha(x')\tilde{P}(x', x_s, \omega)\, dx' = \int_{-\infty}^{\infty} G_0(x, x', \omega)k^2\alpha(x')\tilde{P}(x', x_s, \omega)\, dx',$$

and for $x$ within the interval $(a, b)$, it follows from equations (11a) and (13) that

$$\frac{\tilde{A}(\omega)}{2ik} e^{ik|x - x_s|} = \Big|_a^b \left\{ \tilde{P}(x', x_s, \omega) \frac{dG_0(x, x', \omega)}{dx'} - G_0(x, x', \omega) \frac{d\tilde{P}(x', x_s, \omega)}{dx'} \right\}$$

and

$$\tilde{A}(\omega) = \frac{\Big|_a^b \left\{ \tilde{P}(x', x_s, \omega) \dfrac{dG_0}{dx'} - G_0(x, x', \omega) \dfrac{d\tilde{P}(x', x_S, \omega)}{dx'} \right\}}{\dfrac{e^{ik|x - x_s|}}{2ik}}. \tag{14}$$

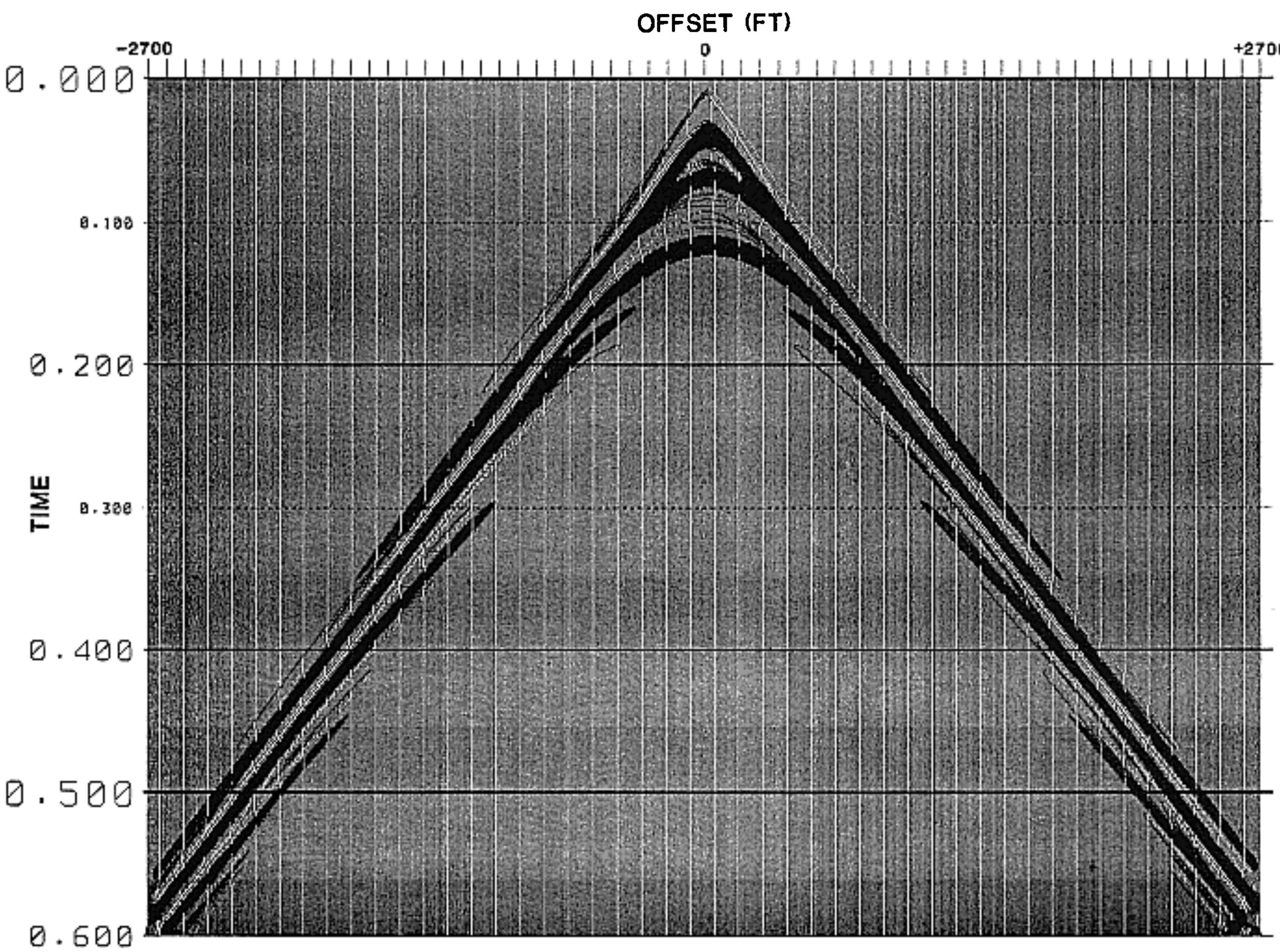


FIG. 7. Shot gather: point source, plane reflector.

This is obviously the 1-D reduction for our wavelet determination scheme, equation (6); i.e.,

$$\tilde{A}(\omega) = \frac{\int_S (\tilde{P}\nabla G_0 - G_0 \nabla \tilde{P}) \cdot \mathbf{n}\, ds}{G_0}.$$

To complete the example, we show that the right-hand side of equation (14) just reduces to $\tilde{A}(\omega)$. Thus we evaluate the right-hand side of equation (14) for the case of a localized scatterer. We need the Green's function,

$$G_0 = \frac{1}{2ik} e^{ik|x-x'|},$$

and its derivative,

$$\frac{dG_0}{dx'} = \frac{e^{ik|x-x'|}}{2} \text{SGN}(x - x').$$

From equation (11b), the wave field is

$$\tilde{P}(x, x_s, \omega) = \tilde{A}(\omega) \left\{ \frac{e^{ik|x-x_s|}}{2ik} - \frac{\lambda}{4} \frac{e^{ik|x-x_0|} e^{ik|x_0-x_s|}}{1 + \frac{i\lambda k}{2}} \right\}$$

and its derivative is

$$\frac{d\tilde{P}(x, x_s, \omega)}{dx} = \tilde{A}(\omega) \left\{ \frac{e^{ik|x-x_s|}}{2} \text{SGN}(x - x_s) - \frac{\lambda}{4} \frac{e^{ik|x-x_0|} e^{ik|x_0-x_s|}}{1 + \frac{i\lambda k}{2}} ik\, \text{SGN}(x - x_0) \right\}.$$

Removing the absolute value signs with $a < x < b$, $a < x_0 < b$, and $x_s < a$, we arrive at

$$\tilde{A}(\omega) = \frac{\tilde{A}(\omega)}{\frac{e^{ik|x-x_s|}}{2ik}} \left\{ \frac{e^{ik|x-x_s|}}{2ik} \right\} = \tilde{A}(\omega),$$

which was to be shown. Notice that although $\tilde{P}$ and $\partial\tilde{P}/\partial n$ depend on the medium through $c_0$, $\lambda$, and $x_0$, the integral

$$\int_S \left\{ \tilde{P} \frac{\partial G_0}{\partial n} - G_0 \frac{\partial \tilde{P}}{\partial n} \right\} ds$$

is only a function of the source signature.

## WAVELET ESTIMATION IN A SPATIALLY VARIANT REFERENCE MEDIUM

As stated in the Introduction, it is apparent that the method for obtaining the wavelet just described will hold for more general cases, in particular for a spatially variant reference medium. In a similar manner, an expression for the wavelet can be derived for the case where the acoustic velocity is characterized in terms of a spatially variant reference velocity. This spatially variant reference velocity will be required for the marine wavelet estimation. As before, the reference medium must agree with the actual medium above the receivers.

### Special case—Marine wavelet estimation

The marine exploration configuration is illustrated in Figure 2. The background medium for this application will consist of two homogeneous half-spaces illustrated in Figure 3.

We now evaluate equation (14) to obtain

$$\tilde{A}(\omega) = \left. \frac{\left\{ \tilde{P}(x', x_s, \omega) \frac{dG_0(x, x', \omega)}{dx'} - G_0(x, x', \omega) \frac{d\tilde{P}(x', x_s, \omega)}{dx'} \right\}}{\frac{e^{ik|x-x_s|}}{2ik}} \right|_a^b$$

and thus, upon substituting for $\tilde{P}$, $G$, and their derivatives,

$$\tilde{A}(\omega) = \frac{\tilde{A}(\omega)}{\frac{e^{ik|x-x_s|}}{2ik}} \Bigg|_a^b \left\{ \left[ \frac{e^{ik|x'-x_s|}}{2ik} - \frac{\lambda}{4} \frac{e^{ik|x'-x_0|} e^{ik|x_0-x_s|}}{1 + \frac{i\lambda k}{2}} \right] \left[ \frac{e^{ik|x-x'|}}{2} \text{SGN}(x' - x) \right] - \frac{e^{ik|x-x'|}}{2ik} \left[ \frac{e^{ik|x'-x_s|}}{2} \text{SGN}(x' - x_s) - \frac{\lambda}{4} \frac{e^{ik|x'-x_0|} e^{ikx_0 - x_s|}}{1 + \frac{i\lambda k}{2}} ik\, \text{SGN}(x' - x_0) \right] \right\}.$$

For this geometry, $G_0$ is given by the equation

$$G_0(x, z, x_s, z_s, \omega) =$$

$$\int_{-\infty}^{\infty} \frac{e^{-ik_x(x-x_s)}}{i\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}} \left\{ e^{-i\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}\,|z-z_s|} \right.$$

$$+ \frac{\frac{\rho_2}{\rho_1}\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2} - \sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2 + \omega^2\Delta}}{\frac{\rho_2}{\rho_1}\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2} + \sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2 + \omega^2\Delta}}$$

$$\left. \times e^{-i\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}\,(z+z_s)} \right\} dk_x,$$

and $dG_0/dz_s$ is given by

$$\frac{dG_0}{dz_s} = \int_{-\infty}^{\infty} \frac{e^{-ik_x(x-x_s)}}{i\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}}$$

$$\times \left\{ e^{-i\left(\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}\,|z-z_s|\right)} \right.$$

$$\times\, i\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}\ \mathrm{SGN}\,(z_s - z)$$

$$+ \frac{\frac{\rho_2}{\rho_1}\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2} - \sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2 + \omega^2\Delta}}{\frac{\rho_2}{\rho_1}\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2} + \sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2 + \omega^2\Delta}}$$

$$\times\, e^{-i\left(\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}\,(z+z_s)\right)}$$

$$\left. \times \left(-i\sqrt{\left(\frac{\omega}{c_1}\right)^2 - k_x^2}\right) \right\} dk_x,$$

where $c_1$, $c_2$ are the acoustic velocities of water and air, respectively, and $\Delta = (1/c_2^2) - (1/c_1^2)$.

## DISTRIBUTED SOURCE

If the source consists of a sequence of localized point sources at $\mathbf{r}_{s_1}, \mathbf{r}_{s_2}, \ldots, \mathbf{r}_{s_n}$, each with same time dependence, then the wavelet $\tilde{A}(\omega)$ is given by

$$\tilde{A}(\omega) = \frac{\displaystyle\int_S \left\{ \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) \frac{\partial_0(\mathbf{r}, \mathbf{r}', \omega)}{\partial n} - G_0(\mathbf{r}, \mathbf{r}', \omega) \frac{\partial \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)}{\partial n} \right\} ds'}{\displaystyle\sum_{j=1}^{n} G_0(\mathbf{r}, \mathbf{r}_{sj}, \omega)}, \tag{15}$$

where $\tilde{P}$ satisfies the differential equation

$$\nabla'^2 \tilde{P}(\mathbf{r}, \mathbf{r}_s, \omega) + \frac{\omega^2}{c_0^2}[1 - \alpha(\mathbf{r}')]\tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)$$

$$= \tilde{A}(\omega) \sum_{j=1}^{n} \delta(\mathbf{r} - \mathbf{r}_{sj})$$

and is the total field due to the $n$ distributed point sources. However, if the localized point sources are each different, we can use the fact that $\mathbf{r}$ is arbitrary to obtain the system of equations

$$\sum_{j=1}^{n} \tilde{A}_j(\omega) G_0(\mathbf{r}_i, \mathbf{r}_{sj}, \omega)$$

$$= \int_S \left\{ \tilde{P}(\mathbf{r}', \mathbf{r}_{s_1} \ldots \mathbf{r}_{s_n}, \omega) \frac{\partial G_0(\mathbf{r}_i, \mathbf{r}', \omega)}{\partial n} \right.$$

$$\left. - G_0(\mathbf{r}_i, \mathbf{r}', \omega) \frac{\partial \tilde{P}(\mathbf{r}', \mathbf{r}_{s_1} \ldots \mathbf{r}_{s_n}, \omega)}{\partial n} \right\} ds', \tag{16}$$

$$i = 1, \ldots, m.$$

This system of equations can be solved in a least-squares sense for the $\tilde{A}_j(\omega)$.

Alternatively, the radiation pattern from a single effective point source could be determined by assuming that $\tilde{A}(\omega)$ is also a function of the radius $\mathbf{r}$, i.e., $\tilde{A}(\omega)\delta(r - r_s)$ becomes the more general $\tilde{A}(\omega, r)$,

$$\int_{-\infty}^{\infty} \tilde{A}(\omega, \mathbf{r}') G_0(\mathbf{r}, \mathbf{r}', \omega)\, dr'$$

$$= \int_S \tilde{\mathbf{p}}(\mathbf{r}', \mathbf{r}_s, \omega) \nabla' G_0(\mathbf{r}', \mathbf{r}, \omega)$$

$$- G_0(\mathbf{r}', \mathbf{r}, \omega) \nabla' \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega)] \cdot \mathbf{n}\, ds, \tag{17}$$

and thus the relative strength of the source field can be obtained by evaluating equation (17) for various $\mathbf{r}$ on a constant radius about the source.

## ELASTIC WAVELET ESTIMATION METHOD

Below we outline the generalization of the wavelet estimation method for elastic waves. As in the acoustic case, a comparison of the Lippmann-Schwinger equation and Green's theorem is used to determine the wavelet.

To derive the Green's theorem for the elastic case, a rank two Green's displacement tensor $\underline{\underline{\mathbf{G}}}_0$ is defined and in addition a rank three Green's stress tensor $\underline{\underline{\underline{\Sigma}}}$ is defined. In an analogous manner to the scalar case, a Green's identity is obtained relating the displacement $\mathbf{u}(x, t)$ to a volume integral involving a body force and an integral over the enclosing surface which involves the displacement and its derivatives through the traction $\mathbf{t}$. This Green's theorem result [e.g., Pao and Varatharajulu (1976), equation (19)] for elastic waves is given by

$$\int_V \rho \mathbf{f}(\mathbf{x}') \cdot \bar{\underset{\sim}{\mathbf{G}}}_0(\mathbf{x}, \mathbf{x}')\, d\mathbf{x}' + \int_S \{\mathbf{t}(\mathbf{x}') \cdot \bar{\underset{\sim}{\mathbf{G}}}_0(\mathbf{x}, \mathbf{x}') - \mathbf{u}(\mathbf{x}') \cdot [\mathbf{n}' \cdot \bar{\bar{\underset{\sim}{\Sigma}}}(\mathbf{x}|\mathbf{x}')]\}\, ds = \begin{cases} \mathbf{u}(\mathbf{x}) & \mathbf{x} \in V \\ 0 & \mathbf{x} \notin V, \end{cases}$$

where $\mathbf{u}(x, t)$ is the displacement vector, $\mathbf{t} = \mathbf{n} \cdot \bar{\bar{\mathbf{T}}}$, where $\mathbf{t}$ is the traction on the surface, $\bar{\bar{\mathbf{T}}}$ is the stress tensor, $\mathbf{n}$ is the unit vector normal to the surface, $\mathbf{f}$ is the body force per unit mass, $\bar{\underset{\sim}{\mathbf{G}}}_0$ is the Green's displacement tensor and $\bar{\bar{\underset{\sim}{\Sigma}}}$ is the third-rank Green's stress tensor.

Let $\bar{\underset{\sim}{\mathbf{G}}}_0$ be the response in a homogeneous medium and $\mathbf{f}$ the passive sources (the scattering centers) of the scattered displacement field originating from the inhomogeneities in the medium. The physical source is outside the volume $V$. In Pao and Varatharajulu, $\mathbf{f}$ represents the body force per unit mass and is inside the volume. They were considering an active physical source inside the homogeneous medium. We

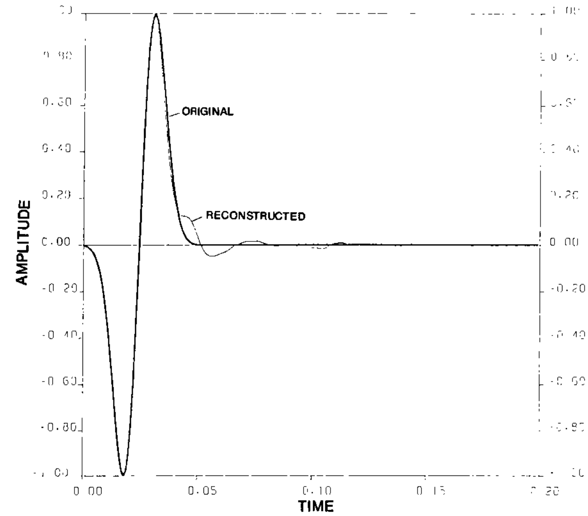


FIG. 9. Wavelet reconstruction for shot gathers in Figures 7 and 8.

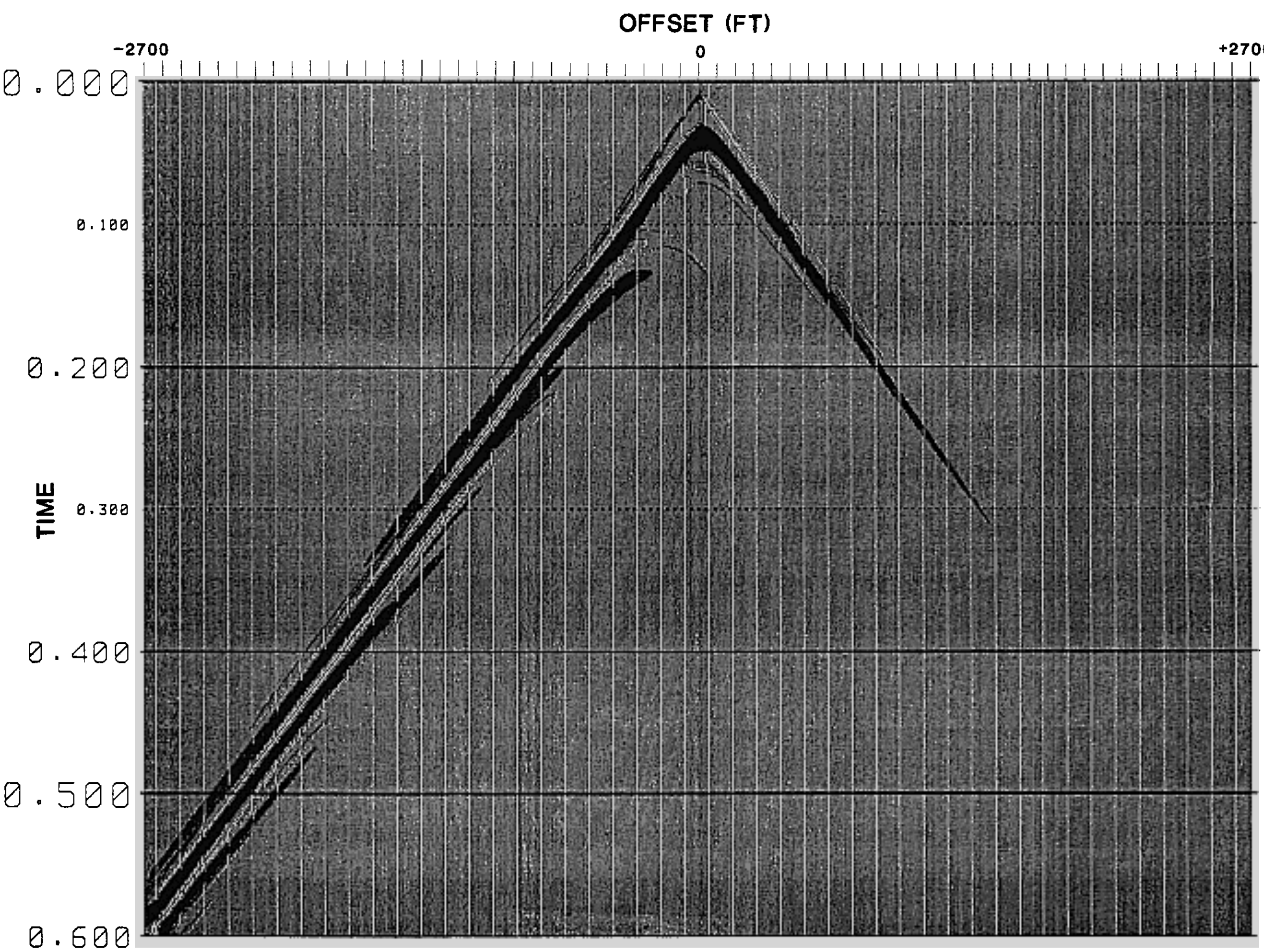


FIG. 8. Shot gather: point source, partial plane reflector.

are considering a physical source outside an inhomogeneous medium with passive scattering sources inside $V$.

Let $u_{mn}(\mathbf{x}, \mathbf{x}')$ be the displacement at $\mathbf{x}$ in the $n$-axis direction due to a source at $\mathbf{x}'$ in the $m$-axis direction. The source is $\tilde{A}_m(\omega)\delta(\mathbf{x} - \mathbf{x}')\mathbf{\Pi}$ where $\mathbf{\Pi}$ is the unit dyadic. In a manner similar to the acoustic case, the wavelet $\tilde{A}_m(\omega)$ can be determined from the Lippmann-Schwinger equation for elastic waves by dividing the surface term in equation (20) of Pao and Varatharajulu (the $mn$th component),

$$\left[\rho c_p^2 \int_S \{(\bar{\mathbf{G}}_0 \cdot \mathbf{n}')(\nabla' \cdot \mathbf{u}) - (\nabla' \cdot \bar{\mathbf{G}}_0)(\mathbf{u} \cdot \mathbf{n}')\}\, ds' - \rho c_s^2 \int_S \{\bar{\mathbf{G}}_0 \cdot (\mathbf{n}' \times \nabla' \times \mathbf{u}) + (\nabla' \times \bar{\mathbf{G}}_0) \cdot (\mathbf{n}' \times \mathbf{u})\}\, ds' \right]_n$$

by $G_{mn}$.

## NUMERICAL EXAMPLES

Numerical examples are now used to investigate the effects of time and spatial sampling, finite aperture, approximations to the derivative of the field and the source array pattern. These examples are provided by solving a 2-D acoustic forward problem with a free surface, and a point source at $\mathbf{x}_S$. The wavelet estimation equation for this case is then

$$\tilde{A}(\omega) = \frac{\displaystyle\int_S \left\{ \tilde{P}(\mathbf{r}', \mathbf{r}_s, \omega) \frac{\partial G_0(\mathbf{r}', \mathbf{r}, \omega)}{\partial n} - G_0(\mathbf{r}', \mathbf{r}, \omega) \frac{\partial \tilde{P}}{\partial n}(\mathbf{r}', \mathbf{r}_s, \omega) \right\} ds}{G_0(\mathbf{r}, \mathbf{r}_s, \omega)}.$$

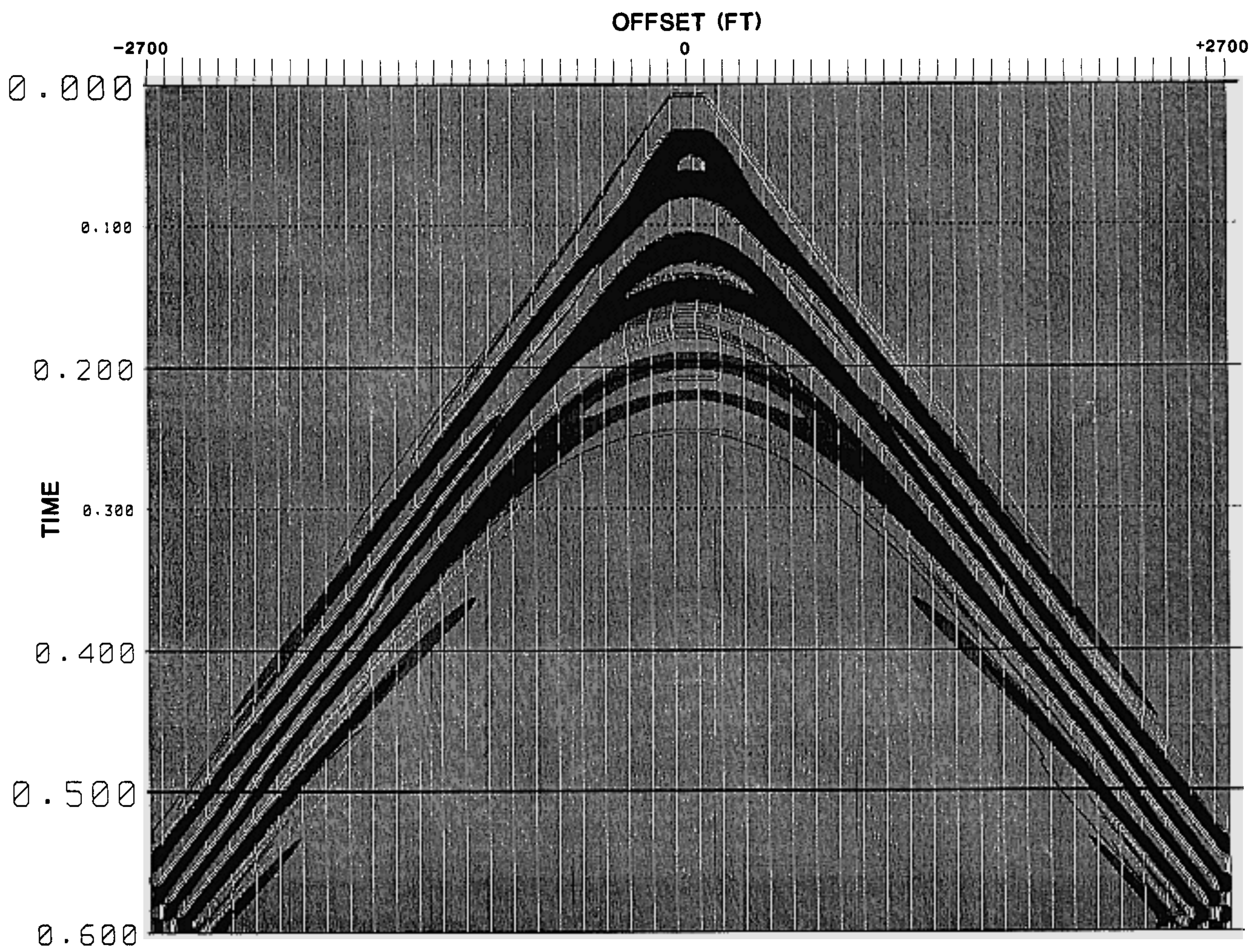


FIG. 10. Shot gather: distributed source, plane reflector.

For the 2-D case with a free surface considered, the Green's function becomes

$$G_0(\mathbf{x}', \mathbf{x}, \omega) = -\frac{i}{4} H_0^{(1)}(k_0 r^p) + \frac{i}{4} H_0^{(1)}(k_0 r^m)$$

where

$$r^p = \sqrt{(x - x')^2 + (z - z')^2}$$

and

$$r^m = \sqrt{(x - x')^2 + (z + z')^2}.$$

The normal derivative of the Green's function becomes

$$\frac{\partial G_0}{\partial n} = -\frac{\partial G_0}{\partial z'} = \frac{ik_0(z - z')}{4r^p} H_1^{(1)}(k_0 r^p) + \frac{ik_0(z + z')}{4r^m} H_1^{(1)}(k_0 r^m)$$

and thus the wavelet estimation equation is

$$\tilde{A}(\omega) = \frac{\int_{x_0}^{x_1} \left\{ \tilde{P}(\mathbf{x}', \mathbf{x}_S, \omega) \frac{\partial G_0(\mathbf{x}', \mathbf{x}, \omega)}{\partial n} - G_0(\mathbf{x}', \mathbf{x}, \omega) \frac{\partial \tilde{P}}{\partial n}(\mathbf{x}', \mathbf{x}_S, \omega) \right\} dx'}{G_0(\mathbf{x}, \mathbf{x}_S, \omega)}. \quad (18)$$

For the first example considered, the field $P$ and its derivative are analytically calculated for the incident wave and the results are shown in Figure 4. Equation (18) was then used to solve for the wavelet and the result is shown in Figure 5. This demonstrates the effect of time and spatial sampling and a finite aperture. For the second example, the field $P$ is again calculated analytically, but now the derivative of the field is computed from two different vertical samplings of the field by using an Eulerian approximation. The results of the wavelet reconstruction are shown in Figure 6. As can be seen, the approximate derivative causes a slight inaccuracy in the reconstruction. Alternate methods for approximating the derivative of the wavefield are available (Secrest amd MacBain, 1990).

To demonstrate that the scattered field is filtered by equation (18), 2-D acoustic finite-difference modeling is used to generate data with a plane reflector extending the full length of the model as shown in Figure 7 and a partial reflector extending from the left edge of the model to beneath the source point as shown in Figure 8. Again, equation (18) is used to solve for the wavelet for both Figures 7 and 8, where the Eulerian approximation is used to solve for the derivative of the field and the results are shown in Figure 9. Since the results are nearly the same as for the incident wave, it has been demonstrated that the scattered energy has been filtered. The slight ripple in the tail has been shown to be due to the finite aperture used.

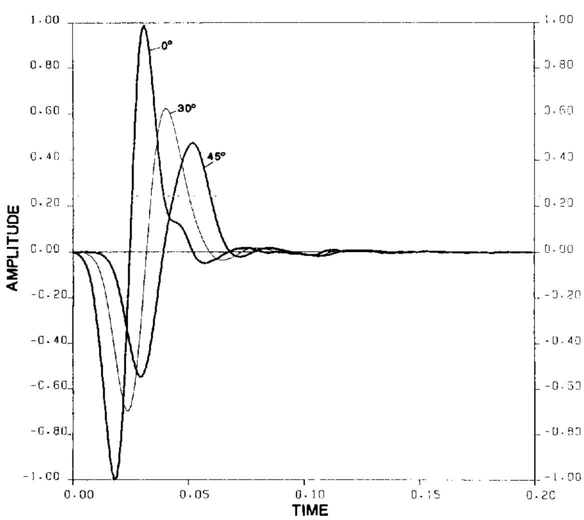


FIG. 11. Wavelet reconstruction for Figure 10 showing source array pattern.

As a final example, a string of 19 sources of equal strength were used as a source with a plane reflector extending the length of the model. The results of the modeling are shown in Figure 10 and the reconstructed wavelet using equation (17) is shown in Figure 11. The variable vector $\mathbf{r}$ was chosen to lie on a constant radius at angles of 0, 30, and 45 degrees from the vertical axis. This demonstrates the usefulness of the method in obtaining the source array pattern.

## CONCLUSIONS

We have presented a multidimensional deterministic wavelet estimation technique for either an acoustic or an elastic earth. We have shown that the method is independent of the earth's structure and acts like a filter to eliminate the scattered wave. A surface integral over the data and its normal derivative are required to produce this estimate. A simple 1-D example was used to illustrate the concept. In addition, the extensions to the marine environment and the elastic generalization were described. A numerical 2-D example was presented which showed the effects of sampled data in time and space, a free surface, a finite aperture, and approximating the derivative of the field. It was also shown that the source array pattern or the radiation pattern of the incident wave can be obtained in the case of a general source distribution. For the latter, no information about the nature of the source was required.

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