A direct inverse solution for AVO/FWI parameter estimation objectives

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Summary

A direct inverse solution is derived from the operator identity relating the change in a medium's properties and the commensurate change in the wavefield. The direct solution is in the form of a series, called the inverse scattering series (ISS). Each term in the series is directly computed in terms of the recorded data, and, without subsurface information. There are isolated task inverse scattering subseries that perform: free surface multiple removal, internal multiple removal, depth imaging, parameter estimation and Q compensation, and each achieves its objective directly and without subsurface information. The general operator identity is combined with the elastic wave equation to form a specific direct solution for changes in elastic properties and density. This paper describes the resulting data requirements and algorithms, a distinct ISS parameter estimation subseries, that provides a fundamental framework and platform for all seismic amplitude analysis and is directly relevant for the objectives of AVO and FWI. A view of a balanced and appropriate role for direct and indirect methods will be presented, as well.

Introduction

Inversion methods can be classified as direct or indirect. An example of a direct solution is given by the solution of the quadratic equation

$$ax^2 + bx + c = 0, \tag{1}$$

as

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

whereas an indirect solution could be to find x such that $(ax^2 + bx + c)^2$ is a minimum. Among indicators, identifiers and examples of "indirect" inverse solutions are: (1) model matching, (2) objective/cost functions, (3) search algorithms, (4) iterative linear inversion and (5) methods corresponding to necessary and not sufficient conditions, e.g., CIG flatness.

The Operator Identity

We begin our discussion of direct inverse solutions with the key operator identity mentioned above. Let L_0, G_0, L, and G be the differential operators and Green's functions for the reference and actual media, respectively, that satisfy:

$$L_0 G_0 = \delta \qquad LG = \delta$$

where δ is a Dirac delta function. Define the perturbation operator, V and the scattered wavefield, as follows:

$$V = L_0 - L \qquad \psi_s = G - G_0.$$

The relationship

$$G = G_0 + G_0 V G \tag{3}$$

is an operator identity that follows from

$$L^{-1} = L_0^{-1} + L_0^{-1}(L_0 - L)L^{-1}.$$

For modeling the wavefield, G, for a medium described by L

$$L \to G \qquad L_0, V \to G$$

where the second form has L entering the modeling algorithms in terms of L_0 and V. Modeling using scattering theory requires a complete and detailed knowledge of medium properties.

Direct Forward and Direct Inverse

The operator identity equation 3 can be solved for G as

$$G = (1 - G_0 V)^{-1} G_0 \tag{4}$$

and

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \dots . \tag{5}$$

Equation 5 has the form of a generalized Geometric series

$$G - G_0 = S = ar + ar^2 + \dots = \frac{ar}{1-r} \tag{6}$$

where we identify $a = G_0$ and $r = VG_0$ in equation 5, and

$$S = S_1 + S_2 + S_3 + \dots . \tag{7}$$

The portion of S that is linear, quadratic, ... in r are:

$$S_1 = ar$$
$$S_2 = ar^2$$
$$\vdots$$

and the sum is

$$S = \frac{ar}{1-r}. \tag{8}$$

Solving equation 8 for r, produces the inverse geometric series,

$$r = \frac{S/a}{1+S/a} = S/a - (S/a)^2 + (S/a)^3 + \dots$$
$$= r_1 + r_2 + r_3 + \dots$$

and is the simplest prototype inverse series, that is, the inverse of the geometric series. For the seismic inverse problem, we associate S with the measured data

$$S = (G - G_0)_{ms} = \text{Data}$$

and the forward and inverse series follow from treating the forward solution as S in terms of V, and the inverse solution from V in terms of S

$$V = V_1 + V_2 + \dots \tag{9}$$

where V_n is the portion of V, that is nth order in the data. Equation 8 is the forward series; and equation 9 is the inverse series. The identity, equation 3, provides a Geometric forward series rather than a Taylor series. In general, a Taylor series doesn't have an inverse series; however, a Geometric series has an inverse series. All conventional current mainstream inversion, including iterative linear inversion and FWI, are based on a Taylor series concept. Solving a forward problem in an inverse sense is not the same as solving an inverse problem directly.

The r_1, r_2, ... terms in

$$r = S/a - (S/a)^2 + (S/a)^3 + \dots$$
$$= r_1 + r_2 + r_3 + \dots$$

generalize for the seismic inverse in terms of $V_1, V_2, \dots,$ and G_0, G, $D = (G - G_0)_m$ as follows (see, e.g., Weglein et al., 2003)

$$\begin{aligned} G_0 V_1 G_0 =& D \\ G_0 V_2 G_0 =& -G_0 V_1 G_0 V_1 G_0 \\ G_0 V_3 G_0 =& -G_0 V_1 G_0 V_1 G_0 V_1 G_0 \\ & - G_0 V_1 G_0 V_2 G_0 - G_0 V_2 G_0 V_1 G_0 \\ & \vdots \end{aligned} \tag{10}$$

The operator identity for the 2D heterogeneous elastic wave equation

We exemplify the method for a 2D elastic heterogeneous earth. The starting point for the 3D generalization is found in Stolt and Weglein (2012). The 2D elastic wave equation for a heterogeneous isotropic medium is

$$L\vec{u} = \begin{pmatrix} f_x \\ f_z \end{pmatrix} \qquad \hat{L}\begin{pmatrix} \phi^P \\ \phi^S \end{pmatrix} = \begin{pmatrix} F^P \\ F^S \end{pmatrix}. \tag{11}$$

$\vec{u}$, f_x, f_z are the displacement and force, in displacement coordinates and ϕ_P, ϕ_S and F^P, F^S are the P and S waves and the force components in P and S coordinates. The operators L, L_0 and V are

$$L = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_x\gamma\partial_x + \partial_z\mu\partial_z & \partial_x(\gamma - 2\mu)\partial_z + \partial_z\mu\partial_x \\ \partial_z(\gamma - 2\mu)\partial_x + \partial_x\mu\partial_z & \partial_z\gamma\partial_z + \partial_x\mu\partial_x \end{pmatrix}\right]$$

$$L_0 = \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0\partial_x^2 + \mu_0\partial_z^2 & (\gamma_0 - \mu_0)\partial_x\partial_z \\ (\gamma_0 - \mu_0)\partial_x\partial_z & \mu_0\partial_x^2 + \gamma_0\partial_z^2 \end{pmatrix}\right] \text{ and}$$

$$V \equiv L_0 - L$$
$$= \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2\partial_x a_\gamma\partial_x + \beta_0^2\partial_z a_\mu\partial_z & \partial_x(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_z + \beta_0^2\partial_z a_\mu\partial_x \\ \partial_z(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_x + \beta_0^2\partial_x a_\mu\partial_z & a_\rho\omega^2 + \alpha_0^2\partial_z a_\gamma\partial_z + \beta_0^2\partial_x a_\mu\partial_x \end{bmatrix}.$$

The quantities $a_\rho \equiv \rho/\rho_0 - 1$, $a_\gamma \equiv \gamma/\gamma_0 - 1$, $a_\mu \equiv \mu/\mu_0 - 1$ are defined in terms of γ_0, μ_0, ρ_0, γ, μ, ρ, the bulk modulus, shear modulus and density in the reference and actual media, respectively.

The forward problem is found from the identity equation 5 and the elastic wave equation 11 (in PS coordinates) as

$$\hat{G} - \hat{G}_0 = \hat{G}_0\hat{V}\hat{G} = \hat{G}_0\hat{V}\hat{G}_0 + \hat{G}_0\hat{V}\hat{G}_0\hat{V}\hat{G}_0 + \dots$$
$$\begin{aligned} \begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} =& \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ &+ \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ &\times \begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} + \dots \end{aligned} \tag{12}$$

and the inverse solution, equation 10, for the elastic equation 11 is

$$\begin{aligned} \begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} =& \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ & \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ =& -\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ &\times \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}. \end{aligned} \tag{13}$$

where, for example, $\hat{V}^{PP} = \hat{V}_1^{PP} + \hat{V}_2^{PP} + \hat{V}_3^{PP} + \dots$ and any one of the four matrix elements of V requires

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}.$$

A few key points

$\hat{D}^{PP}$ can be determined in terms of

$$\begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}$$

and $\hat{V}^{PP}$ or $\hat{V}^{PS}$, $\hat{V}^{SP}$, $\hat{V}^{SS}$ require a series in

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}.$$

That's what the general relationship $G = G_0 + G_0VG$ requires, that is, a direct non-linear inverse solution is a solution order by order in the data matrix (in 2D)

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}.$$

The direct solution is not iterative linear inversion. Iterative linear starts with

$$G_0V_1G_0 = D, \tag{14}$$

solves for V_1, changes the reference medium, finds a new L_0 and G_0 (and require generalized inverses of noisy bandlimited data dependent operators). The next linear step involves V_1',

$$\begin{aligned} G_0'V_1'G_0' &= D' = (G - G_0')_{ms} \\ L_0' &= L_0 - V_1 \\ L_0'G_0' &= \delta \end{aligned}$$

where V_1' is the portion of V linear in the data $(G-G_0')_{ms}$. The direct inverse solution equations 9 and 13 call for a single unchanged reference medium, for computing $V_1, V_2, \ldots$. For a homogeneous reference medium they are obtained by an analytic inverse. The inverse to find V_1 from data, is the same inverse to find $V_2, V_3, \ldots$, from equation 10. There are no numerical inverses, no generalized inverses, no inverses of matrices that contain noisy bandlimited data.

The difference between iterative linear and the direct inverse of equation 13 is much more substantive and serious than merely a different way to solve $G_0V_1G_0 = D$, equation 14, for V_1. If equation 14 is our entire basic theory, you can mistakenly think that $\hat{D}^{PP} = \hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P$ is sufficient to update $\hat{D}^{PP} = \hat{G}_0^P{}'\hat{V}_1^{PP}{}'\hat{G}_0^P{}'$. That step loses contact with and violates the basic operator identity $G = G_0 + G_0VG$ for the elastic wave equation. That's as serious as considering problems involving a right triangle and violating the Pythagorean theorem in your method.

That is, iteratively updating PP data with an elastic model violates the basic relationship between changes in a medium, V and changes in the wavefield, $G - G_0$ for the simplest elastic earth model.

This direct inverse method provides a platform for amplitude analysis, AVO and FWI. It communicates when a "FWI" method should work, in principle. Iteratively inverting multi-component data has the correct data but doesn't corresponds to a direct inverse algorithm. To honor $G = G_0 + G_0VG$, you need both the data and the algorithm that direct inverse prescribes. Not recognizing the message that an operator identity and the elastic wave equation unequivocally communicate is a fundamental and significant contribution to the gap in effectiveness in current AVO and FWI method and application (equation 13). This analysis generalizes to 3D with P, S_h, and S_v data.

There's a role for direct and indirect methods in practical real world application. Indirect methods are to be called upon for recognizing that the world is more complicated than the physics that we assume in our models and methods. For the part of the world that you are capturing in your model (and methods) nothing compares to direct methods for clarity and effectiveness. The listed references provide detail and examples. An optimal indirect method would seek to satisfy a cost function that derives from a property of the direct method. In that way the indirect and direct method would be aligned and cooperative for accommodating the part of the world described by your physical model and the part that is outside.

Conclusions

This paper: (1) describes the direct inverse parameter estimation algorithm (subseries) and its data requirements (2) compares that direct inversion with current FWI approaches; and (3) will provide an application for 4D.

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