A first comparison of the inverse scattering series non-linear inversion and the iterative linear inversion for parameter estimation

Jinlong Yang and Arthur B. Weglein, M-OSRP, University of Houston*

SUMMARY

The inverse scattering series (ISS) direct non-linear inversion and the iterative linear inversion for parameter estimation are examined and compared. The convergence and the rate of convergence of both the ISS inversion subseries and the iterative inversion method are tested for different velocity contrasts on a simple 1D one parameter acoustic model. The rate of convergence of the ISS inversion method is analytically and numerically studied. When the reflection coefficient $R < 0.618$, the ISS inversion subseries monotonically term-by-term improves the estimation of medium properties; when $R > 0.618$, the ISS inversion subseries still converges, but not monotonically. Numerical tests show that when the velocity contrast is small, both inversion methods converge and the ISS inversion method converges faster than the iterative inversion method. When the velocity contrast increases, the iterative inversion method will not converge when $R > 0.5$, while the ISS inversion method always converges.

INTRODUCTION

The objective of seismic inversion is to estimate the medium properties of the subsurface from the recorded wavefield. Typically this begins with a chosen reference medium and the measured wavefield. Then an operator identity is called upon that relates the difference between the medium and reference properties and the difference between the measured total wavefield and the reference wavefield. This identity can be used to find a direct solution to the forward problem or a direct solution to the inverse problem for any type of medium.

If we seek the parameters of an elastic heterogeneous isotropic subsurface, then the differential operator in the operator identity is the differential operator that occurs in the elastic, heterogeneous isotropic wave equation. The elastic isotropic model is the base acceptable earth model-type for amplitude analysis, for example, AVO and FWI. Taking the operator identity (called the Lippmann-Schwinger or scattering theory equation) and the elastic wave equation, we can obtain a direct inverse solution for the changes in elastic properties and density. The direct inverse solution specifies both the data required and the algorithm to achieve a direct solution. The direct inverse (Zhang and Weglein, 2006; Li, 2014) requires multi-component/PS data and prescribes how that data are utilized for a direct parameter estimation solution. The direct solution (Weglein et al., 2003, 2009) provides a solid framework and firm math-physics foundation for the data requirement and algorithms to solve the problem that you are interested. There are many other issues that contribute to the gap in FWI today, e.g., the need for broadband data. But starting with and employing a framework that provides confidence of the data and methods is a significant, fundamental, and practical contribution towards filling the gap (Weglein, 2015). Only a direct solution can provide that clarity, confidence and effectiveness. The current industry standard FWI, using variants of iterative linear inverse, correspondent to model matching procedures, and iteratively linearly updating P data or multi-component data does not correspond to, and will not produce, a direct solution with its clarity and effectiveness.

The direct solution is in the form of a series, referred to as the inverse scattering series (Weglein et al., 2003). It can achieve all processing objectives within a single framework without requiring any subsurface information. There are isolated-task inverse scattering subseries, which can perform free-surface multiple removal, internal multiple removal, depth imaging, parameter estimation, and Q compensation. In this paper, we focus on analyzing and examining the ISS inversion subseries for parameter estimation. The distinct issues of: (1) data requirements, (2) model-type, and (3) inversion algorithm for the direct inverse are all important. For a normal incident wave on a single horizontal reflector in an acoustic medium, we can isolate and focus on the algorithm difference when mode-type agrees and there is the same data, a single reflector acoustic P wave. This allows us to focus on the algorithm issues.

A comparison between the ISS direct non-linear inversion and the iterative inversion will be tested and shown on a 1D, one parameter, and a single horizontal reflector model, where the velocity is assumed to be known above the reflector and unknown below the reflector. Their convergence and the rate of convergence will be discussed and studied. In the ISS inversion subseries, each term of the series works towards the final goal. Sometimes when more terms in the series are included, the estimation may be worse locally, but in fact it is purposeful and essential in the contribution towards convergence and the final goal. This property has also been indicated by Carvalho (1992) in the free-surface multiple elimination subseries, e.g., what appears to make a second-order free-surface multiple larger with a first-order free-surface algorithm is actually preparing the second-order multiple to be removed by the higher-order terms. This simple example provides a guide when we move on to the more complicated elastic world.

THEORY

Starting from the two basic differential equations (Weglein et al., 2003), which govern wave propagation in actual medium and reference medium

$$LG = \delta \quad \text{and} \quad L_0 G_0 = \delta \tag{1}$$

where L, L_0 and G, G_0 are the differential operators and Green's functions in actual and reference medium, respectively. Defining the perturbation $V = L_0 - L$, the forward modeling series

(Born series) can be derived

$$G = G_0 + G_0VG_0 + G_0VG_0VG_0 + \cdots \quad (2)$$

from the Lippmann-Schwinger equation. Expanding V as a series

$$V = V_1 + V_2 + V_3 + \cdots \quad (3)$$

and substituting it into the equation 2, the inverse scattering series is obtained as

$$D = [G_0V_1G_0]_{ms} \quad (4)$$
$$0 = [G_0V_2G_0]_{ms} + [G_0V_1G_0V_1G_0]_{ms} \quad (5)$$
$$\vdots$$

where D is $G - G_0$ on the measurement surface. The inverse scattering series provides a direct method for obtaining the subsurface information by inverting the series order-by-order to solve for the perturbation operator V, using only the measured data D and a reference wave field G_0, for any type of medium.

On the other hand, the iterative linear method for estimating the perturbation operator V starts with equation 4. We solve for V_1 and change the reference medium iteratively. The new differential operator L_0' becomes and the new reference medium G_0' satisfies

$$L_0' = L_0 - V_1 \quad \text{and} \quad L_0'G_0' = \delta. \quad (6)$$

Through the same equation 4 with different reference background

$$G_0'V_1'G_0' = D' = (G - G_0')_{ms}, \quad (7)$$

we can continually update L_0' and G_0', and finally solve the perturbation operator V.

Considering a simple 1D case, the model consists of two half-spaces with acoustic velocities c_0 and c_1 and an interface located at $z = a$ as shown in Figure 1. If we choose an acoustic

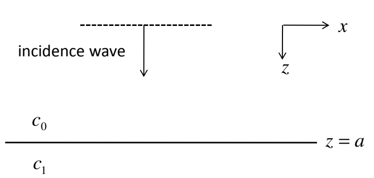


Figure 1: 1D acoustic model with velocities c_0 over c_1

whole-space with velocity c_0 as the reference medium, the perturbation V (Weglein et al., 2003) can be expanded as

$$V(z) = \frac{\omega^2}{c_0^2} - \frac{\omega^2}{c^2(z)} = \frac{\omega^2}{c_0^2}(1 - \frac{c_0^2}{c^2(z)}) = k_0^2\alpha(z), \quad (8)$$

where ω is the angular frequency, $c(z)$ is the local acoustic velocity, and $k_0 = \omega/c_0$. Depending on V, $\alpha(z)$ can be expanded as a series in terms of data, $\alpha(z) = \alpha_1(z) + \alpha_2(z) + \alpha_3(z) + \cdots$. Thus, we have

$$V_1 = k_0^2\alpha_1, \quad V_2 = k_0^2\alpha_2, \quad \cdots. \quad (9)$$

From the inverse scattering series (Equations 4 and 5), Shaw et al. (2004) isolated the leading order imaging subseries and the direct non-linear inversion subseries.

In this paper, we will focus on studying the convergence properties of the ISS inversion subseries. The inversion only terms isolated from the inverse scattering series are

$$\alpha(z) = \alpha_1(z) - \frac{1}{2}\alpha_1^2(z) + \frac{3}{16}\alpha_1^3(z) + \cdots. \quad (10)$$

If the incidence angle is θ, Zhang (2006) showed that α_1 can be expressed as

$$\alpha_1(z) = 4R(\theta)\cos^2\theta H(z-a), \quad (11)$$

where R is the reflection coefficient, and H represents Heaviside function*. For the normal incidence case, we have $R = \frac{c_1-c_0}{c_1+c_0}$. When $z > a$,

$$\alpha_1 = 4R. \quad (12)$$

Substituting α_1 into equation (10), the ISS direct non-linear inversion subseries in terms of R can be written as

$$\alpha = 4R - 8R^2 + 12R^3 + \cdots = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n. \quad (13)$$

After solving α, the inverted velocity $c(z)$ can be obtained through $c_1 = c_0/\sqrt{1-\alpha}$ (equation 8).

Considering the convergence property of the series of α or the inversion subseries, we can calculate the ratio test,

$$\left|\frac{\alpha_{n+1}}{\alpha_n}\right| = \left|\frac{(n+2)(-R)^{n+1}}{(n+1)(-R)^n}\right| = \left|\frac{n+2}{n+1}R\right|. \quad (14)$$

If $\lim_{n\to\infty}\left|\frac{\alpha_{n+1}}{\alpha_n}\right| < 1$, this subseries converges absolutely. That is

$$|R| < \lim_{n\to\infty}\frac{n+1}{n+2} = 1. \quad (15)$$

Therefore, the ISS direct non-linear inversion subseries converges when the reflection coefficient $|R|$ is less than 1, which is always true. Hence, for this example, the ISS inversion subseries will converge under any velocity contrasts between the two media.

For the iterative linear inversion, we will update the reference velocity $c_0' = c_0/\sqrt{1-\alpha_1}$ by using $\alpha_1 = 4R$. Then, the new linear inversion velocity is calculated by $\alpha_1' = 4R'$, where $R' = \frac{c_1-c_0'}{c_1+c_0'}$. The same procedure will be applied iteratively until we achieve the final inversion result.

ANALYTIC EXAMPLE

The rate of convergence of the estimated α or the ISS inversion subseries (equation 13) is analytically examined and studied for a 1D normal incidence case. Since α is always convergent when $R < 1$, the summation of this subseries is

$$\alpha = 4R\sum_{n=0}^{\infty}(n+1)(-R)^n = 4R\frac{1}{(1+R)^2}. \quad (16)$$

*The definition of Heaviside function is: $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$

If the error between the estimated and the actual α is monotonically decreasing, it means the subseries is a term-by-term added value improvement towards determining the actual medium properties. If this error is increasing before decreasing, it means that the estimation becomes worse before it gets better. In other words, the error for the first order and the error for the second order have the relation,

$$|\alpha - \alpha_1 - \alpha_2| > |\alpha - \alpha_1|, \quad (17)$$

i.e.,

$$|4R\frac{3R^2+2R^3}{(1+R)^2}| > |4R\frac{-R^2-2R}{(1+R)^2}|. \quad (18)$$

After simplification, it gives

$$R^2 + R - 1 > 0. \quad (19)$$

We can solve it and obtain the reflection coefficient $R < \frac{-1-\sqrt{5}}{2}$ $= -1.618$ or $R > \frac{-1+\sqrt{5}}{2} = 0.618$. Therefore, when $R > 0.618$, the error increases first. Similarly, if the error for the third order is greater than that for the second order, we get $R > 0.667$; If the error for the fourth order is greater than that for the third order, we obtain $R > 0.721$. In summary, when $R > 0.618$ the error increases and the estimated α gets worse first. The green dash line in Figure 2 shows that when the reflection coefficient R is equal to 0.618, the error for the first order is equal to the error for the second order. The detail of the numerical tests will be discussed in the next section.

NUMERICAL TESTS

In this section, we will examine the convergence property and the rate of convergence of α by using the ISS inversion subseries (equation 13) and the iterative linear inversion methods to the velocity contrast in the 1D acoustic case. In addition, the inversion results by these two methods is discussed and compared.

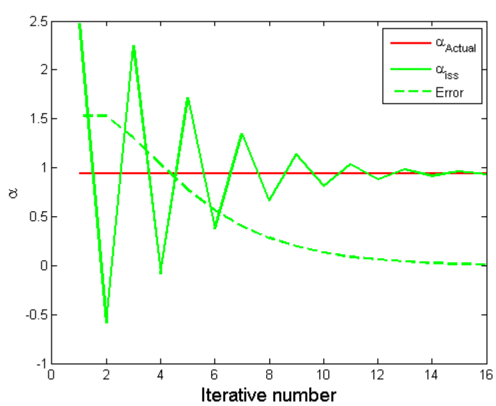
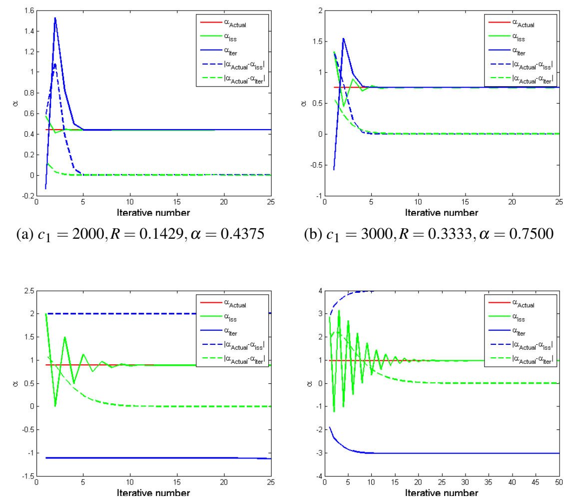


Figure 2: The error (green dash line) of estimated α at $R = 0.618$ and $\alpha = 0.9443$.

In the simple 1D model (Figure 1), only one parameter (velocity) varies and a plane wave propagates into the medium. There is only a single reflector and we assume the velocity is known above the reflector and unknown below the reflector. We will compare the convergence of the perturbation α and the inversion results by using the ISS direct non-linear method and the iterative linear method. With the reference velocity $c_0 = 1500m/s$, four analytic examples with different velocity contrasts for $c_1 = 2000$, 3000, 4500, $9000m/s$ are examined. In Figure 3, the red line represents the actual α that is calculated from our model for each velocity contrast. The horizontal axis represents the orders of the ISS inversion subseries or the iterative numbers. The vertical axis shows the value of α. The green solid line represents the estimated value of α through the ISS inversion method verse the summation of α_n to n^{th} order. The green dash line represents the absolute value of the error between the ISS estimated and the actual value of α. The blue solid line represents the estimated value of α through the iterative inversion method verse the iterative numbers. The blue dash line represents the absolute value of the error between the iterative estimated and the actual value of α.



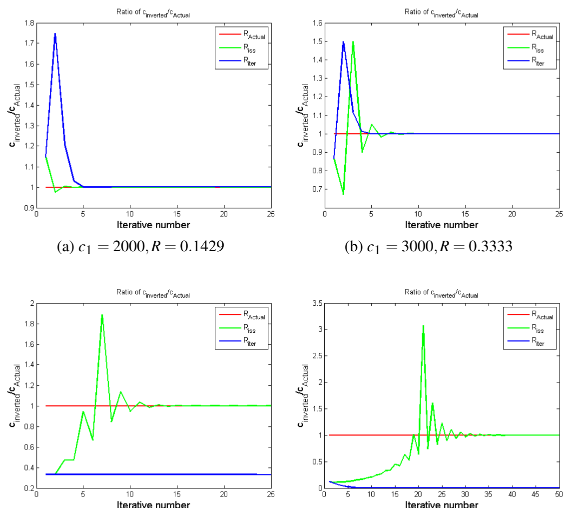
(a) $c_1 = 2000, R = 0.1429, \alpha = 0.4375$ (b) $c_1 = 3000, R = 0.3333, \alpha = 0.7500$ (c) $c_1 = 4500, R = 0.5000, \alpha = 0.8889$ (d) $c_1 = 9000, R = 0.7143, \alpha = 0.9722$

Figure 3: The estimated α: The horizontal axis is the orders of the ISS suberies or the iterative numbers, and the vertical axis shows the value. The red line shows the actual value of α. The green and blue lines show the estimations of α by using the ISS inversion method and the iterative inversion method. The green and blue dash lines are their corresponding absolute difference between the actual value and the estimations.

From the estimated results of α for the different velocity contrasts, we can see that the smaller the contrast, the faster the inversion results will converge as shown in Figure 3. In other words, when the velocity contrast increases, the error of α estimation increases, therefore it takes more terms to deal with the bigger contrast issue as shown in Figure 3d. Another important point is, when the velocity contrast is getting bigger, at some point, the iterative inversion method is not convergent (see the blue solid and dash lines in Figures 3c and 3d). From the analysis, the iterative inversion method can not estimate the correct inversion result when $R > 0.5$, while the ISS inversion method always produces useful results (see the green solid and dash lines in Figures 3c and 3d). For the simple 1D one reflector example, it shows that the ISS direct non-linear inversion subseries converges for all values of R. Comparing the errors of α (green and blue dash lines in Figures 3a and 3b) by the ISS inversion method and the iterative method, we can see that at small contrast, both methods converge and the ISS inversion method converges faster than the iterative inversion method.

From the green dash line in Figure 3, for a small contrast, the error between the estimated and the actual α is monotonically decreasing, in other words, the estimation of α is always a term-by-term added value improvement towards determine c_1; when the contrast is very large (Figure 3d), the error is increasing before decreasing. It means that the estimation of α becomes worse before it gets better. However, when it starts to add value, it getting better when each further term added to the series. The green dash line in Figure 3 also shows that as more terms are captured and added up, the error always approaches zero, which means the correct estimation is always achieved. Figures 3a, 3b and 3c show that when the reflection coefficient R is smaller than 0.618, this inversion subseries is monotonically term-by-term added value improvement towards determining c_1. When the reflection coefficient R is equal to 0.618, the error for the first order equals the error for the second order as shown in Figure 2. When the reflection coefficient is larger than 0.618 (Figure 3d), the series still converges, but the estimation of α will become worse before it gets better. From the analytic and numerical examples, we can see that each term in the series works towards the final goal. Sometimes when more terms in the series are included, the estimation looks worse locally, but once it starts to improve the estimation at a specific order, the approximations never become worse again, every single term after that order will produce an improved estimation.

The convergence results are also presented for the velocity estimation as shown in Figure 4. At small velocity contrast, both methods are convergent very fast and estimate the correct velocity (Figures 4a and 4b). When the contrast increases, the ISS inversion subseries always converges, but the iterative inversion method does not converge (Figures 4c and 4d).



(a) $c_1 = 2000, R = 0.1429$

(b) $c_1 = 3000, R = 0.3333$

(c) $c_1 = 4500, R = 0.5000$

(d) $c_1 = 9000, R = 0.7143$

Figure 4: The ratio of the estimated velocity and the actual velocity: The horizontal axis is the order of the ISS suberies or the iterative numbers, and the vertical axis shows the value. The red line is the actual ratio, which is 1. The green and blue lines show the ratios by using the ISS inversion method and the iterative inversion method.

CONCLUSIONS

The ISS direct non-linear inversion and the iterative inversion are examined and compared in a 1D, one parameter, and a single horizontal reflector case, where the velocity is assumed to be known above the reflector and unknown below the reflector. The rate of convergence of the ISS inversion method is analytically and numerically studied. From the analytic example, we show that when the reflection coefficients $R < 0.618$, the ISS inversion subseries is a term-by-term improvement towards determining medium properties; when $R > 0.618$, the inversion subseries still converges, but the estimation will locally be less accurate before it converges. Numerical results show that when the velocity contrasts are small, i.e., the reflection coefficients are small, both inversion methods converge and the ISS inversion method converges faster than the iterative inversion method. When velocity contrasts increase, the reflection coefficients get larger, the iterative inversion method will not converge when $R > 0.5$, while the ISS inversion method still converges. Hence, for the simplest situation, the iterative linear inversion is not equivalent to the direct non-linear solution provided by the inverse scattering series. For more complicated circumstances (e.g., the elastic non-normal incidence case), the difference is much greater, not just on the algorithms, but also on data requirements and on how the band-limited noisy nature of the seismic data impact the inverse operators in iterative linear inversion but not in the ISS direct inversion.

ACKNOWLEDGMENTS

We are grateful to all M-OSRP sponsors for long-term encouragement and support in this research.

REFERENCES

Carvalho, P. M., 1992, Free-surface multiple reflection elimination method based on nonlinear inversion of seismic data: PhD thesis, Universidade Federal da Bahia.

Li, X., 2014, I. multi-parameter depth imaging using the inverse scattering series; ii. multi-component direct nonlinear inversion for elastic earth properties using the inverse scattering series: PhD thesis, University of Houston.

Shaw, S. A., A. B. Weglein, D. J. Foster, K. H. Matson, and R. G. Keys, 2004, Isolation of a leading order depth imaging series and analysis of its convergence properties: Journal of Seismic Exploration, **2**, 157–195.

Weglein, A. B., 2015, Direct inversion and FWI: Invited keynote address given at the SEG Workshop Full-waveform Inversion: Fillings the Gaps, Abu Dhabi, UAE. (Available at http://mosrp.uh.edu/events/event-news/arthur-b-weglein-will-present-an-invited-key-note-address-on-direct-inversion-at-the-seg-workshop-on-fwi-30-march-1-april-2015-in-abu-dhabi-uae).

Weglein, A. B., F. V. Araújo, P. M. Carvalho, R. H. Stolt, K. H. Matson, R. T. Coates, D. Corrigan, D. J. Foster, S. A. Shaw, and H. Zhang, 2003, Inverse scattering series and seismic exploration: Inverse Problems, **19**, R27–R83.

Weglein, A. B., H. Zhang, A. C. Ramírez, F. Liu, and J. E. M. Lira, 2009, Clarifying the underlying and fundamental meaning of the approximate linear inversion of seismic data: Geophysics, **74**, WCD1–WCD13.

Zhang, H., 2006, Direct nonlinear acoustic and elastic inversion: towards fundamentally new comprehensive and realistic target identification: PhD thesis, University of Houston.

Zhang, H., and A. B. Weglein, 2006, Direct non-linear inversion of multi-parameter 1d elastic media using the inverse scattering series: 76th International Annual Meeting, SEG, Expanded Abstracts, 284–311.