Topical Review: Inverse-scattering series and seismic exploration

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Abstract

This paper presents both an overview and a more detailed description of the key logic steps and mathematical-physics framework behind the development of practical algorithms for seismic exploration derived from the inverse scattering series.

We present both the rationale for seeking and methods of identifying uncoupled task specific subseries that accomplish: (1) free-surface multiple removal; (2) internal-multiple attenuation; (3) imaging primaries at depth; and (4) inverting for earth material properties. A combination of forward series analogue and physical intuition are employed to locate those subseries. We show that the sum of the four task specific subseries does not correspond to the original entire inverse series since terms with coupled tasks are never considered or computed. This aspect of the program, i.e., inversion in stages, with an isolated task followed by restarting the problem, provides tremendous practical advantage, since the achievement of a task is a form of useful information exploited in the redefined problem; and, the latter represents a critically important step in the logic and overall strategy.

There are both tremendous symmetries and critical and subtle differences between the forward scattering series construction and the inverse scattering series processing of seismic events. These similarities and differences help explain the efficiency and effectiveness of different inversion objectives. The individual subseries are analyzed

and their strengths, limitations and prerequisites exemplified with analytic, numerical, and field data examples.

1 Introduction and background

In exploration seismology, a man−made source of energy on (or near) the surface of the earth (or in the ocean, in marine exploration) generates a wave that propagates into the subsurface. When the wave reaches a rapid change in earth material properties, (i.e., a reflector) a portion of the wave is reflected back upward the surface; and, in the marine case, is recorded at numerous receivers along a towed streamer in the water column near the air-water boundary.

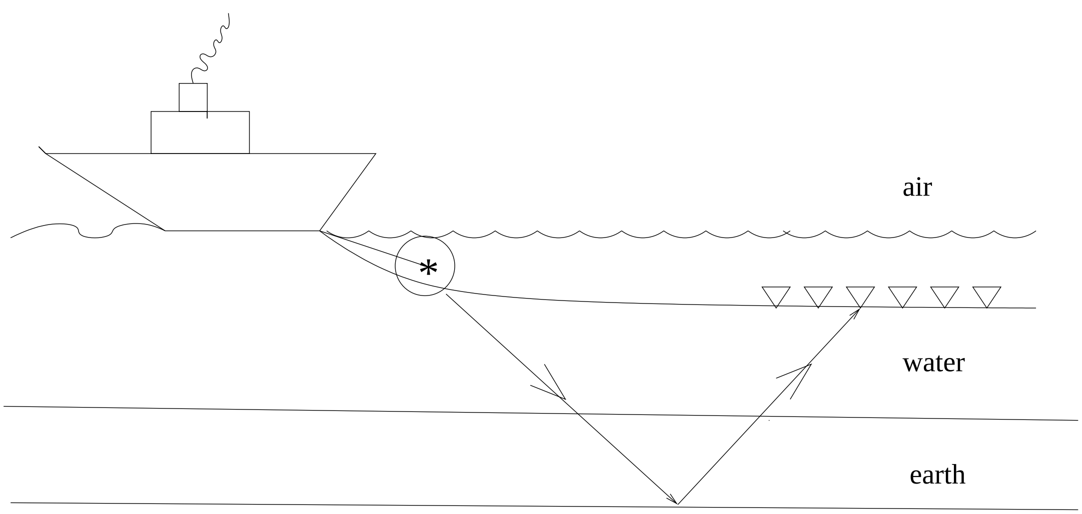


Figure 1: Marine Seismic Exploration Geometry: $*$ and $\bigtriangledown$ indicate source and receiver, respectively. The boat moves through the water with the source and receivers, and the experiment is repeated. The collection of the different source-receiver wave field measurements defines the seismic reflection data.

The objective of seismic exploration is to determine subsurface earth properties from the recorded wavefield. The ultimate objective is to determine subsurface targets and then to estimate the type and extent of rock and fluid properties for hydrocarbon potential.

The need for more effective and liable techniques for extracting information from seismic data is driven by several factors including (1) the higher acquisition and drilling cost and risk associated with the industry trend to explore and produce in deeper water; and (2) the serious technical challenges associated with either deep water, or imaging beneath a complex and ill-defined overburden, above the target.

An event is a distinct arrival of seismic energy. Seismic reflection events are catalogued as primary or multiple depending on whether the energy arriving at the receiver has experienced one or more upward reflections, respectively. Multiples or multiply reflected events are further classified by the location of the downward reflection between two upward reflections. For marine data, multiples that have experienced at least one downward reflection at the air-water (free) surface are called free surface multiples. Multiples that have all of their downward reflections below the free surface are called internal multiples. (See Fig.2). Methods for extracting subsurface information from seismic data typically assume that the

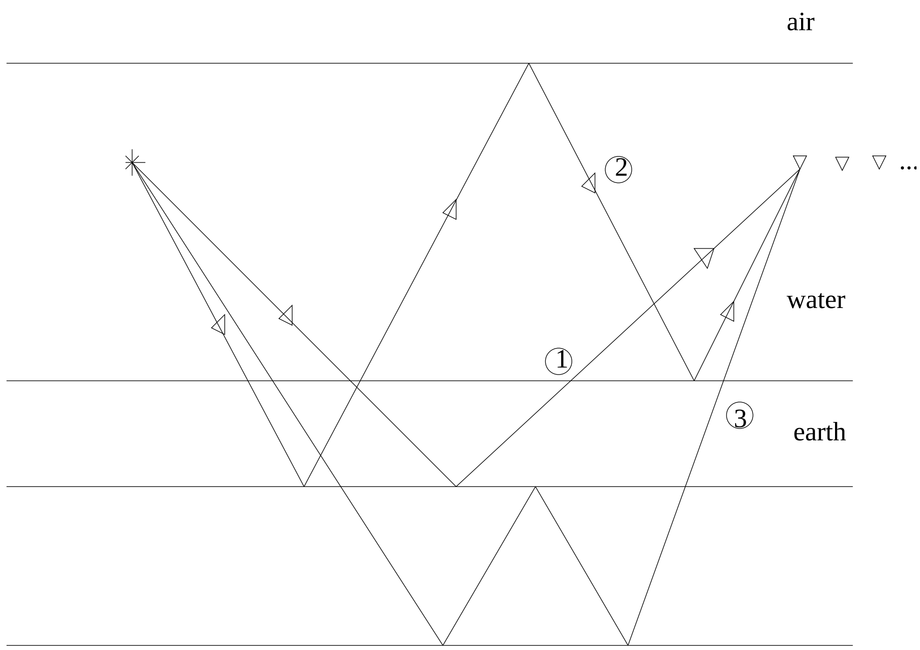


Figure 2: 1, 2 and 3 are examples of primaries, free-surface multiples and internal multiples, respectively.

data consists of primaries, since that model allows essentially one reflection process to be associated with each recorded event. The latter primaries-only assumption simplifies the processing of seismic data for determining the spatial location of reflectors and the local change in earth material properties across a reflector. Hence, multiple removal is a requisite to seismic processing. It is a long-standing problem and while significant progress has been achieved over the past decade, conceptual and practical challenges remain. The inability to remove multiples can lead to multiples masquerading or interfering with primaries causing false or misleading interpretations; and, ultimately poor drilling decisions. The assumption of singly reflected (or scattered) data in seismic data analysis is an assumption shared with other fields of inversion and non-destructive evaluation, e.g., it is common to medical imaging, and ground penetrating radar, environmental hazards, and the violation of this assumptions in practice, can lead to deleterious and serious consequences for medical diagnosis, hazard detection, and buried object and fluid location/identification, for tunnels and caves. Even if multiples were to be removed from seismic reflection data, the challenges for accurate imaging (locating) and inversion across reflectors are significant, especially when the medium of propagation and the geometry of the target are complex and the contrast in earth material properties is large. The latter large contrast property condition is all by itself enough to cause linear inverse methods to bump up hard against their assumptions.

Specifically, the location and definition of hydrocarbon targets beneath salt, basalt, volcanics, and karsted sediments are of high economic moment in the petroleum industry today, and serious challenges to current imaging and inversion techniques that go beyond the daunting issues concerning the removal of multiples. For the latter geologic circumstances the requirement of all current methods for the imaging-inversion of primaries for an accurate (or at least adequate) model of the medium above the target, can often not be achievable in practice, leading to erroneous, ambivalent or misleading prediction. These difficult imaging conditions often occur in, e.g., the deep water Gulf of Mexico, where the confluence of large hydrocarbon reserves beneath salt and the high cost of drilling in deep water, drives the demand for more effective and reliable methods. In this Topical Review, we will describe how the inverse scattering series has provided the promise of an entire new vision and level of seismic capability and effectiveness. That promise has already been delivered for the removal of free surface and internal multiples. We will also describe the recent research progress and results on the inverse series for the processing of primaries. Our objectives in writing this Topical Review are: (1) To provide both an overview and a more comprehensive mathematical-physics description of these new seismic processing concepts and practical industrial production strength algorithms, that derive from the inverse series and (2) To describe and exemplify the strengths and limitations of these seismic processing algorithms; and to discuss open issues and challenges. (3) To explain how this work reflects a general philosophy, and approach (strategy and tactics) to defining, prioritizing and choosing, and then solving significant real-world problems, from developing new fundamental theory, to issues with limitations with field-data, to satisfying practical prerequisites and computational requirements.

The problem of determining earth material properties from seismic reflection data is an inverse scattering problem; and, specifically, a non-linear inverse scattering problem. Although, an overview of all seismic methods is well beyond the scope of this paper, it is accurate to say that prior to the early 1990s, (when non-linear inverse scattering series methods were first applied) (Weglein, Boyse and Anderson (1981) and practical algorithms first demostrated Weglein *et al.* (1997)) all deterministic methods used in practice in exploration seismology could be viewed as different realizations of a linear approximation to inverse scattering, the inverse-Born approximation (see, e.g., Cohen and Bleistein (1997), Stolt and Weglein (1985), Morley and Claerbout (1983)).

All scientific methods assume a model that starts with assumptions that include some (and ignore other) phenomena and components of the reality. Among earth models used in seismic exploration are: acoustic, elastic, heterogeneous, anisotropic and anelastic, and the experimental description model for the characteristics of the man-made source, and the resultant incident field, the character of the receivers, the dimension of variability of the earth and geometry of reflectors. The configuration and extent of the experiment, and the sampling rate of sources and receivers that comprise the recorded seismic wave field need to be included in the model and subsequent theory and algorithms.

Although 2D, 3D closed form complete integral equation solutions exist for the Schrodinger equation (see, Newton (2002)) - there is no analogous closed form complete multi-dimensional inverse solution for the acoustic or elastic equation. The push to develop complete multi-dimensional non-linear seismic inversion methods came from a several directions: (1) The need to remove multiples from a complex multidimentional earth and (2) the interest in a more realistic model for primaries.

This absence of a closed form exact inverse (for a 2D acoustic or elastic earth) shifted attention to non-closed or series forms. An inverse series can be written, at least formally, for any differential equation expressed in a perturbative form.

This paper describes and illustrates the development of concepts and practical methods from the inverse scattering series for multiple attenuation and, provides some recent new promising conceptual and algorithmic results for primaries.

2 Seismic data and scattering theory

2.1 The scattering equation

Scattering theory is a form of perturbation analysis. In broad terms it describes how a perturbation in the properties of a medium relates a perturbation to a wave field that experiences that perturbed medium. It is customary to consider the original unperturbed medium as the reference, and, to consider the perturbation or alteration of properties turning the reference into the actual. The difference between the actual and reference media is characterized by the perturbation operator. The corresponding difference between the actual and reference wavefields is called the scattered field. Forward scattering takes as input the reference medium, the reference field, and the perturbation operator, and outputs the actual wavefield. Inverse scattering takes as input the reference medium, the reference field and values of the actual field on the measurement surface, and outputs the difference between actual and reference medium properties, through the perturbation operator. Inverse-scattering-theory methods typically assume the support of the perturbation to be on one side of the measurement surface. In seismic application this condition translates to a requirement that the difference between actual and reference media be non-zero only below the source-receiver surface. Consequently, inverse scattering methods require, for seismic application, that the reference medium agrees with the actual at and above the measurement surface.

For marine application the sources and receivers are located within the water column and the simplest reference medium is a half-space of water bounded by a free surface at the air-water interface. Since scattering theory relates the difference between actual and reference wavefields to the difference between their medium properties, it is reasonable that the mathematical description begin with the differential equations governing wave propagation in these media. Let

$$\mathbf{L}\mathbf{G} = -\delta(\mathbf{r} - \mathbf{r}_s) \tag{1}$$

and

$$\mathbf{L}_0\mathbf{G}_0 = -\delta(\mathbf{r} - \mathbf{r}_s) \tag{2}$$

where $\mathbf{L}$, $\mathbf{L}_0$ and $\mathbf{G}$, $\mathbf{G}_0$ are the actual and reference differential operators and Greens functions, respectively, for a single temporal frequency. Equations (1) and (2) assume that the source and receiver signatures have been deconvolved. The impulsive source is ignited at $t = 0$. $\mathbf{G}$ and $\mathbf{G}_0$ are the matrix elements of the Greens operator, $\mathbf{G}$ and $\mathbf{G}_0$, in the spatial coordinates and temporal frequency representation. $\mathbf{G}$ and $\mathbf{G}_0$ satisfy $\mathbf{L}\mathbf{G} = -\mathbf{1}$ and $\mathbf{L}_0\mathbf{G}_0 = -\mathbf{1}$, where $\mathbf{1}$ is the unit operator. The perturbation operator, $\mathbf{V}$, and the scattered field operator, Ψ_s, are defined as follows:

$$\mathbf{V} = \mathbf{L} - \mathbf{L}_0 \ , \tag{3}$$

$$\Psi_s = \mathbf{G} - \mathbf{G}_0 \ . \tag{4}$$

Ψ_s is not itself a Green's operator. The Lippmann-Schwinger equation, the fundamental equation of scattering theory, is an operator identity that relates Ψ_s, $\mathbf{G}_0$, $\mathbf{V}$ and $\mathbf{G}$, (see, e.g., Taylor, 1972),

$$\Psi_s = \mathbf{G} - \mathbf{G}_0 = \mathbf{G}_0 \mathbf{V} \mathbf{G} \ . \tag{5}$$

In the coordinate representation, Eq. (5) is valid for all positions of $\mathbf{r}$ and $\mathbf{r}_s$ whether or not they are outside the support of $\mathbf{V}$. A specific simple example of $\mathbf{L}$, $\mathbf{L}_0$, and $\mathbf{V}$, when $\mathbf{G}$ corresponds to a pressure field in an inhomogeneous acoustic medium (see, e.g., Clayton and Stolt, (1981))

$$\mathbf{L} = \frac{\omega^2}{\kappa} + \nabla \cdot \left(\frac{1}{\rho} \nabla \right) \ ,$$

$$\mathbf{L}_0 = \frac{\omega^2}{\kappa_0} + \nabla \cdot \left(\frac{1}{\rho_0} \nabla \right) \ ,$$

and

$$\mathbf{V} = \omega^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa_0} \right) + \nabla \cdot \left[\left(\frac{1}{\rho} - \frac{1}{\rho_0} \right) \nabla \right] \ , \tag{6}$$

where κ, κ_0, ρ and ρ_0 are the actual and reference bulk modulus and densities, respectively; ω is the temporal frequency. Other forms that are appropriate for elastic isotropic media and a homogeneous reference, begin with the generalization of equations Eqs. (1), (2) and (5) where matrix operators e.g.:

$$G = \begin{pmatrix} G_{pp} & G_{ps} \\ G_{sp} & G_{ss} \end{pmatrix}$$

and

$$G_0 = \begin{pmatrix} G_p^0 & 0 \\ 0 & G_s^0 \end{pmatrix}$$

express the increased channels available for propagation and scattering and

$$V = \begin{pmatrix} V_{pp} & V_{ps} \\ V_{sp} & V_{ss} \end{pmatrix}$$

is the perturbation in an elastic world. (see,e.g., Stolt and Weglein, 1985).

2.2 Forward series in operator form

Equation (5) can be expanded in an infinite series

$$\Psi_s \equiv \mathbf{G} - \mathbf{G}_0 = \mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \dots \tag{7}$$

for Ψ_s in orders of the perturbation operater, $\mathbf{V}$. Then Eq. (7) can be rewritten as

$$\Psi_s = (\Psi_s)_1 + (\Psi_s)_2 + (\Psi_s)_3 + \dots \tag{8}$$

where $(\Psi_s)_n \equiv \mathbf{G}_0(\mathbf{V}\mathbf{G}_0)^n$, and is the portion of Ψ_s that is n-th order in $\mathbf{V}$. The inverse series of Eq. (7) (or Eq. (8)) is an expansion for $\mathbf{V}$ in orders (or powers) of the measured values of $\Psi_s \equiv (\Psi_s)_m$. The measured values of $\Psi_s = (\Psi_s)_m$ constitute the data, D. Expand $\mathbf{V}$ as a series

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \dots \tag{9}$$

where $\mathbf{V}_n$ is the portion of $\mathbf{V}$ that is nth order in the data, D.

To find $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, ... and, hence, $\mathbf{V}$, first substitute the inverse form Eq. (9) into the forward Eq. (5)

$$\begin{aligned}\Psi_s &= \mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \dots)\mathbf{G}_0 + \mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \dots)\mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \dots)\mathbf{G}_0 \\ &\quad + \mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \dots)\mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \dots)\mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \dots)\mathbf{G}_0 \\ &\quad + \dots .\end{aligned} \tag{10}$$

Evaluate both sides of Eq. (10) on the measurement surface and set terms of equal order in the data equal. The first order terms are

$$(\Psi_s)_m = D = (\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0)_m , \tag{11}$$

the second order terms

$$0 = (\mathbf{G}_0\mathbf{V}_2\mathbf{G}_0)_m + (\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0)_m \tag{12}$$

and the third order

$$\begin{aligned}0 &= (\mathbf{G}_0\mathbf{V}_3\mathbf{G}_0)_m + (\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_2\mathbf{G}_0)_m \\ &\quad + (\mathbf{G}_0\mathbf{V}_2\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0)_m + (\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0)_m + \dots\end{aligned} \tag{13}$$

and to n-th order

$$\begin{aligned}0 &= (\mathbf{G}_0\mathbf{V}_n\mathbf{G}_0)_m + (\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_{n-1}\mathbf{G}_0)_m + \dots \\ &\quad + (\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1 \dots \mathbf{G}_0\mathbf{V}_1\mathbf{G}_0)_m .\end{aligned} \tag{14}$$

$(\Psi_s)_m$ are the measured values of the scattered field Ψ_s. To solve these equations, start with Eq. (11) and invert the $\mathbf{G}_0$ operators that sandwich $\mathbf{V}_1$. Then substitute $\mathbf{V}_1$ into Eq. (12) and perform the same inversion operation as in Eq. (11) to invert the $\mathbf{G}_0$ operators that sandwich $\mathbf{V}_2$. Now substitute $\mathbf{V}_1$ and $\mathbf{V}_2$ found from equations (11) and (12), into Eq. (13) and again invert the $\mathbf{G}_0$ operators that bracket $\mathbf{V}_3$ and in this manner continue this program to compute any $\mathbf{V}_n$. This method for determining $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$, $\ldots$ and, hence, $\mathbf{V} = \sum_{n=1}^{\infty} \mathbf{V}_n$ is an explicit direct inversion formalism that, in principle, can accommodate a wide variety of physical phenomena, and concomitant differential equations, including multidimensional acoustic, elastic, an certain forms of anelastic wave propagation. Because a closed or integral equation solution is currently not available for the multidimensional forms of the latter equations, and a multidimensional earth model is a minimal required realism to develop relevant and differential technology, the inverse scattering series became a focus of attention for those seeking significant added completeness and effectiveness beyond linear, 1D, or small contrast techniques.

In the derivation of the inverse series equations (11)–(14) there is no assumption about the closeness of $\mathbf{G}_0$ to $\mathbf{G}$, nor of the closeness of $\mathbf{V}_1$ to $\mathbf{V}$, nor is $\mathbf{V}$ or $\mathbf{V}_1$ assumed to be small in any sense. $\mathbf{V}_1$ is the portion of $\mathbf{V}$ that is linear in the data. That is all that is assumed. Equation (11) is an exact equation for $\mathbf{V}_1$. If one were to assume that $\mathbf{V}_1$ is close to $\mathbf{V}$, and then treat Eq. (11) as an approximate solution for $\mathbf{V}$, then that would then correspond to the inverse Born approximation. The latter assumption of $\mathbf{V} \approx \mathbf{V}_1$ is never made in the formalism of the inverse scattering series. The inverse Born approximation inputs the data D, and $\mathbf{G}_0$, and outputs $\mathbf{V}_1$ which is then treated as $\mathbf{V}$.

All of current seismic processing methods for imaging and inversion are different incarnations of using Eq. (11) to find an approximation for $\mathbf{V}$(see Stolt and Weglein (1985)), hence, the understandable and sustained effort to build ever more realism and completeness into the reference differential operator, $\mathbf{L}_0$ and its impulse response, $\mathbf{G}_0$. As with all technical approaches, the latter road (and current mainstream seismic thinking) eventually leads to a stage of maturity where further sustained effort will no longer bring a commensurate benefit. The inverse series methods provides a vennato achieve objectives beyond the reach of linear methods for a given level of a-priori information. Several additional comments: (1) the forward and inverse Born approximations are two separate and distinct methods: the forward Born approximation for the scattered field, Ψ_s, uses a linear truncation of Eq. (7) (and Eq. (8)) to estimate Ψ_s

$$\Psi_s \cong \mathbf{G}_0 \mathbf{V} \mathbf{G}_0$$

and inputs $\mathbf{G}_0$ and $\mathbf{V}$ to find an approximation to Ψ_s; the inverse Born approximation inputs

D and $\mathbf{G}_0$ and solves for $\mathbf{V}_1$ as $\mathbf{V}$ which it approximates by inverting

$$(\Psi_s)_m = D \cong (\mathbf{G}_0\mathbf{V}\mathbf{G}_0)_m$$

(2) the inverse series is a separate and distinct procedure from iterative linear inversion.

Iterative linear inversion would start with Eq. (11) and solve for $\mathbf{V}_1$. Then a new reference operator, $L'_0 = L_0 + \mathbf{V}_1$; impulse response, $\mathbf{G}'_0$ (where $\mathbf{L}'_0\mathbf{G}'_0 = -\delta$); and data, $D' = (\mathbf{G} - \mathbf{G}'_0)_m$ are input to a new linear inverse form

$$D' = (\mathbf{G}'_0\mathbf{V}'_1\mathbf{G}'_0)_m$$

where a new operator,$\mathbf{G}'_0$, has to now be removed (inverted) from both sides of $\mathbf{V}'_1$. These linear steps are iterated, note that at each step a new and, in general, more complicated operator (or matrix or frechetderivative) is required to be inverted. In contrast, the inverse-scattering series Eqs. (11)–(14) inverts the same and original input operator, $\mathbf{G}_0$, at each step. The inverse-scattering series methods were first developed by Moses (1956), Prosser (1969), Razavy (1975), and transformed for application to a multi-dimensional Earth and exploration seismic reflection data by Weglein, Boyse and Anderson (1981) and Stolt and Jacobs (1980). The first question in considering a series solution is the issue of convergence and if encouraging, followed closely by the question of rate of convergence. The important pioneering work on convergence criteria for the inverse series by Prosser (1969) is given as a condition which is difficult to translate into a statement on the size and duration of the contrast between actual and reference media. Faced with that lack of theoretical guidance, empirical tests of the inverse series were performed (Carvalho (1992)) for a 1D acoustic medium, which indicated that, starting with no a-priori information, convergence was observed but appeared to be restricted to small contrasts and duration (e.g., $< 11\%$ difference between actual earth acoustic velocity and water (reference) speed).

Since the acoustic wave speed in the earth quickly gets further than 11% from the acoustic wave speed in water (1500 m/sec) the practical value of the entire series, without a priori information, appeared to be quite limited.

A reasonable response might seem to be to use seismic methods that estimate the velocity trend of the earth to try to get the reference medium proximal to the actual, and that in turn could allow the series to possibly converge.

The problem with that thinking was that velocity trend estimation methods assumed that multiples were removed prior to that analysis. Furthermore, concurrent with these technical strategic decisions (around 1990) was the loud and clear message heard from petroleum industry operating units that multiple removal was on increasingly prioritized and serious problem and impediment to effectiveness.

Methods for removing multiples (at that time) assumed either: (1) the earth was 1D, (2), the velocity model was known, (3) reflectors generating the multiples could be defined, different patterns could be identified in waves from primaries and multiples, and (5) primaries were random and multiples periodic. All of these assumptions were seriously violated in deep-water and/or complex geology, and the methods based upon them often enough outright failed to perform, or produced erroneous or misleading results.

This interest in multiples came in large part from the industry trend to explore in deep water where the depth alone ($> 1km$) can cause, e.g., multiple removal methods based on periodicity arguments to seriously violate their assumptions. Complex multidimensional heterogeneous and hard to estimate geologic conditions and targets provided additional challenges for multiple removal methods that relied on having 1D assumptions or access to inaccessible details about the reflectors that were the source of these multiples.

The inverse scattering series was (and remains) the only multi-dimensional direct inversion formalism that could accommodate arbitrary heterogeneity directly in terms of $\mathbf{G}_0$ with estimated rather than actual propagation properties.

The confluence of these factors lead to the development of thinking that viewed inversion as a series of tasks or stages, and to view one of these as multiple removal.

2.3 Subseries

A combination of factors: (1) that the inverse series represent the only multidimensional direct seismic inversion; (2) numerical tests that suggested an apparent lack of robust convergence of the overall series, (when starting with no a-priori information), and, (3) the interest in extracting something of value from this only formalism for complete multi-dimensional inversion; and, (4) the industry need for more effective methods for removing multiply reflected events (multiples) from data collected over an unknown heterogeneous earth, all came together to imagine inversion in terms of steps or stages with intermediate objectives towards the ultimate goal of identifying earth material properties. The stages were each defined as achieving a task or objective: (1) removing free-surface multiples; (2) removing internal multiples; (3) imaging (locating) reflectors in space; and (4) determining the changes in earth material properties across those reflectors. The idea was to seek to identify within the overall series, and specific distinct subseries that performed these focused tasks and to evaluate those subseries for convergence, rate of convergence, data requirements and theoretical and practical prerequisites. Perhaps a subseries for one specific task would have a more favorable attitude towards, e.g., convergence in comparison to the entire series. These tasks, if achievable, could bring practical benefit on their own terms, and, if achievable, could be realized from the inverse-scattering series directly in terms of the data, D, and reference wave propagation, $\mathbf{G}_0$ and where $\mathbf{G}_0$ is not assumed to be proximal to the actual. At the outset, many important issues were open (and some remain open) that could cause a pause or hesitation in pursuing such a new task separation strategy. Among them are (1) does the series in fact uncouple in terms of tasks, (2) if it does uncouple, then how to identify those uncoupled task-specific subseries; (3) will the inverse series view multiples as noise to be removed, or as signal to be used for helping to image/invert the target; and (4) will the subseries derived require different algorithms (and computer codes) for different earth model types (e.g., acoustic version and elastic version) how can you know or determine, in a given application, how many terms in a subseries will be required to achieve a certain degree of effectiveness. We will address and respond to these questions in this paper and list others that are outstanding or the subject of current investigation. How to identify a task specific subseries? The pursuit of task specific subseries used several different types of analysis with testing of new concepts to evaluate, refine and develop embryonic thinking largely based on analogues and physical intuition. To begin, the forward and inverse series Eqs. (7) (8), and Eqs. (11)–(14) have a tremendous symmetry. The forward series produces the scattered wavefield, Ψ_s from a sum of terms each of which is composed of the operator, $\mathbf{G}_0$ acting on $\mathbf{V}$. When evaluated on the measurement surface, the forward series creates all of the data, $(\Psi_s)_m = D$ and contains all recorded primaries and multiples. The inverse series produces $\mathbf{V}$ from a series of terms each of which can be interpreted as the operator $\mathbf{G}_0$ acting on the recorded data, D. Hence, in scattering theory the same engine, $\mathbf{G}_0$, that acts on $\mathbf{V}$ to create data, acts on D to invert data. If we consider

$$< \mathbf{G}_0\mathbf{V}\mathbf{G}_0 >_m = < \mathbf{G}_0(\mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \dots)\mathbf{G}_0 >_m$$

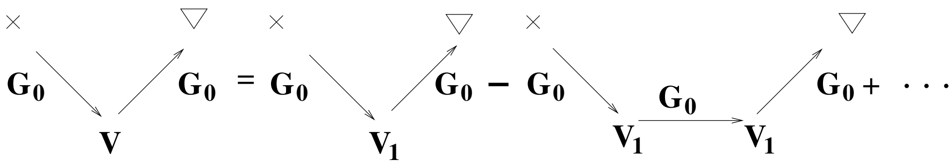
and use equations (12)–(14) we find

$$< \mathbf{G}_0\mathbf{V}\mathbf{G}_0 >_m = < \mathbf{G}_0\mathbf{V}_1\mathbf{G}_0 >_m - < \mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0 >_m + \dots \tag{15}$$

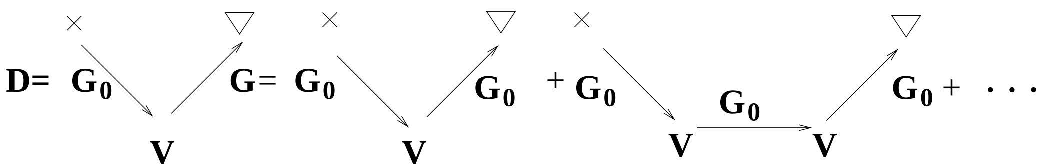
there is a remarkable symmetry between the inverse series Eq. (15) and the forward series Eq. (7)

$$(\Psi_s)_m = < \mathbf{G}_0\mathbf{V}\mathbf{G}_0 >_m + < \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 >_m + \dots \tag{7}$$

In terms of diagrams, the inverse series for $\mathbf{V}$, Eq. (15) can be represented as



while the forward series, Eq. (7) for the data, $(\Psi_s)_m \equiv D$, can be represented as



and, therefore, this diagram comparison indicates further opportunities for relating forward and inverse processes. The symbols $\times$ and $\bigtriangledown$ indicate a source and receiver, respectively. However, we know that the forward and inverse problems are not "inverses" in some more formal sense - meaning that the forward creates data but the inverse doesn't annihilate data, it inverts data. Never-the-less, the inverse scattering task specific subseries were thought to act on only specific subsets of the data, e.g., free surface multiples, internal multiples, and, imaging and inverting primaries. Hence, the guess was that if we could figure out how those events were created in the forward series in terms of $\mathbf{G}_0$ and $\mathbf{V}$, perhaps we could figure out how those events were processed in the inverse series when once again $\mathbf{G}_0$ was acting on D. That intuitive leap was later provided with a somewhat rigorous basis for free surface multiples, but the more challenging internal multiple attenuation subseries and the distinct subseries that image and invert primaries at depth without the velocity model while having attracted some insightful mathematical-physics rigor (Ten Kroode (2002)), remain with certain key steps in their logic based on plausibility, empirical tests, and physical intuition. In fact, for internal multiples understanding how the forward scattering series produces an event only hints at where the inverse process might be located. That 'hint' required and presently remains indebted to intuition, testing and subtle refinement of concepts to go to the location of the inverse operation. This is further internal. This last statement is neither an apology nor an expression of hubris, but a normal and expected stage in the development and evolution of new fundamentally concepts.

3 The marine case

For the marine case, with sources and receivers in the water column, the simplest reference medium is a half-space of water bound by a free surface at the air-water interface. The reference Green's function, $\mathbf{G}_0$, consists of two parts

$$\mathbf{G}_0 = {\mathbf{G}_0}^d + {\mathbf{G}_0}^{FS} \ , \tag{16}$$

where ${\mathbf{G}_0}^d$ is the direct propagating, causal, whole-space Green's function in water and ${\mathbf{G}_0}^{FS}$ is the additional part of the Green's function due to the presence of the free surface (see Fig. 3). ${\mathbf{G}_0}^{FS}$ corresponds to a reflection off the free surface. In the absence of a free

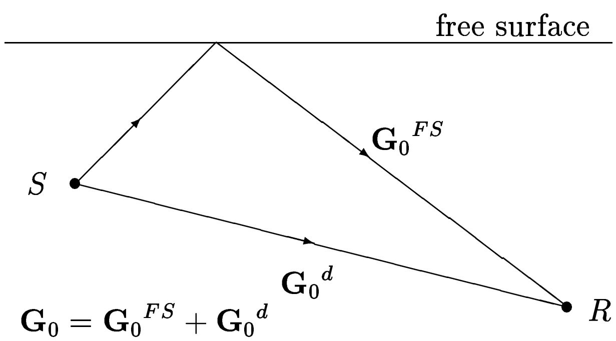


Figure 3: The marine configuration and reference Green's function.

surface, the reference medium is a whole-space of water and ${\mathbf{G}_0}^d$ is the reference Green's function. In this case, the forward series equation (7) describing the data is constructed from the direct propagating Green's function, ${\mathbf{G}_0}^d$, and the perturbation operator, $\mathbf{V}$. With our choice of reference medium, the perturbation operator characterizes the difference between earth properties and water; hence, the support of $\mathbf{V}$ begins at the water bottom. With the free surface present, the forward series is constructed from $\mathbf{G}_0 = {\mathbf{G}_0}^d + {\mathbf{G}_0}^{FS}$ and the same perturbation operator, $\mathbf{V}$. Hence, ${\mathbf{G}_0}^{FS}$ is the sole difference between the forward series with and without the free surface; therefore ${\mathbf{G}_0}^{FS}$ is responsible for generating those events that owe their existence to the presence of the free surface, i.e., ghosts and free-surface multiples. Ghosts are events that either start their history propagating from the source up to (and reflecting down from) the free-surface or end their history as the downgoing portion of the recorded wavefield at the receiver, having its last experience as a downward reflection at the free surface (see Fig.3).

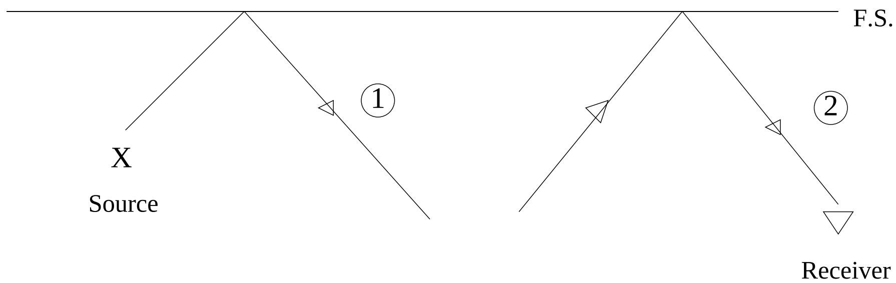


Figure 4: 1 and 2 correspond to source and receiver ghosts, respectively.

In the inverse series, equations (11)–(14), it is reasonable to infer that $\mathbf{G_0}^{FS}$ will be responsible for all the extra tasks that inversion needs to perform when starting with data containing ghosts and free-surface multiples rather than data without those events. Those extra inverse tasks include deghosting and the removal of free-surface multiples. In the section on the free-surface demultiple subseries that follows, we describe how the extra portion of the reference Green's function due to the free surface, $\mathbf{G_0}^{FS}$, performs deghosting and free-surface multiple-event removal.

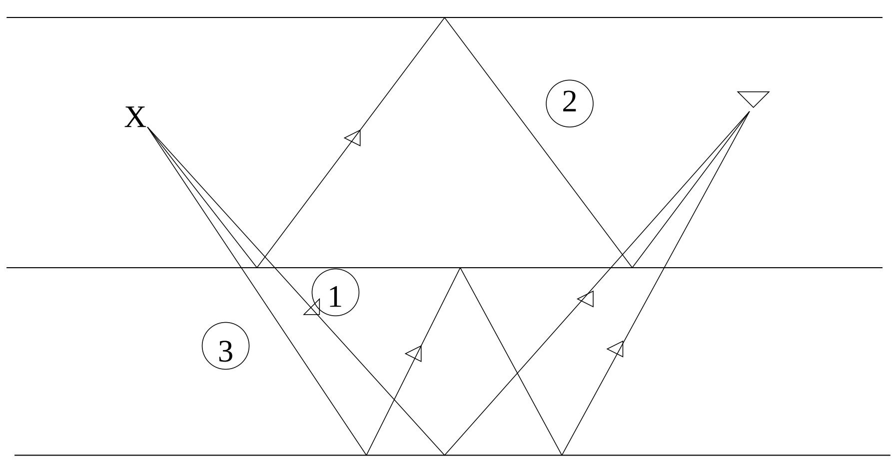


Figure 5: Deghosted Marine Data: 1, 2 and 3 represent deghosted primaries, free surface multiples and internal multiples, respectively.

Once the events associated with a free surface are removed, the remaining measured field consists of primaries and internal multiples. For a marine experiment absent of a free surface, the scattered field, Ψ'_s, can be expressed as a series in terms of a reference medium consisting of a whole-space of water, the reference Green's function, $\mathbf{G_0}^d$, and the perturbation, $\mathbf{V}$, as follows:

$$\begin{aligned}\Psi'_s &= \mathbf{G_0}^d\mathbf{V}\mathbf{G_0}^d + \mathbf{G_0}^d\mathbf{V}\mathbf{G_0}^d\mathbf{V}\mathbf{G_0}^d + \mathbf{G_0}^d\mathbf{V}\mathbf{G_0}^d\mathbf{V}\mathbf{G_0}^d\mathbf{V}\mathbf{G_0}^d + \ldots \\ &= (\Psi'_s)_1 + (\Psi'_s)_2 + (\Psi'_s)_3 \ldots \end{aligned} \tag{17}$$

The values of Ψ'_s on the measurement surface, D', are the data, D, absent of free-surface events; i.e., D' consists of primaries and internal multiples

$$D' = D'_1 + D'_2 + D'_3 + \ldots + D'_n + \ldots$$

where D'_m is the projection of $(\Psi'_s)_m$ on the measurement surface. Unfortunately, the free-space Green's function, $\mathbf{G_0}^d$, doesn't separate into a part responsible for primaries and a part responsible for internal multiples; a new concept was necessary to be introduced to separate the tasks associated with $\mathbf{G_0}^d$ (Weglein *et al.*, (1997)).

A seismic event represents the measured arrival of energy that has experienced a specific set of actual reflections, R, and transmissions, T, at reflectors and propagation, p, governed by medium properties between reflectors. A complete description of an event would typically consist of a single-term expression with all the actual episodes of R, T, and p in its history. The classification of an event in D' as a primary or an internal multiple depends on the number and type of actual reflections it has experienced.

In contrast, forward scattering describes data, D', in terms of a series. Each term of the series corresponds to a sequence of *reference* medium propagations, $\mathbf{G_0}^d$, and scatterings off the perturbation, $\mathbf{V}$. The scattering theory description of any specific event in D' also requires an infinite series necessary to build the actual R, T, and p's in terms of reference propagation, $\mathbf{G_0}^d$, and the perturbation operator, $\mathbf{V}$. That is, R, T, and p are nonlinearly related to $\mathbf{G_0}^d$ and $\mathbf{V}$. We will illustrate this with a simple example later in this section. Hence two chasms need to be bridged to determine e.g., the subseries that removes internal multiples. The first requires a map between primary and internal multiples in D' and their description in the language of forward scattering theory, $\mathbf{G_0}^d$ and $\mathbf{V}$; the second requires a map between the construction of internal multiple events in the forward series and the removal of these events in the inverse series.

The internal multiple attenuation concept requires the construction of these two dictionaries: one relates seismic events to a forward-scattering description, the second relates forward construction to inverse removal. The task separation strategy requires that those two maps to be determined . Both of these multidimensional maps were originally derived using arguments of physical intuition and mathematical reasonableness. Subsequently, Matson

(1996) provided, for 1-D constant-density acoustic media, a mathematically rigorous map of the relationship between seismic events and the forward scattering series. Recent work by Nita *et al.* (2003), and Innanan and Weglein (2003), extend that work to prestack analysis and absorptive media, respectively. Within the context of the important Matson paper, his results agree with and confirm the original intuitive arguments. The second map, relating forward construction and inverse removal, remains largely based on its original foundation of reasonableness. Recently, Ten Kroode (2002) presented a formal mathematical map for certain important aspects of the forward to inverse internal multiple map based on a leading order definition of internal multiples. For the purpose of this paper, we present only the key logical steps of the original arguments that lead to the required maps; the argument of the first map is presented here; the second map, relating forward construction and inverse removal, is presented in the next section.

To understand how the forward scattering series describes a particular event, it is useful to recall that the forward series for D' is a generalized Taylor series in the scattering operator, $\mathbf{V}$ (Keys and Weglein, 1983). But what is the forward scattering subseries for a given event in D'? Since a specific event consists of a set of actual R, T, and p factors, it is reasonable to start by asking how these individual factors are expressed in terms of the perturbation operator. Consider the simple example of one-dimensional acoustic medium consisting of a single interface and a normal-incidence plane wave, e^{ikz}, illustrated in Fig. 6.

Let the reference medium be a whole-space with acoustic velocity, c_0. The actual and reference differential equations describing the actual and reference wave fields, P and P_0, are:

$$\left[\frac{d^2}{dz^2}+\frac{\omega^2}{c^2(z)}\right]P(z,\omega)=0\ ,$$

and

$$\left[\frac{d^2}{dz^2}+\frac{\omega^2}{c_0^2}\right]P_0(z,\omega)=0\ ,$$

where $c(z)$ is the actual velocity.

The perturbation operator, $\mathbf{V}$, is

$$\mathbf{V}=L-L_0=\frac{\omega^2}{c^2(z)}-\frac{\omega^2}{c_0^2}\ .$$

Characterize $c(z)$ in terms of c_0 and the variation in index of refraction, α,

$$\frac{1}{c^2(z)}=\frac{1}{c_0^2}[1+\alpha(z)]\ .$$

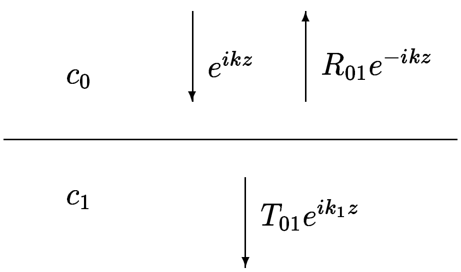


Figure 6: The 1-D plane-wave normal-incidence acoustic example.

In the lower half-space

$$\frac{1}{c_1^2} = \frac{1}{c_0^2}[1+\alpha] \ ,$$

α_1 essentially represents (within a constant factor of ω^2/c_0^2) the change in the perturbation operator at the interface. The reflection and transmission coefficients and the transmitted wave propagating in the lower half-space are

$$R_{01} = \frac{c_1 - c_0}{c_1 + c_0} \ ,$$
$$T_{01} = \frac{2c_1}{c_1 + c_0} \ ,$$

and

$$P_1 = T_{01} e^{i\frac{\omega}{c_1}z} = T_{01} p_1$$

Using

$$c_1 \equiv \frac{c_0}{(1+\alpha)^{\frac{1}{2}}} \cong c_0 \left[1 - \frac{1}{2}\alpha + h.p.(\alpha)\right] \ ,$$

these R, T, and p quantities are expandable as power series in the perturbation, α_1 ($h.p.$ denotes "higher powers of").

$$\begin{aligned} R_{01} &= -\frac{1}{4}\alpha + h.p.(\alpha) \ , \\ T_{01} &= 1 + h.p.(\alpha) \ , \\ p_1 &= e^{i\frac{\omega}{c_1}z} = e^{i\frac{\omega}{c_0}z} + h.p.(\alpha) \\ &= p_0 + h.p.(\alpha) \ . \end{aligned}$$

Thus, to lowest order in an expansion in the local perturbation, the actual reflection is proportional to the local change in the perturbation, the transmission is proportional to 1, and the actual propagation is proportional to the reference propagation. An event in D' consists of a combination of R, T and p episodes. The first term in the series that contributes to this event is determined by collecting the leading-order contribution (in terms of the local change in the perturbation operator) from each R, T and p factor in its history. Since the mathematical expression for an event is a *product* of all these actual R, T and p factors, it follows that the lowest order contribution, in the powers of the perturbation operator, will equal the number of R factors in that event. The fact that the forward series, Eq. (17), is a power series in the perturbation operator then allows us to identify the term in Eq. (3) that provides the first contribution to the construction of an event. Since by definition all primaries have only one R factor, their leading contribution comes with a single power of the perturbation operator; hence, from the first term of the series for D'. First-order internal multiples, with three factors of reflection, have their leading contribution with three factors of the perturbation operator; hence, the leading-order contribution to a first-order internal multiple comes from the third term in the series for D'. All terms in the series beyond the first make second-order and higher contributions for the construction of the R, T and p's of primaries; similarly, all terms beyond the third provide higher-order contributions for constructing the actual reflections, transmissions and propagations of first-order internal multiples. How do we separate the part of the third term in the forward series that provides a third-order contribution to primaries from the portion providing the leading-term contribution to first-order internal multiples?

The key to the separation resides in recognizing that the three perturbative contributions in D'_3 are located at the spatial *location* of reflectors. For a first-order internal multiple the leading-order contribution (illustrated on the right-hand-member of Fig. 7), consists of perturbative contributions located at the spatial location (depth) of the three reflectors where reflections occur. Specifically for the example in Fig. 7, the three linear approximations to R_{12}, R_{10}, R_{12}, that is, $\alpha_2 - \alpha_1$, α_1, $\alpha_2 - \alpha_1$ are located at depths z_1, z_2, z_3 where $z_1 > z_2$ and $z_3 > z_2$. In this single layer example $z_1 = z_3$. In general, D'_3 consists of the sum of all three perturbative contributions from any three reflectors at depths z_1, z_2, and z_3. The portion of D'_3 where the three reflectors satisfy $z_1 > z_2$ and $z_3 > z_2$ corresponds to the leading order construction of a first-order internal multiple involving those three reflectors. The part of D'_3 corresponding to the three perturbative contributions at reflectors that do not satisfy both of these inequalities, provide third-order contribution to the construction of primaries. A simple example is illustrated in Fig. 8.

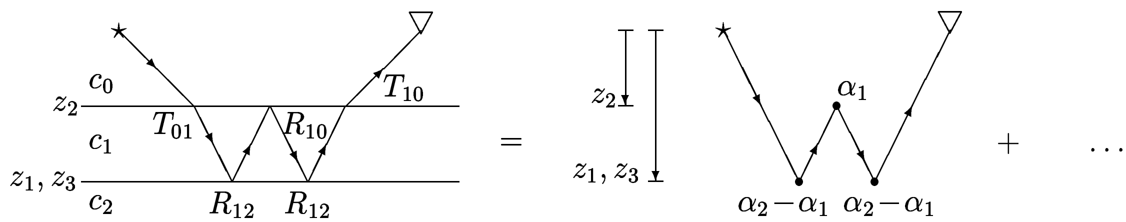


Figure 7: The left-hand-member of this diagram represents a first-order internal multiple; and the right-hand-member illustrates the first series contribution, from D_3', to its construction. α_1 and $\alpha_2 - \alpha_1$ are the perturbative contributions at the two reflectors; c_0, c_1 and c_2 are the acoustic velocities and $1/c_2^2 = 1/c_0^2(1+\alpha_2)$, $1/c_1^2 = 1/c_0^2(1+\alpha_1)$.

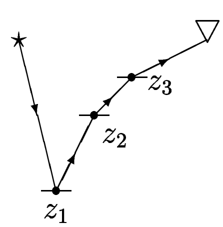


Figure 8: Diagram representing a portion of D_3' that makes a third-order contribution to the construction of a primary.

The *sum* of *all* the contributions in D_3' that satisfy $z_1 > z_2$ and $z_3 > z_2$ for locations of the three successive perturbations, is the sum of the leading contribution term for *all* first-order internal multiples. Similarly, second, third, $\cdots$, n-th order internal multiple find their initial contribution in the fifth, seventh, $\cdots$, $(2n+1)$-th term of the forward series. We use this identified leading-order contribution to all internal multiples of a given order in the forward series to suggest a map to the corresponding leading-order *removal* of all internal multiples of that order in the inverse series.

The forward map between the forward scattering series Eqs. (7) and (8) for $(\Psi_s)_m$ and the primaries and multiples of seismic reflection data works as follows. The scattering series builds the wavefield as a sum of terms with propagations $\mathbf{G}_0$ and scattering off $\mathbf{V}$. Scattering occurs in all directions from the scattering point $\mathbf{V}$ and the relative amplitude in a given direction determined by the isotropy (or anisotropy) of the scattering operator. An scattering operator being anisotropic is distinct from physical anisotropy, the latter of course means that the wave speed in the actual medium at a point is a function of the direction of propagation of the wave at that point. A two-parameter acoustic (isotropic) medium has an anisotropic scattering operator. (see Eq. (6)). In any case, since primaries and multiples are defined in terms of reflections, we imagine that primaries and internal multiples will be distinguished by the number of reflection like scatterings in their forward description. A reflection-like scattering is where the incident wave moves away from and the wave emerging from the scattering point moves towards the measurement surface. Every reflection event in

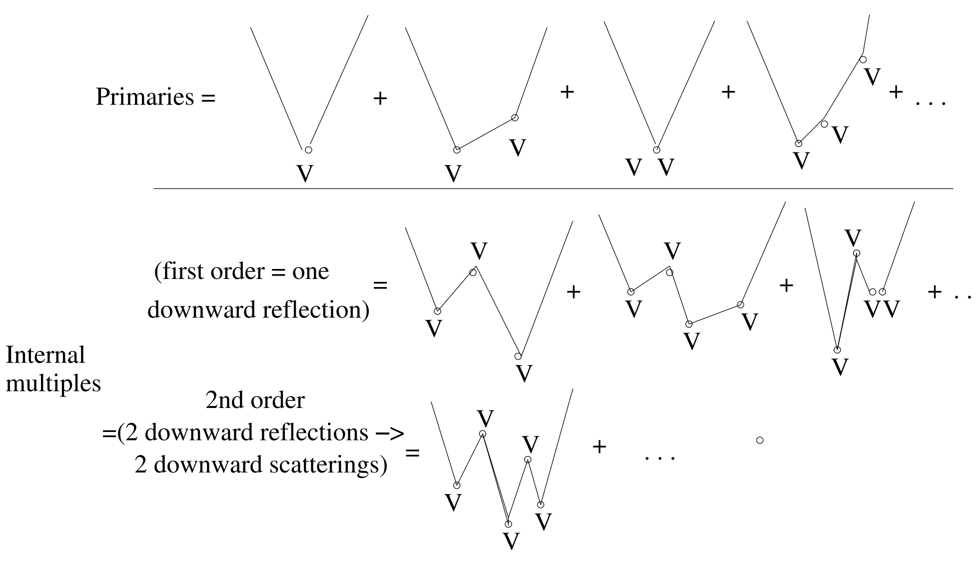


Figure 9: A scattering series description of primaries and internal multiples.

seismic data requires contributions from an infinite number of terms in the scattering theory description. Even with water as reference speed the simplest primaries, i.e., the water bottom reflection, requires an infinite number of contributions to take the scattering ingredients $\mathbf{G}_0$ and $\mathbf{V}$ into $\mathbf{G}_0$ and $\mathbf{R}$ where $\mathbf{V}$ and $\mathbf{R}$ correspond to the perturbation operator change and reflection coefficient at the water bottom, respectively. For a sub-water bottom primary the series has further issues to deal with beyond turning the local value of $\mathbf{V}$ into the local reflection coefficient, $\mathbf{R}$. For the latter case the reference Green's function, $\mathbf{G}_0$, no longer corresponds to the propagation down to and back from the reflector $(\mathbf{G} \neq \mathbf{G}_0)$, and the terms in the series beyond the first, $\mathbf{G}_0\mathbf{V}\mathbf{G}_0$, are required to correct, e.g., for the timing errors, and for ignoring transmission coefficients, in addition to taking $\mathbf{V}$ into $\mathbf{R}$.

The remarkable fact is that all primaries are constructed in the forward series by portions of every term in the series. The contributing part has one and only one upward reflection-like scattering. Furthermore, internal multiples of a given order (order is defined by the number of downward reflections, independent of the location of that reflector) have contributions from all terms that have exactly a number of reflection-like scatterings corresponding to the order of that internal multiple.

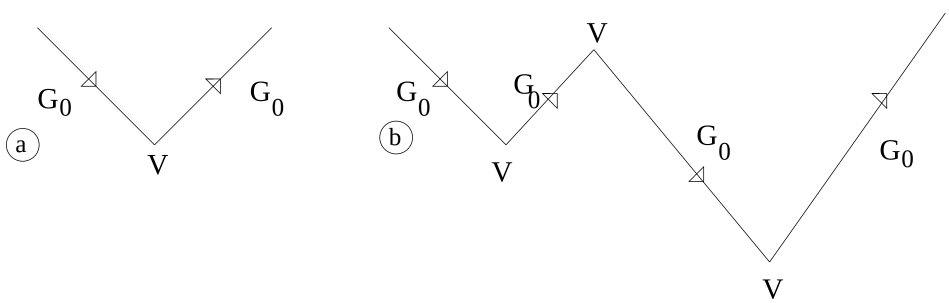


Figure 10: a)A reflection-like scattering for a primary. b)Three reflection-like scatterings contributing to a first order internal multiple.

The first term in the forward series for the data equations Eqs. (7) and (8),

$$((\Psi_s)_m)_1 = (\mathbf{G}_0\mathbf{V}\mathbf{G}_0)_m$$

where

$$(\Psi_s)_m = \sum_{n=1}^{\infty}[(\Psi_s)_m]_n$$

is an integral over the entire subsurface, where $\mathbf{V}$ resides, and approximates all primaries at once, as well as a single scattering model case, independent of depth. Of course the quality of the approximation represented by $(\Psi_s)_1$, depends on how many issues (e.g., phase, transmission coefficients and reflection coefficient) starting with $\mathbf{G}_0$, $\mathbf{V}$ into $\mathbf{G}$ and $\mathbf{R}$ are required by any particular event.

All internal multiples of first order begin their creation in the scattering series in the portion of the third term of $[(\Psi)_s]_m$, $[[(\Psi)_s]_m]_3$, with three reflection like scatterings. All terms in the fourth and high terms of $(\Psi_s)_m$ that consist of three and only three reflection−like scattering, plus any number of transmission−like and self−interactions also contribute to the construction of first order internal multiples.

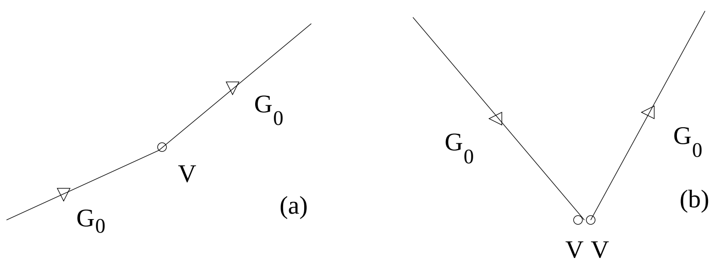


Figure 11: Examples of transmission (a) and self-interaction (b) scattering diagrams.

As mentioned, all of these conclusions were originally deduced based on physically intuitive arguments and later confirmed by analysis of the relationship between seismic events and the forward series for 1D media (Matson, 1996). Further research in the scattering theory descriptions of seismic events is warranted, and underway, and will strengthen the first of the two key logic links (maps) required for developments of more effective and better understood task specific inversion procedures.

Map I takes data into scattering series forward description. Map II takes scattering series description of seismic events to the inverse scattering series processing that are performed on those events. If you know how $\mathbf{G}_0$ and $\mathbf{V}$ make primaries and multiples, then perhaps you can figure out how $\mathbf{G}_0$ and $D(t)$ processes those same events.

4 The inverse series and task separation: terms with coupled and uncoupled tasks

As we mentioned, the fact that: (1) in the forward series ${\mathbf{G}_0}^{FS}$ is the agent that creates all events that come into existence due to the presence of the free surface (i.e., ghosts and free-surface multiples) and (2) that the inverse series starting with data that includes those free-surface related events - has additional tasks to perform (i.e., deghosting and free surface

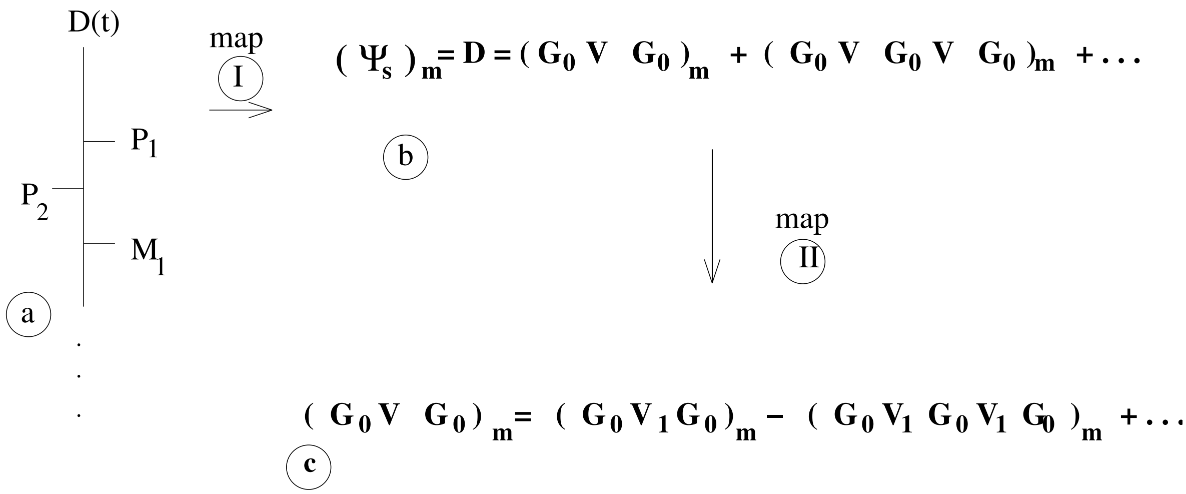


Figure 12: Two maps for inverse scattering subseries. a) $D(t) = (\Psi_s)_m$ consisting of primaries P_1, P_2, ... and multiples, M_1, ... b) $(\Psi_s)_m = D(t)$ as a forward series in terms of $\mathbf{G}_0$ and $\mathbf{V}$. c) The inverse series for $< \mathbf{G}_0\mathbf{V}\mathbf{G}_0 >_m = < \mathbf{G}_0\mathbf{V}_1\mathbf{G}_0 >_m - < \mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0 >_m + \ldots$ Map (I) takes seismic events to a scattering series description. Map (II) takes forward construction of events to inverse processing of those events.

multiple removal) on the way to constructing the perturbation, $\mathbf{V}$, and (3) that the forward and inverse engine, the reference Green's function, $\mathbf{G}_0$, consists of ${\mathbf{G}_0}^d$ plus ${\mathbf{G}_0}^{FS}$, for the marine case, would be taken together imply that ${\mathbf{G}_0}^{FS}$ would, in the inverse series, be the removal machine for the events it is responsible for having created in the forward series.

How to go from that thought to a deghosting and free surface multiple removal subseries? The inverse series expansions (11)-(14), in the marine case, consists of terms $< \mathbf{G}_0\mathbf{V}_n\ \mathbf{G}_0 >_m$ with $\mathbf{G}_0 = {\mathbf{G}_0}^d + {\mathbf{G}_0}^{FS}$. Deghosting is realized by removing the two outside $\mathbf{G}_0 = {\mathbf{G}_0}^d + {\mathbf{G}_0}^{FS}$ functions and replacing them with ${\mathbf{G}_0}^d$, a downgoing wave from source to $\mathbf{V}$ and an upgoing wave from $\mathbf{V}$ to the receiver. Details are provided in section 10.

After that deghosting operation $D =< \mathbf{G}_0\mathbf{V}_1\ \mathbf{G}_0 >_m$ to $\tilde{D} =< {\mathbf{G}_0}^d\mathbf{V}_1\ {\mathbf{G}_0}^d >_m$ where D and $\tilde{D}$ are $(\Psi_s)_m = (G - \mathbf{G}_0)_m$ and the source and receiver deghosted data, respectively. The objective is to remove free surface multiples from the deghosted data, $\tilde{D}$.

The terms in the series equations (11)-(14), with input $\tilde{D}$ replacing D, contain both ${\mathbf{G}_0}^d$ and ${\mathbf{G}_0}^{FS}$ between the operators $\mathbf{V}_1$. The terms in the series are of three types, e.g.,

$$
\begin{aligned}
Type1 &: \left({\mathbf{G}_0}^d\mathbf{V}_1{\mathbf{G}_0}^{FS}\mathbf{V}_1{\mathbf{G}_0}^{FS}\mathbf{V}_1{\mathbf{G}_0}^d\right)_m; \\
Type2 &: \left({\mathbf{G}_0}^d\mathbf{V}_1{\mathbf{G}_0}^{FS}\mathbf{V}_1{\mathbf{G}_0}^d\mathbf{V}_1{\mathbf{G}_0}^d\right)_m;
\end{aligned}
$$

and

$$Type3 \; : \; \left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\right)_m .$$

From an isolated task point of view we interpret these types of terms as: Type (1) when only $\mathbf{G_0}^{FS}$ appears between two $\mathbf{V}_1$ contributions then the term removes free surface multiples (when added to $\tilde{D}$) and no other task is to be performed; Type (2): when both $\mathbf{G_0}^d$ and $\mathbf{G_0}^{FS}$ appear between two $\mathbf{V}_1$ contributions, then a free surface multiple removal plus a task associated with $\mathbf{G_0}^d$ are *both* to be performed and; Type (3), when only $\mathbf{G_0}^d$ appears between two $\mathbf{V}_1$ contributions, then no free surface multiples are removed by that term. The two outside G_0^d merely denotes that the data has been deghosted.

The idea behind task separated subseries is two fold: (1) isolate the terms in the overall series that perform a given task *as if no other task exists* (i.e., Type 1 above) and (2) do not return to the original inverse series with its coupled tasks involving $\mathbf{G_0}^{FS}$ and $\mathbf{G_0}^d$, but rather restart the problem with an input data, D', (equation (11”)), absent of free surface multiples. Collecting all Type 1, $\mathbf{G_0}^{FS}$ terms we have

$$\begin{aligned}
D'_1 &\equiv \tilde{D} = \left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\right)_m && (11')\\
D'_2 &= -\left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^{FS}\mathbf{V}_1\mathbf{G_0}^d\right)_m && (12')\\
D'_3 &= +\left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^{FS}\mathbf{V}_1\mathbf{G_0}^{FS}\mathbf{V}_1\mathbf{G_0}^d\right)_m && (13')\\
&\vdots
\end{aligned}$$

and $D' = \sum_{i=1}^{\infty} D'_i$ is the deghosted and free-surface multiple removed data. The data D' consists of primaries and internal multiples and an inverse series for $\mathbf{V} = \sum_{i=1}^{\infty}\mathbf{V}'_i$ where $\mathbf{V}'_i$ is the portion of $\mathbf{V}$ first order in primaries and internal multiples

$$\begin{aligned}
D' &= \left(\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\right)_m && (11'')\\
\left(\mathbf{G_0}^d V'_2\mathbf{G_0}^d\right)_m &= -\left(\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\right)_m && (12'')\\
\left(\mathbf{G_0}^d V'_3\mathbf{G_0}^d\right)_m &= -\left(\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\right)_m \\
&\quad -\left(\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d V'_2\mathbf{G_0}^d\right)_m \\
&\quad -\left(\mathbf{G_0}^d\mathbf{V_2}'\mathbf{G_0}^d\mathbf{V_1}'\mathbf{G_0}^d\right)_m && (13'')\\
&\vdots \quad .
\end{aligned}$$

$\mathbf{G_0}^d$ creates primaries and internal multiples in the forward series and is responsible for inverse tasks on the same events in the inverse.

We repeat this process for removing internal multiples seeking to isolate terms that only care about this one and only responsibility of ${\mathbf{G}_0}^d$. No coupled task terms (e.g., that involve tasks concerned with both internal multiples and primaries) are included.

After that is accomplished and internal multiples are attenuated, restart the problem, once again, to write an inverse series whose input consists only of primaries. This task isolation and restarting the definition of the inversion procedure strategy has several advantages over a rigid fixation with the original series. Those advantages includes the recognition that a task has already been accomplished is a form of new information and makes subsequent tasks in our list that are often progressively more difficult, considerably less daunting, especially compared to the original all-inclusive data series approach. For example, after removing multiples with a reference medium of water speed, it is easier to estimate a variable background to aid. Transforming to a simpler data with fewer tasks to perform has serious advantages over the strict adherence to the original series for

$$\mathbf{V} = \sum_{i=1}^{\infty} \mathbf{V}_i.$$

Note that the $\mathbf{V}$ the difference between water and earth properties is the same in $\mathbf{V} = \sum_{i=1}^{\infty} \mathbf{V}_i$ and $\mathbf{V} = \sum_{i=1}^{\infty} \mathbf{V}'_i$, but $\mathbf{V}_i \neq \mathbf{V}'_i$ since $\mathbf{V}_i$ assumes the data is D (primaries and all multiples) and ${\mathbf{V}_i}'$ assumes the data is D' (primaries and only internal multiples),e.g., $\mathbf{V}_1$ is linear in all primaries, free surface and internal multiples, while $\mathbf{V}'_1$ is linear in all primaries and internal multiples.

5 An analysis of the Earth model-type and the inverse series and sub-series

To invert for medium properties requires choosing a set of parameters that you seek to identify. The chosen set of parameters (e.g. P and S wave velocity and density) defines an Earth model-type (e.g. acoustic, elastic, isotropic, anisotropic earth), and the details of the inverse series will depend on that choice. Choosing an earth model-type defines the form of L, L_0 and $\mathbf{V}$.

On the way towards identifying the earth properties, (for a given model type), intermediate tasks are performed, such as the removal of free surface and internal multiplies and the location of reflectors in space.

It will be shown below that the free surface and internal multiple attenuation sub-series not only do not require subsurface information for a given model type, they are even independent of earth model type for a very large class of models. The meaning of model type independent task specific subseries is that the defined task is achievable with precisely the same algorithm for an entire class of earth model-types. The members of the model type class we are considering satisfy the convolution theorem, and include acoustic, elastic and certain anelastic media.

In this section, we provide a more general and complete formalism for the inverse series and especially the sub-series that has appeared in the literature to-date. That formalism allows us to examine the issue of model-type and inverse scattering objectives. Finally, when we discuss the imaging and inversion subseries in §7, we use this general formalism as a framework for defining and addressing the new challenges we face in developing subseries that perform imaging at depth without the velocity and inverting large contrast complex targets. All inverse methods for identifying an objective function or medium properties require specification of the parameters to be determined, i.e., of the assumed earth-model type that has generated the scattered wavefield.

To understand how the free surface multiple removal and internal multiple attenuation task specific subseries avoid this requirement, it is instructive to examine the mathematical-physics and logic behind the classic inverse series and see precisely the role model type plays in the derivation.

References for the inverse series include: *Moses, H. E. 1956, Razavy, M., 1975, Weglein, A.B., Boyse, W.E., and Anderson, J.E., 1981, Stolt, R.H., and Jacobs, B., 1980.* In an outline: the inverse series paper by Razavy (1975) is a lucid and important paper relevant to seismic exploration. In that paper, Razavy considers a normal plane wave incident on a one dimensional acoustic medium. We follow the Razavy (1975) development to see precisely how model type enters, and, to glean further physical insight from the mathematical procedure. Then we introduce a perturbation operator, $\mathbf{V}$, general enough in structure to accommodate the entire class of earth model types under consideration.

Finally, if a process (i.e., a subseries) can be performed without specifying how $\mathbf{V}$ depends on the earth property changes, (i.e., what set of earth properties are assumed to vary inside $\mathbf{V}$), the process itself is independent of earth model type.

5.1 Inverse series for a 1-D acoustic constant density medium

Start with the 1-D variable velocity, constant density acoustic wave equation, where $c(x)$ is the wave speed and $\Psi(x,t)$ is a pressure field at location x at time t. The equation that $\Psi(x,t)$ satisfies is

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2(x)}\frac{\partial^2}{\partial t^2}\right)\Psi(x,t) = 0 \tag{18}$$

and after a temporal Fourier transform, $t \to \omega$,

$$\left(\frac{d^2}{dx^2} + \frac{\omega^2}{c^2(x)}\right)\Psi(x,\omega) = 0. \tag{19}$$

Characterize the velocity configuration $c(x)$ in terms of a reference velocity, c_0, and perturbation, V

$$\frac{1}{c^2(x)} = \frac{1}{c_0^2}\left(1 - V(x)\right)\ . \tag{20}$$

The experiment consists of a plane wave e^{ikx} where $k = \frac{\omega}{c_0}$ incident upon $V(x)$ from the left (see Fig. (13)). Assume here that V has compact support and that the incident wave approaches $V(x)$ from the same side of $V(x)$ that the scattered field is measured.

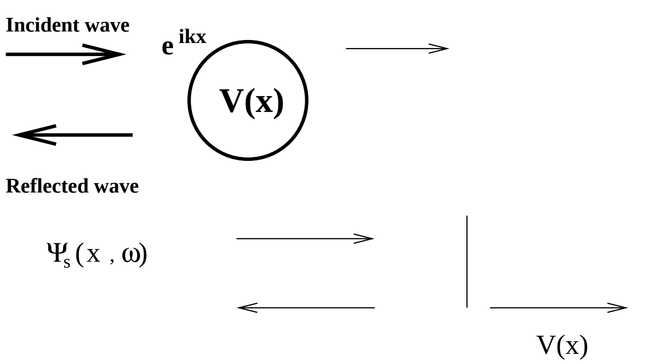


Figure 13: The scattering experiment: a plane wave incident upon the perturbation, V.

Let $b(k)$ denote the overall reflection coefficient for $V(x)$. It is determined from the reflection data at a given frequency ω. e^{ikx} and $b(k)e^{-ikx}$ are the incident and the reflected waves respectively. Rewrite (19) and (20) and the incident wave boundary condition as an integral equation,

$$\Psi(x,\omega) = e^{ikx} + \frac{1}{2ik}\int e^{ik|x-x'|}k^2 V(x')\Psi(x',\omega)dx' \tag{21}$$

and define the scattered field Ψ_s

$$\Psi_s(x,\omega) \equiv \Psi(x,\omega) - e^{ikx}.$$

Also, define the T matrix

$$T(p,k) \equiv \int e^{-ipx} V(x) \Psi(x,k) dx \tag{22}$$

and the Fourier sandwich of the parametrization, V

$$V(p,k) \equiv \int e^{-ipx} V(x) e^{ikx} dx \ .$$

The scattered field, Ψ_s takes the form

$$\Psi_s(x,\omega) = b(k) e^{-ikx} \tag{23}$$

for values of x less than the support of $V(x)$.

From equations (21), (22) and (23) it follows that

$$T(-k,k)\frac{k}{2i} = b(k). \tag{24}$$

Multiply Eq. (21) by $V(x)$ and then Fourier transform over x to find

$$T(p,k) = V(p,k) - k^2 \int_{-\infty}^{\infty} \frac{V(p,q)T(q,k)}{q^2 - k^2 - i\epsilon} dq \tag{25}$$

where p is the Fourier conjugate of x. Razavy (1975) also derives another integral equation by exchanging the roles of unperturbed and perturbed with L_0 viewed as a perturbation of $-V$ upon L,

$$V(p,k) = T(p,k) + k^2 \int_{-\infty}^{\infty} \frac{T^*(k,q)T(p,q)}{q^2 - k^2 - i\epsilon} dq \ . \tag{26}$$

Finally, define $W(k)$ as essentially the Fourier transform of the sought after perturbation, V

$$W(k) \equiv V(-k,k) = \int_{-\infty}^{\infty} e^{2ikx} V(x) dx \tag{27}$$

and recognize that predicting $W(k)$ for all k produces $V(x)$.

From Eq. (26) we find after setting $p = -k$,

$$W(k) = V(-k,k) = T(-k,k) + k^2 \int_{-\infty}^{\infty} \frac{T^*(k,q)T(-k,q)}{q^2 - k^2 - i\epsilon} dq \ . \tag{28}$$

The left hand member of Eq. (28) is the desired solution, $W(k)$, but the right hand member requires both $T(-k,k)$ (that we determine from $\frac{2i}{k}b(k)$) and $T^*(k,q)T(-k,q)$ for all q.

We cannot directly determine $T(k,q)$ for all q from measurements outside V.

If we could determine $T(k,q)$ for all q, then (28) would represent **a closed form** solution to the (multidimensional) inverse problem. If $T(-k,k)$ relates to the reflection coefficient, then what does $T(k,q)$ mean for all q?

Let us start with the integral form for the scattered field

$$\Psi_s(x,k) = \frac{1}{2\pi}\int\int \frac{e^{ik'(x-x')}}{k^2-k'^2}dk'k^2V(x')\Psi(x',k)dx' \tag{29}$$

and Fourier transform (29) going from the configuration space variable, x, to the wave number p to find

$$\Psi_s(p,k) = \int\int \frac{\delta(k'-p)e^{-ik'x'}}{k^2-k'^2-i\epsilon}dk'k^2V(x')\Psi(x',k)dx' \tag{30}$$

and if integrate over k' to find

$$\Psi_s(p,k) = \frac{k^2}{k^2-p^2-i\epsilon}\int e^{-ik'x'}V(x')\Psi(x',k)dx'. \tag{31}$$

The integral in Eq. (31) is recognized from equation (22) as

$$\Psi_s(p,k) = k^2\frac{T(p,k)}{k^2-p^2-i\epsilon}. \tag{32}$$

Therefore to determine $T(p,k)$ for all p for any k is to determine $\Psi_s(p,k)$ for all p and any k ($k=\frac{w}{c_0}$).

But to find $\Psi_s(p,k)$ from $\Psi_s(x,k)$ you need to compute

$$\int_{-\infty}^{\infty} e^{-ipx}\Psi_s(x,k)dx \tag{33}$$

i.e. it requires $\Psi_s(x,k)$ at every x, (not just at the measurement surface, i.e. a fixed x value outside of V).

Hence (28) would provide $W(k)$ and therefore $V(x)$, if we provide not only reflection data $b(k)=T(-k,k)\frac{2i}{k}$ but the scattered field, Ψ_s, at all depths, x.

Since knowledge of the scattered field, Ψ_s (and, hence, the total field), at all x could be used in equation (19) to directly compute $c(x)$, at all x, there is not much point or value in treating Eq. (28) in its pristine form as a complete and direct inverse solution.

Moses (1956) first presented a way around this dilemma. His thinking resulted in the inverse scattering series and consisted of two necessary and sufficient ingredients:

(1) model type combined with (2) A solution for $V(x)$, (and all quantities that depend on V) order by order in the data, $b(k)$.

Expand $V(x)$ as series in orders of the measured data

$$V = V_1 + V_2 + V_3 + \cdots = \sum_{n=1}^{\infty} V_n \tag{34}$$

where V_n is $n-$th order in the data D. When the inaccessible $T(p,k)$, $|p| \neq |k|$ are ignored, Eq. (28) becomes the Born-Heitler approximation and a comparison to the inverse Born approximation (the Born approximation ignores the entire second term of the right hand member of Eq. (28)) was analyzed in Devaney and Weglein (1985).

It follows that all quantities that are power series (starting with power one) in V are also power series in the measured data.

$$T(p,k) = T_1(p,k) + T_2(p,k) + \ldots \tag{35}$$

$$W(k) = W_1(k) + W_2(k) + \ldots \tag{36}$$

$$V(p,k) = V_1(p,k) + V_2(p,k) + \ldots . \tag{37}$$

The model type, in this simple acoustic case, provides a key relationship

$$V(p,k) = W\left(\frac{k-p}{2}\right) \tag{38}$$

that constrains the Fourier sandwich, $V(p,k)$, to be a function of only the difference between k and p. This model-type (acoustic constant density model), combined with order by order construction of the $T(p,k)$ for $p \neq k$ required by the series, provides precisely what we need to solve for $V(x)$.

Start with the measured data, $b(k)$, and substituting $W = \sum W_n$, $T = \sum T_n$ from equations (35) and (36) into Eq. (28) we find

$$\sum_{n=1}^{\infty} W_n(k) = \frac{2i}{k} b(k) + k^2 \int \frac{dq}{q^2 - k^2 - i\epsilon} \left(\sum_{n=1}^{\infty} T_n^* \sum_{n=1}^{\infty} T_n \right) . \tag{39}$$

To first order in the data, $b(k)$, $k > 0$ (note that $b^*(+k) = b(-k)$, $k > 0$)

$$W_1(k) = \frac{2i}{k} b(k) \tag{40}$$

and Eq. (40) determines $W_1(k)$ for all k. From Eq. (40) together with Eq. (27) to first order in the data

$$W_1(k) = V_1(-k, k) = \int_{-\infty}^{\infty} V_1(x) e^{2ikx} dx \tag{41}$$

we find $V_1(x)$. The next step towards our objective of constructing $V(x)$ is to find $V_2(x)$.

From $W_1(k)$ we can determinate $W_1(\frac{k-p}{2})$ for all k and p and from Eq. (38) to first order in the data

$$V_1(p, k) = W_1\left(\frac{k-p}{2}\right) \tag{42}$$

which in turn provides $V_1(p, k)$ for all p, k.

(Here is model type in action: The acoustic model with variable velocity and constant density).

Next we go to Eq. (26) to first order $V_1(p, k) = T_1(p, k)$ for all p and k, and substituting in (28) we get the second order in the data

$$W_2(k) = k^2 \int_{-\infty}^{\infty} \frac{dq}{q^2 - k^2 - i\epsilon} T_1^*(k, q) T_1(-k, q) \tag{43}$$

and

$$W_2(k) = \int_{-\infty}^{\infty} e^{2ik} V_2(x) dx. \tag{44}$$

After finding $V_2(x)$ we can repeat the steps to determine the total V order by order

$$V = V_1(x) + V_2(x) + \ldots$$

Order-by-order arguments and model type allow

$$T_1(p, k) = V_1(p, k)$$

although

$$T(p, k) \neq V(p, k) \ .$$

From a physics and information content point-of-view what has happened? The data D collected at e.g. $x = 0$, $\Psi_s(x = 0, \omega)$ determines $b(k)$. This in turn allows the construction of $T(p, k)$, $(k = \omega/c_0)$ for all p order by order in the data. Hence the required scattered wavefield at depth, represented by $T(p, k)$, $k = \omega/c_0$, for all p, for Eq. (28) is constructed order-by-order, for a single temporal frequency, ω, using the model type constraint. The data at one depth for all frequencies is traded for the wavefield at all depths at one frequency. This observation, that in constructing the perturbation, $V(x)$, order-by-order in the data, the actual wavefield at depth is constructed, represents an alternate path or strategy for seismic inversion (see Weglein *et al.* (2000)).

If the inverse series makes these model type requirements for its construction how do the free surface and internal multiple sub-series work independent of earth model type? What can we anticipate about the attitude of the imaging and inversion at depth sub-series with respect to these model type dependence issues?

5.2 The operator V for a class of earth-model types

Consider, once again, the variable velocity, variable density acoustic wave equation

$$\left(\frac{\omega^2}{K} + \nabla \cdot \frac{1}{\rho}\nabla\right) P = 0 \tag{45}$$

where K and ρ are the bulk modulus and density, and can be written in terms of reference values and perturbations a_1 and a_2

$$\frac{1}{K} = \frac{1}{K_r}(1 + a_1) \qquad \frac{1}{\rho} = \frac{1}{\rho_r}(1 + a_2)$$

$$L_0 = \frac{\omega^2}{K_r} + \nabla \cdot \frac{1}{\rho_r}\nabla \tag{46}$$

$$\mathbf{V} = \frac{\omega^2}{K_r} a_1(\vec{r}) + \left(\nabla \cdot \frac{a_2(\vec{r})}{\rho_r}\nabla\right) . \tag{47}$$

We will assume a 2-D earth with line sources and receivers, (the 3-D generalization is straightforward). A Fourier sandwich of this $\mathbf{V}$ is

$$V(\vec{p}, \vec{k}; \omega) = \int e^{-i\vec{p}\cdot\vec{r}} V e^{i\vec{k}\cdot\vec{r}} d\vec{r} = \frac{\omega^2}{K_r} a_1(\vec{k} - \vec{p}) + \frac{\vec{k}\cdot\vec{p}}{\rho_r} a_2(\vec{k} - \vec{p}) \tag{48}$$

where $\vec{p}$ and $\vec{k}$ are arbitrary 2D vectors. Green's Theorem and the compact support of a_1 and a_2 help Eq. (47) to Eq. (48). For an isotropic elastic model, equation (48) generalizes for $\mathbf{V}_{pp}$ (see Stolt and Weglein (1985), Boyse (1986), and Boyse and Keller (1986))

$$V_{pp}(\vec{p},\vec{k};\omega) = \frac{\omega^2}{K_r}a_1(\vec{k}-\vec{p}) + \frac{\vec{k}\cdot\vec{p}}{\rho_r}a_2(\vec{k}-\vec{p}) - \frac{2\beta_0^2}{\omega^2}|\vec{k}\times\vec{p}|^2 a_3(\vec{k}-\vec{p}) \tag{49}$$

where a_3 is the relative change in shear modulus and β_0 is the shear velocity in the reference medium.

The inverse series procedure can be extended for perturbation operators (48) or (49), but the detail will differ for these two models. The model-type and order-by-order arguments still hold. Hence 2-D (3-D) general perturbative form will be

$$V(\vec{p},\vec{k};\omega) = V_1(\vec{p},\vec{k};\omega) + \cdots$$

where $\vec{p}$ and $\vec{k}$ are two dimensional (or 3D) independent wave-vectors (that can accommodate a set of earth model types that include acoustic, elastic and certain anelastic forms).

- **ACOUSTIC**

$$\mathbf{V} = \frac{\omega^2}{\alpha_0^2}a_1$$

- **ACOUSTIC** (density variable)

$$\mathbf{V} = \frac{\omega^2}{\alpha_0^2}a_1 + \vec{k}\cdot\vec{k}'a_2$$

- **ELASTIC** isotropic (p-p)

$$\mathbf{V} = \frac{\omega^2}{\alpha_0}a_1 + \vec{k}\cdot\vec{k}'a_2 - 2\frac{\beta_0^2}{\omega^2}|\vec{k}\times\vec{k}'|^2 a_3$$

$a_1 \equiv$ Relative change in the bulk modulus.

$a_2 \equiv$ Relative change in density.

$a_3 \equiv$ Relative change in shear modulus. What can we compute in the inverse series without specifying how **V** depends on $(a_1), (a_1, a_2), \cdots$? If we can achieve a task in the inverse series without specifying what parameters **V** depends on, then that task can be attained with the same identical algorithm **independent** of earth-model type.

5.3 Free surface and internal multiple subseries and model-type independence

In equations (11)–(14), we presented the general inverse scattering series without specifying the nature of the reference medium that determines $\mathbf{L}_0$ and $\mathbf{G}_0$ and the class of earth model types that relate to the form of $\mathbf{L}$, $\mathbf{L}_0$ and $\mathbf{V}$. In this section, we present the explicit inverse scattering series for the case of marine acquisition geometry. This will also allow the issue of model-type independence to be analyzed in the context of marine exploration.

The reference medium is a half-space, with the acoustic properties of water, bounded by a free surface at the air-water interface, located at $z = 0$. We consider a 2-D medium, and assume that a line source and receivers are located at (x_s, ϵ_s) and (x_g, ϵ_g), where ϵ_s and ϵ_g are the depths below the free surface of the source and receivers, respectively.

The reference operator, $\mathbf{L}_0$, satisfies

$$\begin{aligned} \mathbf{L}_0\mathbf{G}_0 &= \left(\frac{\nabla^2}{\rho_0} + \frac{\omega^2}{\kappa_0}\right)\mathbf{G}_0(x, z, x', z'; \omega) \\ &= -\delta(x - x')\big\{\delta(z - z') - \delta(z + z')\big\}\ , \end{aligned} \tag{50}$$

where ρ_0 and κ_0 are the density and bulk modulus of water, respectively. The two terms on the right member of Eq. (10), correspond to the actual source located at (x', z') and the image of this source, across the free surface, at $(x', -z')$, respectively; (x, z) is any point in 2-D space.

The actual medium is a general earth model with associated wave operators, $\mathbf{L}$, and Green's function, $\mathbf{G}$. Fourier transforming Eq. (10) with respect to x, we find:

$$\left[\frac{1}{\rho_0}\frac{d^2}{dz^2} + \frac{q^2}{\kappa_0}\right] G_0(k_x, z, x', z'; \omega) = \\ -\frac{1}{(2\pi)^{1/2}} e^{-ik_x x'}\big\{\delta(z - z') - \delta(z + z')\big\}\ . \tag{51}$$

The causal solution of Eq. (11) is

$$G_0(k_x, z, x', z'; \omega) = \frac{\rho_0}{\sqrt{2\pi}}\frac{e^{-ik_x x'}}{-2iq}\left(e^{iq|z-z'|} - e^{iq|z+z'|}\right)\ , \tag{52}$$

where the vertical wave number, q, is defined as

$$q = \text{sign }(\omega)\sqrt{(\omega/c_0)^2 - k_x^2}\ ,$$

and c_0 is the acoustic velocity of water

$$c_0 = \sqrt{\kappa_0/\rho_0} \ .$$

With G_0 given by Eq. (12), the linear form equation (11) can be written as

$$D(k_g, \epsilon_g, k_s, \epsilon_s; \omega) = \frac{\rho_0^2}{q_g q_s} \sin(q_g \epsilon_g) \sin(q_s \epsilon_s) V_1(k_g, q_g, k_s, q_s, \omega) \ , \tag{53}$$

where $V(\vec{k_g}, \vec{k_s}, \omega) = V_1(\vec{k_g}, \vec{k_s}, \omega) + V_2(\vec{k_g}, \vec{k_s}, \omega) + \dots$ and $\vec{k_g}, \vec{k_s}$ are arbitrary two dimensional vectors. The variable k_z is defined as

$$k_z = -(q_g + q_s) \ ,$$

where

$$q_g = \text{sign } (\omega) \sqrt{(\omega/c_0)^2 - k_g^2} \ , \tag{54}$$

and

$$q_s = \text{sign } (\omega) \sqrt{(\omega/c_0)^2 - k_s^2} \ . \tag{55}$$

The first term in the inverse series (in two dimensions) in equation (11') in terms of deghosted data, $\tilde{D}$ is

$$\frac{D}{(e^{2iq_g \epsilon_g} - 1)(e^{2iq_s \epsilon_s} - 1)} = \mathbf{G_0}^d \mathbf{V}_1 \mathbf{G_0}^d = \tilde{D}(k_g, \epsilon_g, k_s, \epsilon_s; \omega) \tag{56}$$

Using the bilinear form for $\mathbf{G_0}^d$ on both sides of $\mathbf{V}_1$ in Eq. (56) and Fourier transforming both sides of this equation with respect to x_s and x_g we find

$$e^{iq_g \epsilon_g} e^{iq_s \epsilon_s} \frac{V_1(\vec{k_g}, \vec{k_s}; \omega)}{q_g q_s} = D(k_g, \epsilon_g, k_s, \epsilon_s; \omega) \tag{57}$$

where $\vec{k_g}$ and $\vec{k_s}$ are now constrained by $|\vec{k_g}| = |\vec{k_s}| = \frac{\omega}{c_0}$ in the left-hand member of Eq. (57).

In a 2D world only the three dimensional projection of the five dimensional $V_1(\vec{p}, \vec{k}; \omega)$ is recoverable from the surface measurements $D(k_g, \epsilon_g, k_s, \epsilon_s; \omega)$ which is a function of three variables, as well.

It is important to recognize that you cannot determine $\mathbf{V}_1$ for a general operator $\mathbf{V}_1(\vec{r_1}, \vec{r_2}; \omega)$ or $\mathbf{V}_1(\vec{k'}, \vec{k}; \omega)$ from surface measurements on m, and only the three dimensional projection of $\mathbf{V}_1(\vec{k'}, \vec{k}; \omega)$ with $|\vec{k}| = |\vec{k'}| = \frac{\omega}{c_0}$ is recoverable.

However this three dimensional projection of $\mathbf{V}_1$ is more than enough to compute the first order changes in any number of two dimensional earth model parameters; $a_i^1(\vec{r})$ for a given earth model type. ($a_i^1 \equiv$ First order approximation to a_1, a_2, a_3 $\ldots$).

After solving for $a_1^1(\vec{r})$, $a_2^1(\vec{r})$, $a_3^1(\vec{r})$ $\ldots$ $\ldots$, you could then use $a_1^1, a_2^1, a_3^1, \ldots$ to compute $\mathbf{V}_1(\vec{k'}, \vec{k}, \omega)$ for all $\vec{k'}, \vec{k}, \omega$ where $a_i'(\vec{r}) \equiv$ linear (or Born) inversion of $a_i(\vec{r})$.

This is the direct extension of the first step of the Moses (1956) procedure where model type is exploited.

Consider $\mathbf{V}_2$ for the operator $\mathbf{V}$ and its linear approximate $\mathbf{V}_1$ (from equation (12))

$$(\mathbf{G}_0\mathbf{V}_2\mathbf{G}_0)_m = -(\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0\mathbf{V}_1\mathbf{G}_0)_m \tag{12}$$

written for the general $\mathbf{V}_1$ form

$$\begin{aligned} V_2(\vec{k'_g}, \vec{k_s}, \omega) = &-\int\int\int\int e^{-i\vec{k'_g}\cdot\vec{r_1}} V_1(\vec{r_1}, \vec{r_2}, \omega) G_0(\vec{r_2}, \vec{r_3}; \omega) \times \\ &V_1(\vec{r_3}, \vec{r_4}; \omega) e^{i\vec{k_s}\cdot\vec{r_4}} d\vec{r_1} d\vec{r_2} d\vec{r_3} d\vec{r_4} \\ = \quad &-\int\int V_1(\vec{k'_g}, \vec{r_2}, \omega) G_0(\vec{r_2}, \vec{r_3}, \omega) V_1(\vec{r_3}, \vec{k_s}, \omega) d\vec{r_2} d\vec{r_3}. \end{aligned} \tag{58}$$

Expressing G_0 as a Fourier transform over $x_2 - x_3$ we find

$$\mathbf{G}_0(x_2 - x_3, z_2, z_3; \omega) = \int dk \mathbf{G}_0(k, z_2, z_3; w) e^{ik(x_2 - x_3)} \tag{59}$$

and

$$\mathbf{G}_0(k, z_2, z_3; \omega) = \int e^{-ikx} dx \mathbf{G}_0(x, z_2, z_3; \omega) \tag{60}$$

for $\mathbf{G}_0 = \mathbf{G}_0{}^d$, Eq. (60) reduces to

$$\mathbf{G}_0{}^d(k, z_2, z_3; \omega) = \frac{e^{iq|z_2 - z_3|}}{2iq} \tag{61}$$

where

$$q = \sqrt{\left(\frac{\omega}{c_0}\right)^2 - k^2}.$$

For the marine case where there is a free surface, the Green's function $\mathbf{G}_0$ satisfies:

$$\left(\nabla^2 + \frac{\omega^2}{c_0^2}\right) \mathbf{G}_0 = \delta(\vec{r_2} - \vec{r_3}) - \delta(\vec{r_2} - \vec{r_3^i}) \tag{62}$$

$$\left(\frac{d^2}{dz_2^2} - k_x^2 + \frac{\omega^2}{c_0^2}\right)\mathbf{G}_0 = \delta(z_2 - z_3) - \delta(z_2 - z_3^i) \tag{63}$$

where z_3^i is the image across the free surface of z_3 (with the free surface at $z = 0, z_3^i = -z_3$. The solution to Eq. (63) is

$$\mathbf{G}_0(k_x, z_2, z_3, \omega) = \frac{e^{iq|z_2 - z_3|} - e^{iq|z_2+z_3|}}{2iq} = G_0^d + G_0^{FS}$$

The contribution to $\mathbf{V}_2$ from the additional portion of the Green's function due to the free surface, ${\mathbf{G}_0}^{FS}$, $-\frac{e^{iq|z_2+z_3|}}{2iq}$, will be from equation (58)

$$\int \mathbf{V}_1(\vec{k_g}, \vec{r_2}; \omega) d\vec{r_2} \cdot \int\int dk \frac{e^{iq|z_2+z_3|}}{2iq} e^{ik(x_2 - x_3)} \mathbf{V}_1(\vec{r_3}, \vec{k_s}; \omega) d\vec{r_3} \tag{64}$$

Using the convention

$$\mathbf{V}_1(\vec{k_1}, \vec{k_2}; \omega) \equiv \int e^{-i\vec{k_1}\cdot\vec{r_1}} \mathbf{V}_1(\vec{r_1}, \vec{r_2}; \omega) e^{i\vec{k_2}\cdot\vec{r_2}} d\vec{r_1} d\vec{r_2}$$

where

$$\vec{k_1} \equiv \vec{k_{out}}$$

and

$$\vec{k_2} \equiv \vec{k_{in}}$$

. The portion of $\mathbf{V}_2$ due to ${\mathbf{G}_0}^{FS}$ has the form

$$\int dk \mathbf{V}_1(k_g, -q_g, k, q, \omega) \mathbf{V}_1(k, -q, k_s, q_s, \omega) = -\mathbf{V}_2(k_g, -q_g, k_s, q_s, \omega) \tag{65}$$

where $\vec{k'} \equiv \vec{k_{out}} and \vec{k} \equiv \vec{k_{in}}$ (Fig.14).

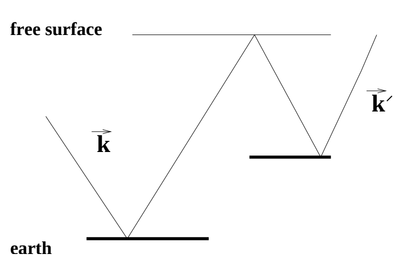


Figure 14: $\vec{k}$ and $\vec{k'}$.

The computation of the portion of $\mathbf{V}_2$ only due to ${\mathbf{G}_0}^{FS}$, ${\mathbf{V}_2}^{FS}$, is computable with $\mathbf{V}_1(\vec{k_g}, \vec{k_s}; \omega)$ where $|\vec{k_g}| = |\vec{k_s}| = \frac{\omega}{c_0}$; which is directly related to $\tilde{D}$ without assumption concerning the relationship between $\mathbf{V}_1$ and relative changes in earth material properties. It is that portion of the inverse series that forms the free surface de-multiple sub-series.

Therefore the free surface demultiple algorithm is independent of the earth model type for the class of models we are considering.

The summary of the free surface demultiple algorithm (from Weglein *et al.*, 1997 and Carvalho, 1992) is as follows:

1. The data, D, is computed by subtracting the reference field, $\mathbf{G}_0 = \mathbf{G}_0{}^d + \mathbf{G}_0{}^{FS}$, from the total field, G, on the measurement surface.

2. Compute the deghosted data, $\tilde{D}$ where
$\tilde{D} = D/[(e^{2iq_g\epsilon_g} - 1)(e^{2iq_s\epsilon_s} - 1)]$
from D and the source and receiver deghosting factors in the $k-\omega$ domain, $\mathbf{G}_0{}^d/\mathbf{G}_0 = 1/(e^{2iq\epsilon} - 1)$. q_s, q_g and ϵ_s, ϵ_g are the vertical wavenumbers and the depths below the free-surface of the source and receiver, respectively.

3. The series for deghosted and free-surface demultipled data, D', is given in terms of the deghosted data D'_1 as follows:

$$\begin{aligned} D'_n(k_g, k_s, \omega) &= \frac{1}{i\pi\rho_r B(\omega)} \int_{-\infty}^{\infty} dkqe^{iq(\epsilon_g+\epsilon_s)} D'_1(k_g, k, \omega) \\ &\quad \times D'_{n-1}(k, k_s, \omega) \quad n = 2, 3, 4, \ldots \end{aligned} \tag{66}$$

and

$$D'(k_g, k_s, \omega) = \sum_{n=1}^{\infty} D'_n(k_g, k_s, \omega) \ . \tag{67}$$

where $D'(k_g, k_s, \omega) \equiv D'(k_g, \epsilon_g, k_s, \epsilon_s, \omega)$, $B(\omega)$ and ρ_r are the source signature and reference density, respectively. The data D' consist of deghosted primaries and internal multiples only and $D'_1 = \tilde{D}$. Hence, D' represents the deghosted data without free-surface multiples. The mathematical details of equations (66) and (67) follow from equations (11′), (12′) and (13′) and are provided in Carvalho *et al.* (1992) and Weglein *et al.* (1997). Equations (66) and (67) are the prestack multidimensional generalizations of the one-dimensional, normal-incidence free-surface-elimination map presented in the appendix of Ware and Aki (1969).

6 Internal multiple attenuation

In the previous section, we described how to achieve the goal of separating the removal of surface multiples from the other three tasks of inversion. We now address the more difficult issue of separating the task of attenuating internal multiples from the last two goals of migration and inversion.

When we separated surface multiples from the other three goals we were able to isolate a portion of the Green's function, $\mathbf{G}_0$, namely ${\mathbf{G}_0}^{FS}$, whose purpose in the forward and inverse series was to produce and remove, respectively, events due to the presence of the free surface. Unfortunately, for internal multiples, we don't have that relatively straightforward road to follow.

If we attempt to repeat the reasoning that proved useful with surface multiples, we seek an example that has neither surface *nor* internal multiples. We can imagine a problem where we have two half-spaces; that is, we wish to invert a model that has only a single horizontal reflector. In that case, the scattered field, the primary, requires for its description a complete forward scattering series in terms of ${\mathbf{G}_0}^d$ and the exact perturbation, $\mathbf{V}$. The inverse series for $\mathbf{V}$ in terms of the data, the primary, requires the full series and ${\mathbf{G}_0}^d$. The lesson, from this single reflector example, is that the complete ${\mathbf{G}_0}^d$ is required in the inverse series when the only tasks are locating reflectors and estimating parameters. Hence, we *cannot* separate ${\mathbf{G}_0}^d$ into an extra part that exists only in the presence of internal multiples, but which is not present when internal multiples are absent. Thus, a fundamentally different approach is required for the attenuation of internal multiples.

We next present the logical path that leads to this new approach. The forward series generates primaries and internal multiples through the action of ${\mathbf{G}_0}^d$ on $\mathbf{V}$. The inverse series constructs $\mathbf{V}$ from the action of ${\mathbf{G}_0}^d$ on the recorded data. The action of ${\mathbf{G}_0}^d$ on data must remove internal multiples on the way to constructing $\mathbf{V}$. In an earlier section, we presented an analysis and interpretation of the forward series and specifically, how ${\mathbf{G}_0}^d$ generates primaries and internal multiples of a given order. However, before we focus on the internal-multiple issue, it is important to note an essential difference between the scattering theory pictures of free-surface and internal multiple generation.

Given data, D', without free-surface events, the forward series generates data, D, with free-surface events by the action of ${\mathbf{G}_0}^{FS}$ on D'. Each term in that series generates one order of free-surface multiple; that is, all events that have reflected from the free surface a given number of times. The modelling that ${\mathbf{G}_0}^{FS}$ provides is an exact description of a wave propagating in the water and reflecting from the free surface. Hence, $\mathbf{G}_0{}^{FS}$ generates in the forward series, and removes in the inverse series, one order of free-surface multiple with each term.

The situation for primaries and internal multiples is quite different. For those events, we adopt a point-scatterer model, and every term in that forward series contributes to (but does not by itself fully describe) either primary or internal multiples. Each primary or internal multiple requires an infinite series for its construction. We adopt the simpler surface reflection model when describing wave phenomena associated with reflectors at or above the measurement surface; we adopt the point-scatterer model for waves associated with sub-receiver/source structure. The former is our model of choice when we have accurate or nearly accurate information about velocities and structure, and the latter is our model when that information is unavailable or unreliable.

The location and properties of the free surface are captured in $\mathbf{G}_0{}^{FS}$ and it is that specific and well defined experience (or its absence) which allows free surface multiples to be separated from primaries and internal multiples with one term creating (in the forward series) and one term removing (in the inverse series) all events that have experienced the free surface a given number of times. The number of $\mathbf{G}_0{}^{FS}$ factors in a term in the subseries equations (12′-13′) correspond to the order of free surface multiples it removes. The internal multiples have (by definition) all of their downward reflections below the free surface and since we assume absolutely no subsurface information those reflectors are assumed to be completely unknown in both location and character.

This makes the problem of distinguishing the generation (and removal) of internal multiples from primaries more difficult in terms of direct propagation through water $\mathbf{G}_0{}^{d}$ and the difference between earth and water properties, $\mathbf{V}$. As mentioned earlier a series is required to generate any primary or any internal multiple in terms of $\mathbf{G}_0{}^{d}$ and $\mathbf{V}$ and new concepts required to distinguished this forward subseries.

It is no surprise that the first term in the generation and first term in the removal of internal multiples are approximate. The efficiency of the first term in the removal subseries of internal multiples is remarkably higher than the first term in the forward creation, e.g., it takes an infinite series to get the important time prediction (phase) of any internal multiple in the forward series (in terms of $\mathbf{G}_0{}^{d}$ and $\mathbf{V}$) whereas, as we will demonstrate, the first term in the removal series (of an internal multiple of a given order) predicts the time precisely and well approximates the amplitude (in terms of $\mathbf{G}_0{}^{d}$ and D') of all multiples of that order - from all reflectors at all depths at once. The efficiency of the inverse subseries accounts for its practical value and impact.

The fact that generating primaries and internal multiples of a given order requires an infinite series suggests that an infinite series of terms, each involving operations with ${\bf G_0}^d$ on D', is required to remove internal multiples of a given order. The particular inverse-scattering subseries for attenuating all internal multiples, described here, chooses only the leading and most significant contribution from the removal series of each order of multiple, forming a series that well attenuates, rather than eliminates, all internal multiples.

In our earlier discussion of the forward series for primaries and internal multiples we argued that primaries are constructed starting with the first term in the series, and first-order internal multiples have their leading contribution in the third term. Similarly, second-order internal multiples are generated by contributions starting with the fifth term in the forward series. In general, n-th order internal multiples have contributions from all terms starting at term $2n+1$. In addition, the portion of the third term that starts to build the first-order internal multiple was distinguished from the part that has a third-order contribution to constructing primaries. The leading-term contribution to constructing a class of multiples in the forward series suggests the leading-term contribution for their removal in the inverse series (Fig. 15).

The first two terms in the forward series don't contribute to generating first-order internal multiples. Similarly, it's argued that the first two terms in the inverse series don't contribute to their removal. The mathematical realization of Fig. 15a, is the leading contribution to the generation of first-order internal multiples; it suggests the corresponding mathematical expression for the leading-order attenuation of those multiples. To realize Fig. 15b select the portion of the third term of the inverse series with $z_1 > z_2$ and $z_3 > z_2$.

With this purpose in mind we examine ${\bf V}_3$, the third term in the inverse series. In contrast with the subseries generated by ${\bf G_0}^{FS}$, for free-surface multiple attenuation, the three terms in ${\bf V}_3$ do not sum to a single term when the inverse series is generated with the direct propagating Green's function, ${\bf G_0}^d$. From the fact that ${\bf G_0}^{FS}$ can be viewed as the Green's function due to an image source above the free surface and is therefore *outside* the volume, it follows that for all z, z' inside the volume (i.e., below the free surface) ${\bf G_0}^{FS}$ satisfies the homogeneous differential equation

$$\left(\frac{\nabla^2}{\rho_0} + \frac{\omega^2}{\kappa_0}\right) {\bf G_0}^{FS} = 0 \ .$$

The fact that ${\bf G_0}^{FS}$ satisfies a homogeneous differential equation leads in turn to the math-

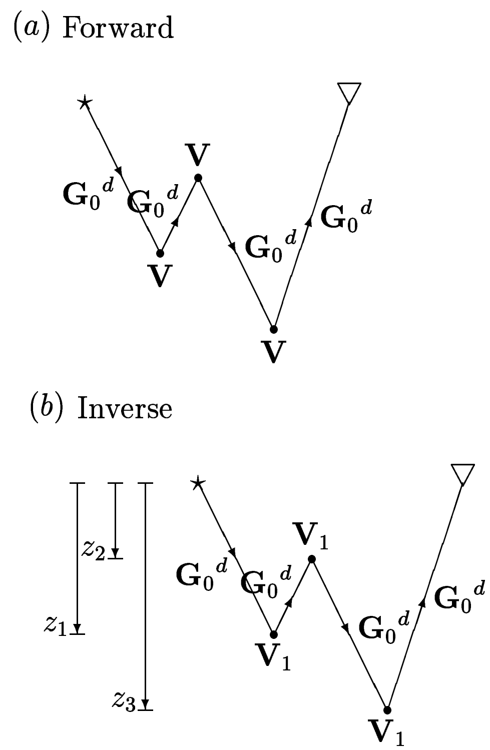


Figure 15: The leading term contribution to the generation of first-order internal multiples is represented in (a) and suggests the leading term contribution, in the inverse series, to the removal of first-order internal multiples represented in (b). $\mathbf{G_0}^d$, $\mathbf{V}$ and $\mathbf{V}_1$ are the whole-space Green's function, the perturbation operator and the "migrated data-like" first-order approximation to $\mathbf{V}$, respectively.

ematical simplification

$$\begin{aligned}\left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^{FS}\mathbf{V}_1\mathbf{G_0}^{FS}\mathbf{V}_1\mathbf{G_0}^d\right)_m &= -\left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^{FS}\mathbf{V}_2\mathbf{G_0}^d\right)_m \\ &= -\left(\mathbf{G_0}^d\mathbf{V}_2\mathbf{G_0}^{FS}\mathbf{V}_1\mathbf{G_0}^d\right)_m \ . \qquad (68)\end{aligned}$$

$\mathbf{G_0}^{FS}$ doesn't require off-shell $\vec{k} \neq \vec{k'}$ contribution as long it is in integrals over the volume V_1, and its effective source, the image source is outside the volume. This is another way to understand why the free-surface demultiple algorithm is automatically model-type independent. Model type was needed in Razavy (1975), to provide $T_1(k,p)$ for $k \neq p$. Since $\mathbf{G_0}^{FS}$ never requires $\vec{k} \neq \vec{p}$ in its integrations with V_1 it doesn't depend upon the inverse series model type argument to generate this subseries. Hence the free-surface multiple removal subseries is independent of earth model type. In contrast, $\mathbf{G_0}^d$ satisfies the inhomogeneous differential equation

$$\left(\frac{\nabla^2}{\rho_0} + \frac{\omega^2}{\kappa_0}\right)\mathbf{G_0}^d = -\delta(z-z')\delta(x-x') \ .$$

From Eq. (13"), and using $\mathbf{G_0}^d$, we have:

$$\begin{aligned}\left(\mathbf{G_0}^d\mathbf{V}_3\mathbf{G_0}^d\right) &= -\left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_2\mathbf{G_0}^d\right) \\ &\quad -\left(\mathbf{G_0}^d\mathbf{V}_2\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\right) - \left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\right) \\ &= \left(\mathbf{G_0}^d\mathbf{V}_{31}\mathbf{G_0}^d\right) + \left(\mathbf{G_0}^d\mathbf{V}_{32}\mathbf{G_0}^d\right) + \left(\mathbf{G_0}^d\mathbf{V}_{33}\mathbf{G_0}^d\right) \ , \qquad (69)\end{aligned}$$

where

$$\mathbf{V}_{31} \equiv -\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_2 \ , \qquad (70)$$

$$\mathbf{V}_{32} \equiv -\mathbf{V}_2\mathbf{G_0}^d\mathbf{V}_1 \ , \qquad (71)$$

and

$$\mathbf{V}_{33} \equiv -\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1 \ . \qquad (72)$$

In contrast to the case of $\mathbf{G_0}^{FS}$, these three terms $\mathbf{V}_{31}, \mathbf{V}_{32}$ and $\mathbf{V}_{33}$ make distinct contributions. The first two terms, $\left(\mathbf{G_0}^d\mathbf{V}_{31}\mathbf{G_0}^d\right)$ and $\left(\mathbf{G_0}^d\mathbf{V}_{32}\mathbf{G_0}^d\right)$, on the right member can be shown (Araújo, 1994) to consist of a refraction-like scattering component, and are thus not chosen for the task of removing internal multiples. These contribute to the other inversion tasks (migration and inversion) that act on primaries. The third term on the right hand side,

$$\left(\mathbf{G_0}^d\mathbf{V}_{33}\mathbf{G_0}^d\right) = -\left(\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\mathbf{V}_1\mathbf{G_0}^d\right) \ ,$$

Figure 16: Diagrams corresponding to different portions of $\Lambda_s \mathbf{G}_0{}^d \mathbf{V}_1 \mathbf{G}_0{}^d \mathbf{V}_1 \mathbf{G}_0{}^d \mathbf{V}_1 \mathbf{G}_0{}^d \Lambda_g$. Only (d), with $z_1 > z_2$ and $z_2 < z_3$, contributes to the attenuation of first-order internal multiples (see also Fig. 15b).

can be broken up into four parts corresponding to the four diagrams in Fig. 16.

Choose the portion of $\left(\mathbf{G_0}^d\mathbf{V}_{33}\mathbf{G_0}^d\right)_m$ corresponding to Fig. 16d; a diagram that represents a contribution to multiple *reflection* attenuation. $\left(\mathbf{G_0}^d\mathbf{V}_{31}\mathbf{G_0}^d\right)_m$ and $\left(\mathbf{G_0}^d\mathbf{V}_{32}\mathbf{G_0}^d\right)_m$ do not support a diagram of the Fig. 16d variety, and therefore were not selected for that task. The mathematical and algorithmic realizations of Fig. 16d takes place by requiring a lower-higher-lower relationship between the successive vertical locations of the data in the integral. Using this criterion, the appropriate portion of each of the odd terms in the series is selected. The generalization of the diagram found in Fig. 16d is used to select the appropriate portion of the leading-order contribution to removing higher-order internal multiples.

7 Internal multiple attenuation and model-type dependence

For $\mathbf{G_0}^d$, the direct propagating Green's function we have from Eq. (61)

$$\frac{e^{iq|z_2-z_3|}}{2iq}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{iq'(z_2-z_3)}}{q^2-q'^2+i\epsilon}dq'$$

and separating the integral into a principal value and a contribution from contours around the poles $q'=\pm q$ as

$$\begin{aligned}\frac{1}{q^2-q'^2+i\epsilon} &= P.V.\left(\frac{1}{q^2-q'^2}\right)+i\pi\delta\left(q'^2-q^2\right)\\ &= P.V.\left(\frac{1}{q^2-q'^2}\right)+\frac{1}{2\pi}\left(i\pi\frac{1}{2\,|q|}\left(\delta(q'-q)+\delta(q'+q)\right)e^{iq'(z_2-z_3)}\right)\end{aligned}$$

This contour around the pole contribution leads to:

$$\int_{-\infty}^{\infty}dk\left[\frac{V_1\left(k_g,-q_g,k,q\right)V_1\left(k,q,k_s,q_s\right)}{2iq}+\frac{V_1\left(k_g,-q_g,k,-q\right)V_1\left(k,-q,k_s,q_s\right)}{2iq}\right]$$

and is computable directly from $\mathbf{V}_1\left(k_g,q_g,k_s,q_s\right)$.

The portion of V_2 that depends on the principal value part of the contribution to G_0^d is not computable from $\Psi_s\left(x_g,\epsilon_g,x_s,\epsilon_s,\omega\right)$ without assuming a model-type. Since the internal multiple algorithm derives from the analogous $i\pi\delta$ contributions from the $\mathbf{V}_1\mathbf{V}_1\mathbf{V}_1$ or V_{33} (equation (72)) contribution from the third term in the series (equation (13”))

$$\int dkdk'\left[\frac{\mathbf{V}_1\left(k_g,-q_g,k,q\right)\mathbf{V}_1\left(k,q,k',-q'\right)}{2iq}.\frac{\mathbf{V}_1\left(k',-q',k_s,q_s\right)}{2iq'}+\ldots\right]$$

is once again computable directly from surface data without assumption of model type.

An important point to recognize in deriving the internal multiple algorithm, not emphasized in previous publication, is that although the "W" or lower-higher-lower relationship from the forward series provides a guide for the examination of a similar diagram in the inverse, to actually realize an internal multiple algorithm the quantity taken through the diagram was not $\mathbf{V}_1$ but rather

$$b_1\left(k_g, \epsilon_g, k_s, \epsilon_s, q_g + q_s\right) = \left(-2iq_s\right) D'\left(k_g, \epsilon_g, k_s, \epsilon_s, \omega\right) = \frac{\mathbf{V}_1\left(k_g, q_g, k_s, q_s, \omega\right)}{-2iq_g}$$

the effective data generated by a single frequency plane-wave incident field. This was originally deduced (see Araujo (1994) and Weglein *et al.* (1997)) through empirical evaluation and testing of different candidate quantities (e.g. a first and natural guess of taking $\mathbf{V}_1$ through "W" doesn't lead to an attenuation algorithm) that, in turn, allow different subdivisions of the $\mathbf{V}_{33}$ term in terms of a "W" diagram.

The fact that this quantity, b_1, results in a localized incident wavefront in every dimension (without the wake behind the front that the impulse response in 1D and 2D experience) is the only and best (although meager) 'understanding' or hint we have for this fact, to-date. Hence, the forward construction and inverse removal symmetry for the internal multiple went only so far and the fact that, $b_1 = \frac{\mathbf{V}_1(k_g,q_g,k_s,q_s,\omega)}{-2iq_g}$ is the quantity that when transformed to (k_g, q_g, z) and broken into lower-higher-lower contributions results in the internal multiple algorithm remains partly intuitive and empirical in its foundation and invites further analysis for better understanding. That algorithm operates in a 1D, 2D or 3D earth. A deeper awareness and comprehension of the workings of the inverse series will, of course, also benefit the current research on imaging and inverting primaries.

The internal multiple algorithm is independent of model type because if derives from a algorithm depending on the portion of $\mathbf{V}_1$ that only requires $|\vec{k_g}| = |\vec{k_s}| = \frac{\omega}{c_0}$.

8 Internal multiple algorithm

The first term in the internal multiple-attenuation subseries is the data, D', consisting of primaries and internal multiples. The second term in the attenuation series comes from a portion of the third term in the series (equations (69) and (72)). This portion of the third term,

$$\begin{aligned} b_3 =& \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 e^{-iq_1(\epsilon_g - \epsilon_s)} dk_2 e^{iq_2(\epsilon_g - \epsilon_s)} \\ & \times \int_{-\infty}^{\infty} dz_1 e^{i(q_g + q_1)z_1} b_1(k_g, k_1, z_1) \\ & \times \int_{-\infty}^{z_1} dz_2 e^{i(-q_1 - q_2)z_2} b_1(k_1, k_2, z_2) \\ & \times \int_{z_2}^{\infty} dz_3 e^{i(q_2 + q_s)z_3} b_1(k_2, k_s, z_3) \ , \end{aligned} \tag{73}$$

is chosen to satisfy $z_1 > z_2$ and $z_2 < z_3$. b_1 is defined in terms of the original prestack data with free-surface multiples eliminated, D', and is defined by

$$D'(k_g, k_s, \omega) = (-2iq_s)^{-1} B(\omega) \ b_1(k_g, k_s, q_g + q_s) \ . \tag{74}$$

b_1 is the data that would result from a single-frequency incident plane wave and $B(w)$ is the source signature. The data with internal multiples attenuated, D^{IM}, is

$$D^{IM}(k_g, k_s, \omega) = (-2iq_s)^{-1} B(\omega) \sum_{n=0}^{\infty} b_{2n+1}(k_g, k_s, q_g + q_s). \tag{75}$$

A recursive relationship that generalizes Eq. (75) and provides b_{2n+1} in terms of b_{2n-1} for $n = 1, 2, 3, \ldots$ is given in Araújo (1994) as

$$\begin{aligned} b_{2n+1}(k_g, k_s, q_g + q_s) =& \frac{1}{(2\pi)^{2n}} \int_{-\infty}^{\infty} dk_1 e^{-iq_1(\epsilon_g - \epsilon_s)} \\ & \times \int_{-\infty}^{\infty} dz_1 e^{i(q_g + q_1)z_1} b_1(k_g, k_1, z_1) A_{2n+1}(k_1, k_s, z_1) \\ & n = 1, 2, 3, \ldots \end{aligned} \tag{76}$$

where

$$\begin{aligned} A_3(k_1, k_s, z_1) =& \int_{-\infty}^{\infty} dk_2 e^{iq_2(\epsilon_g - \epsilon_s)} \int_{-\infty}^{z_1} dz_2 e^{i(-q_1 - q_2)z_2} \\ & \times \ b_1(k_1, k_2, z_2) \int_{z_2}^{\infty} dz_3 e^{i(q_2 + q_s)z_3} b_1(k_2, k_s, z_3) \end{aligned}$$

and

$$\begin{aligned} A_{2n+1}(k_1,k_s,z_1) = & \int_{-\infty}^{\infty} dk_2 e^{iq_2(\epsilon_g-\epsilon_s)} \\ & \times \int_{-\infty}^{z_1} dz_2 e^{i(-q_1-q_2)z_2} b_1(k_1,k_2,z_2) \\ & \times \int_{-\infty}^{\infty} dk_3 e^{-iq_3(\epsilon_g-\epsilon_s)} \int_{z_2}^{\infty} dz_3 e^{i(q_2+q_3)z_3} \\ & \times \; b_1(k_2,k_3,z_3) A_{2n-1}(k_3,k_s,z_3) \\ & \qquad\qquad\qquad\qquad n = 2,3,4,\ldots\ . \end{aligned}$$

As we mentioned, the full series for **V** can have restrictive convergence properties and a sensitivity to missing low-frequency information, (see, e.g., Carvalho, 1992). In contrast, tests indicate (see Araújo, 1994; Araújo et al., 1994a; Araújo et al., 1994b) that the multiple attenuation subseries in Eq. (76) always converges and is insensitive to missing low frequency information.

Free-surface multiple attenuation methods operate one temporal frequency at a time (see equations (66) and (67)); in contrast, the attenuation of an internal multiple from a single frequency of data requires data at all frequencies [see equations 75 − 76]. This requirement derives from the integral over temporal frequency in the transform of $q_g + q_s$ to z. With bandlimited data this transform is only approximate; nevertheless, the truncated integral remains effective in attenuating multiples. As in the case of the surface-removal algorithm each term in the series, Eq. (75), attenuates a given order of internal multiple, and prepares the higher order internal multiples for the higher demultiple terms in the series. Since $e^{ik_z z} b_1(k_g,k_s,\omega)$ is a downward continuation of shots and receivers to depth z in the reference medium, and subsequent integration over k_z is a simple constant Jacobian away from integration over ω ($t = 0$ imaging condition), it follows that $b_1(k_g,k_s,z)$ corresponds to uncollapsed-migration (Stolt and Weglein, 1985; Weglein and Stolt, 1999). Indeed, the algorithm can be interpreted as a sequence of these uncollapsed migrations restricted to lower-higher-lower, pseudo-depth, which is essentially vertical travel time, since the reference is constant water speed. Uncollapsed-migration is a generalization of the original migration concept; sources and receivers are downward continued to a common-depth level z, time is evaluated at zero, and information at $x_g \neq x_s$ is retained. The latter retention of $x_g \neq x_s$ distinguishes uncollapsed-migration from migration; it provides local angle-dependent reflection coefficients rather than the angle-averaged reflection coefficient of the traditional $x_g = x_s$ imaging condition(see Weglein and Stolt, 1999).

When free-surface and internal multiples are present: (1) apply the free-surface demultiple algorithm to D and output D', then (2) input D' to the internal multiple attenuation algorithm, to produce primaries.

9 Purposeful perturbation concept and examples of free surface and internal multiple attenuation

9.1 Purposeful perturbation

As we have described, the response to the apparent lack of robust convergence of the entire inverse series, without a-priori information, and the recognition that it nevertheless represented the only complete inversion formalism for the multidimensional acoustic and elastic waves combined to encourage seeking task specific subseries that would have more favorable properties. However, another issue that these task specific (well-converging) subseries faced was how many terms would you require in practice to achieve a certain level of effectiveness. The concept of purposeful perturbation was developed to address the latter issue.

The idea was to identify the specific purpose or role that each term within a task specific subseries performs independent of the subsurface or target over which the recorded data was collected.

The terms of the series perform tasks and coupled tasks; the task specific subseries perform isolated, uncoupled tasks; and, we define the purposeful perturbation concept to know precisely what each term within a given task specific subseries is designed to accomplish. For example, a term in the inverse scattering subseries for eliminating free surface multiples removes precisely one order of free-surface multiple completely independent of the depth of the water or any other property or characteristic of the Earth.

9.2 1D free surface demultiple algorithm

For the simplest illustration of this purposeful perturbation concept, consider the generation of free-surface multiples for a 1-D Earth, whose primary reflections and internal multiples have a response $R(\omega)$; and, where the free surface is characterized by a reflection coefficient of -1 (see Fig. 17).

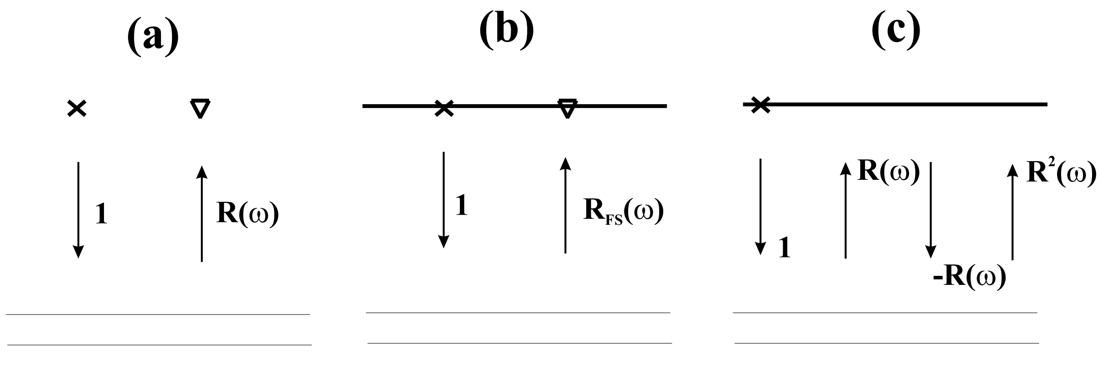


Figure 17: Illustration of the free surface multiple removal series: (a) Data without a free surface, (b) the total upgoing field in the presence of a free surface R_{FS}, and (c) the series for R_{FS}.

If the subseries we isolate are defined for accomplishing one of the four broad tasks we earlier defined, purposeful perturbation seeks to determine, or further define, the specific role or sub-task that individual terms in the sub-series perform. Then, e.g., if you estimate the range of depths of potential hydrocarbon reservoirs in a given setting, and the depth to the water-bottom, then you have a good way to determine the highest order of water-bottom multiple you need to be concerned with and precisely the number of terms in the free surface demultiple subseries (equations (66) and (67)) that can accomplish that objective.

For source and receiver deghosted data, and a source wavelet with unit amplitude, the upgoing field in the presence of a free surface R_{FS} is able to be written in terms of $R(\omega)$ by imagining (see Fig (17)c) the wave first leaving the source moving down into the Earth; that incident unit pressure wave generates a reflected response from the Earth, $R(\omega)$, consisting of primaries and internal multiples. This in turn propagates as a train up in the water column until it hits the free-surface, where it experiences a (-1) reflection coefficient and heads down through the water columns as $-R(\omega)$. The impulse response of the Earth $R(\omega)$ times this effective downgoing "wavelet", $-R(\omega)$ produces a new wave moving up from the Earth through the water column towards the free surface. This process continues and results in the total upgoing wave in the presence of the free surface, $R_{FS}(\omega)$ in terms of the primary and internal multiple wavefield, $R(\omega)$ as follows

$$\begin{aligned} R_{FS} &= R - R^2 + R^3 - \dots \\ &= \frac{R}{1+R} \end{aligned} \tag{77}$$

Each term in Eq. (77) generates all free surface multiples of a given order independent of any detail of the subsurface. The order of a free surface multiple corresponds to the number of times that event has experienced a reflection at the free surface. Since each successive term in Eq. (77) comes from one additional reflection at the free-surface, it generates one additional order of free-surface multiple. Solving Eq. (77) for the data without free-surface multiples, R, we have

$$\begin{aligned} R &= \frac{R_{FS}}{1 - R_{FS}} \\ &= R_{FS} + R_{FS}^2 + R_{FS}^3 + \dots \end{aligned} \tag{78}$$

The first term in Eq. (78), R_{FS}, is the upgoing portion of the reflection data that contains all free-surface multiples. When the second term, R_{FS}^2, is added to R_{FS} two things happen: (1) all free-surface multiples that have reflected once (and only once) from the free-surface are removed and (2) all higher-order free-surface multiples are altered in preparation for higher terms, e.g., R_{FS}^3, to remove them order-by-order, as well. This well-defined action of the terms in the free-surface demultiple series is totally independent of any water-bottom or subsurface detail (of course, within an assumed 1D, 2D or 3D dimension of the Earth variation).

This is an example of purposeful perturbation; and, it has enormous practical significance. For example, if you estimate that for a given depth of water and target, that only a certain order of multiples could be troublesome, then you know precisely how many terms in the series you need to use in your processing algorithm for that data. Eq. (78) is the 1-D normal incidence special case of the general multi-dimensional inverse scattering subseries for free-surface multiple removal equations (66) and (67) (see also Carvalho *et al.*, 1992 and Weglein *et al.*, 1997).

Equation (78) is the 1D antecedent of equations (11′- 13′) and (66) and (67) for free surface multiple removal. Several observations about equations (77) and (78) are worth noting.

First, the role of $\mathbf{G}_0{}^{FS}$, the extra portion of $\mathbf{G}_0$ due to the free-surface, is played by the (-1) reflection coefficient in deriving (77) and its inverse (78). Second, the forward construction series was a guide (and in this simple instance, more than a guide) to the inverse process. Only the free-surface reflection coefficient (-1) i.e., $\mathbf{G}_0{}^{FS}$ terms enter confirms the forward and removal series (77), (78). Focussing on the one task, and only one task, this simple consistent strategy we described, i.e., no $\mathbf{G}_0{}^{FS}, \mathbf{G}_0{}^{D}$ coupled terms appear in the analogous and transparently simple equations (77) and (78).

Regarding some practical issues, exemplified by equations (77) and (78), if instead of 1, a unit incident pulse, a wavelet $A(w)$ was the source signature, then equation (77) would become

$$R_{FS} = \frac{A(w)R}{1+R} \tag{79}$$

and Eq. (78) becomes

$$R = \frac{\frac{R_{FS}}{A(w)}}{1 - \frac{R_{FS}}{A(w)}} = \frac{R_{FS}}{A(w)} + (\frac{R_{FS}}{A(w)})^2 + \dots \tag{80}$$

and, hence, the wavelet is a critical requirement for the free-surface removal and all subseries application.

A similar process of purposeful perturbation occurs (and has been identified) for the internal multiple removal series. Understanding the specific purpose, within an overall task, that each term accomplishes not only reveals what has (and has not) been achieved for a given finite number of terms and this significantly mitigates issues of overall convergence and rate of convergence.

9.3 1D analytic example of the internal multiple attenuation algorithm

The 2D internal multiple algorithm described in equations (73-76)

$$b(k_g, k_s, q_g + q_s) = -2iq_s D'(k_g, k_s, \omega)$$

where D' is the free-surface multiple removed data resulting from an impulsive source. The second term in the internal multiple attenuation series, b_3, is given by

$$\begin{aligned} b_3(k_g, k_s, q_g + q_s) \quad = \quad & \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_1 e^{iq_1(\epsilon_s - \epsilon_g)} dk_2 e^{iq_2(\epsilon_g - \epsilon_s)} \\ & \times \int_{-\infty}^{\infty} dz_1 e^{i(q_g+q_1)z_1} b_1(k_g, -k_1, z_1) \\ & \times \int_{-\infty}^{z_1} dz_2 e^{i(-q_1-q_2)z_2} b_1(k_1, -k_2, z_2) \\ & \times \int_{z_2}^{\infty} dz_3 e^{i(q_2+q_s)z_3} b_1(k_2, -k_s, z_3) \end{aligned}$$

The first two terms in the multiple attenuation series, $(-2iq_s)(b_1 + b_3)$, Eq. (75), attenuate all first order internal multiples. For a 1D earth and a normal incidence plane wave, Eq.

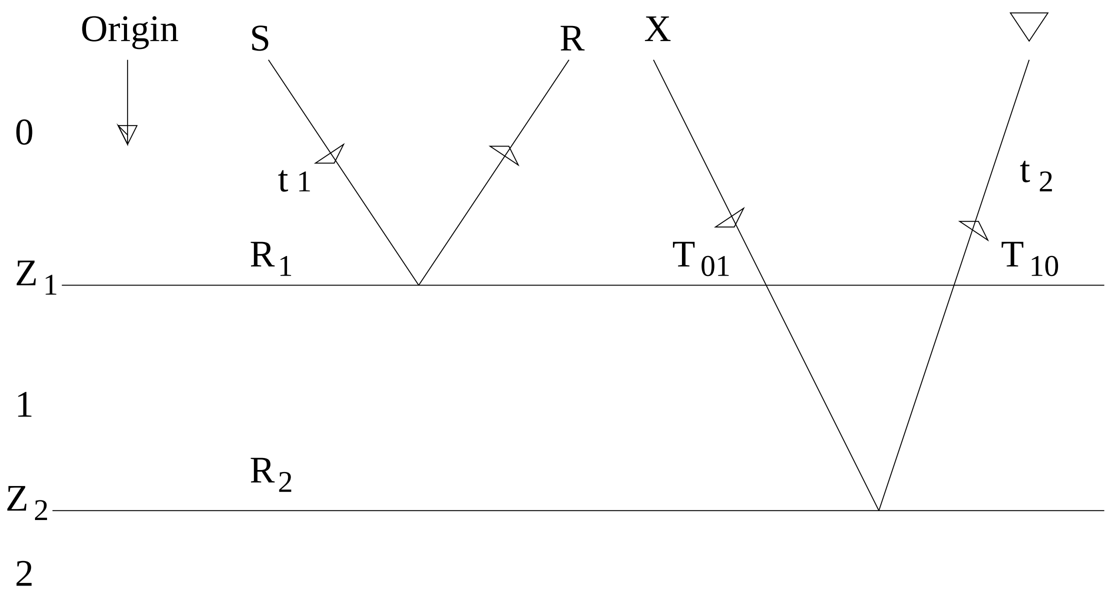


Figure 18: One dimensional model with 2 interfaces.

(73) reduces to:

$$b_3(k) = \int_{-\infty}^{\infty} dz_1 e^{ikz_1} b(z_1) \int_{\infty}^{z_1} dz_2 e^{-ikz_2} b(z_2) \int_{z_2}^{\infty} dz_3 e^{ikz_3} b(z_3) \tag{81}$$

To explicitly demonstrate how the internal multiple attenuation algorithm works and to examine its properties, we will consider the simplest model that can produce an internal multiple. For the model shown in Fig. (18) the reflection data due to an impulsive incident wave $\delta(t - \frac{z}{c})$ is

$$D(t) = R_1 \delta(t - t_1) + T_{01} R_2 T_{10} \delta(t - t_2) + \ldots$$

where t_1, t_2, R_1, R_2 are the two way times and reflection coefficients from the two reflectors and T_{01} and T_{10} are the transmission coefficients between 0 and 1 and 1 and 0, respectively.

$$D(\omega) = R_1 e^{i\omega t_1} + T_{01} R_2 T_{10} e^{i\omega t_2} + \ldots \tag{82}$$

where $D(w)$ is the temporal former transform of $D(t)$.

Note that the $(-2iq_s)$ factor that multiples D' in the internal multiple theory is not required in this example since we assume that the incident wave is an impulsive plane wave. The role of the $(-2iq_s)$ is to transform an incident (or reference field) $\mathbf{G}_0$, into a plane wave in the Fourier domain. The internal multiple algorithm inputs the data with primaries and internal multiples, b and is the first term in the multiple attenuated series, b_1.

$$b^{IM}(k) = b_1(k) + b_3(k) + b_5(k) + \ldots$$

where $b_1 = b$, the input data.

The vertical wave number

$$k_z = \sqrt{(\omega/C_0)^2 - k_{x_g}^2} + \sqrt{(\omega/C_0)^2 - k_{x_s}^2}$$

for 1D medium and normal incident wave is $k_z = 2\frac{\omega}{C_0}$ and

$$b(k_z) = D(\omega) \ . \tag{83}$$

The reflection data from Eq. (82) and (83) is expressed in terms of k_z

$$b(k_z) = R_1 e^{i(\frac{2\omega}{C_0})(\frac{C_0 t_1}{2})} + T_{01} R_2 T_{10} e^{i(\frac{2\omega}{C_0})(\frac{C_0 t_2}{2})} + \ldots \tag{84}$$

and define the pseudo-depths z_1 and z_2 in the reference medium as:

$$z_1 \equiv \frac{C_0 t_1}{2}$$

$$z_2 \equiv \frac{C_0 t_2}{2} \ .$$

The input data is now expressed in terms of $k = k_z$ and z_1 and z_2 as

$$b(k) = R_1 e^{ikz_1} + T_{01} R_2 T_{10} e^{ikz_2} + \ldots \tag{85}$$

ready for the internal multiple algorithm.

Substitute the data from Eq. (85) into the algorithm (b_3, equation (81)). After transforming from $k = k_z$ to z.

$$b(z) = \int_{-\infty}^{\infty} e^{-ikz} b(k) dk \tag{86}$$

The first integral in Eq. (81) towards computing b_3 is

$$\int_{z_2'+\epsilon_1}^{\infty} dz_3 e^{ikz_3} (R_1 \delta(z_3' - z_1) + R_2' \delta(z_3' - z_2) + \ldots) \tag{87}$$

where

$$R_2' \equiv T_{01} R_2 T_{10} \ .$$

ϵ_1 is a small positive parameter chosen to insure that the $'W'$ diagram is strictly lower-higher-lower and avoids the lower than or equal to contribution. In actual seismic field data application the parameter ϵ is chosen to be the width of the source wavelet and speaks two the fact that subresolution (i.e., thin bed multiples) will not be attenuated. The integral (87) evaluates to:

$$H(z_1-(z_2'+\epsilon_1))R_1e^{ikz_1}+H(z_2-(z_2'+\epsilon_1))R_2'e^{ikz_2}\ .$$

The second integral in Eq. (81) is:

$$\begin{aligned}&\int_{-\infty}^{z_1'-\epsilon_2}(R_1\delta(z_2'-z_1)+R_2'\delta(z_2'-z_2))(H(z_1-(z_2'+\epsilon_1))R_1e^{ikz_1}+\\&H(z_2-(z_2'+\epsilon_1))R_2'e^{ikz_2})e^{-ikz_2'}dz_2'\\&=R_1^2H((z_1'-\epsilon_2)-z_1)\underline{H(z_1-(z_1+\epsilon_1))}e^{ikz_1}e^{-ikz_1}\\&+R_1R_2'H((z_1'-\epsilon_2)-z_2)\underline{H(z_1-(z_2+\epsilon_1))}e^{ikz_1}e^{-ikz_2}\\&+R_1R_2'H((z_1'-\epsilon_2)-z_1)H(z_2-(z_1+\epsilon_1))e^{ikz_2}e^{-ikz_1}\\&+(R_2')^2H((z_1'-\epsilon_2)-z_2)\underline{H(z_2-(z_2+\epsilon_1))}e^{ikz_2}e^{-ikz_2}\end{aligned}\tag{88}$$

where ϵ_2 is a positive parameter with the same function as G_1 and all the underlined terms are zero.

The third (and last) integral is:

$$\begin{aligned}b_3(k)=&\int_{-\infty}^{\infty}dz_1'e^{-ikz_1'}(R_1\delta(z_1'-z_1)+R_2'\delta(z_1'-z_2))\\&(R_1R_2'H((z_1'-\epsilon_2)-z_1)H(z_2-(z_1+\epsilon_1))e^{ikz_2}e^{-ikz_1})\\=&e^{ikz_1}R_1^2R_2'\underline{H(-\epsilon_2)}H(z_2-z_1+\epsilon_1)e^{ikz_2}e^{-ikz_1}\\&+e^{ikz_2}R_1(R_2')^2H(z_2-z_1-\epsilon_2)H(z_2-z_1-\epsilon_1)e^{ikz_2}e^{-ikz_1}\end{aligned}$$

and the underlined function is zero. Then, since

$$R_2'=T_{01}R_2T_{10},$$

the prediction is:

$$b_3(k)=R_1R_2^2T_{01}^2T_{10}^2\ e^{2ikz_2}\ e^{2ikz_2}e^{-ikz_1}$$

and

$$\rightarrow b_3(t)=R_1R_2^2T_{01}^2T_{10}^2\delta(t-(2t_2-t_1))$$

From the example it is easy to compute the actual first order internal multiple precisely

$$-R_1 R_2^2 T_{01} T_{10} \delta(t - (2t_2 - t_1)) \ .$$

Hence the key prediction of time is perfect and the amplitude of the prediction has an extra power of $T_{01}T_{10}$ thus defining exactly the difference between the attenuation represented by b_3 and elimination.

Since $T_{01}T_{10}$ is less than one, the method always attenuates and, furthermore, the residual after adding b_1 to b_3 has the same sign as the multiple. Therefore, the internal multiple algorithm has well-defined amplitude prediction properties. If $R = 1/4$ (a large reflection coefficient) then $R^2 = 1/16$ and $T^2 = 15/16(R^2 + T^2 = 1)$, and $T = \sqrt{15/16} \sim 31/32$.

So even with large R, T^2 is still not far from 1 and that explains the remarkable efficiency of the leading order term for removing first order multiples. It produces the precise timing of all internal multiples of first order, independent of where the upward and downward reflections occur and well approximates their amplitudes, always less than the actual, the precise relationship between the internal multiple amplitude and the b_3 prediction is quantified. Since the difference in amplitude is related to transmission information the internal multiple predictor can provide indirect useful effective, overbunden transmission estimation, as well. Hence, while it is precise to say that the internal multiple algorithm doesn't predict the exact amplitude it is not accurate to say that no significant useful amplitude information is predicted by the internal multiple algorithm.

9.4 Synthetic and field data examples

Figure 19 shows an example of the internal multiple attenuation series algorithm applied to a 2-D synthetic dataset. From left to right, the three panels show the input data, the predicted internal multiples and the result of inverse scattering internal multiple attenuation, respectively.

Figures 20–22 illustrate the free surface and internal multiple attenuation algorithms applied to a dataset from the Gulf of Mexico over a complex salt body. Seismic imaging beneath salt is a challenging problem due to the complexity of the resultant wavefield. In Figure 20, the left panel is a stack section of the input data, and the right panel shows the result of the inverse scattering free surface multiple removal algorithm. Figure 21 is a cartoon that illustrates the events that are used by the algorithm to predict the free surface multiples in the data. Figure 22 illustrates the internal multiple attenuation method applied to the

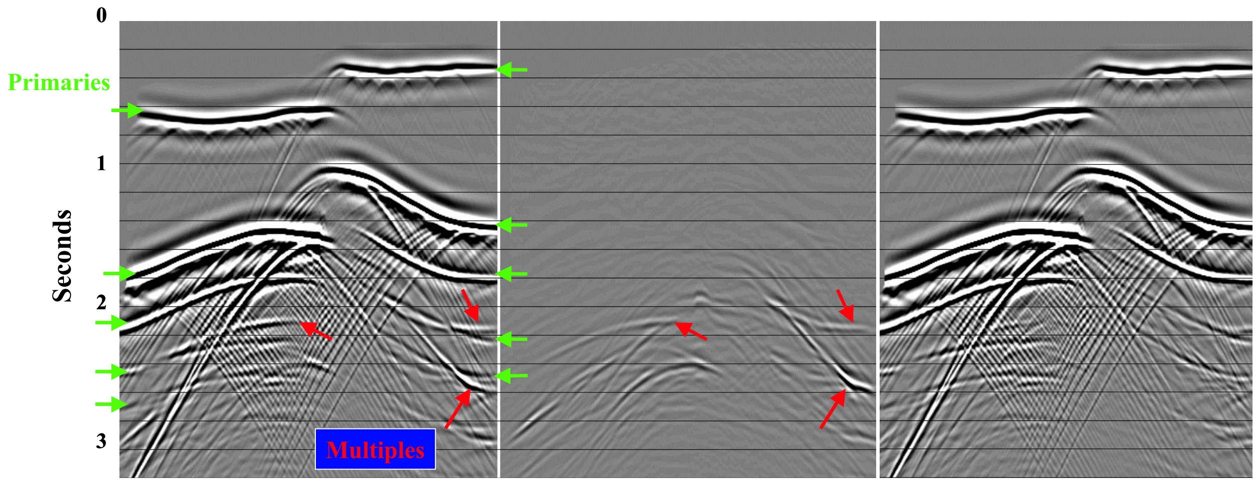


Figure 19: The left panel shows a common offset display from a 2-D synthetic dataset. The middle panel shows the predicted internal multiples. The right-hand panel is the result of subtracting the multiples from the input dataset.

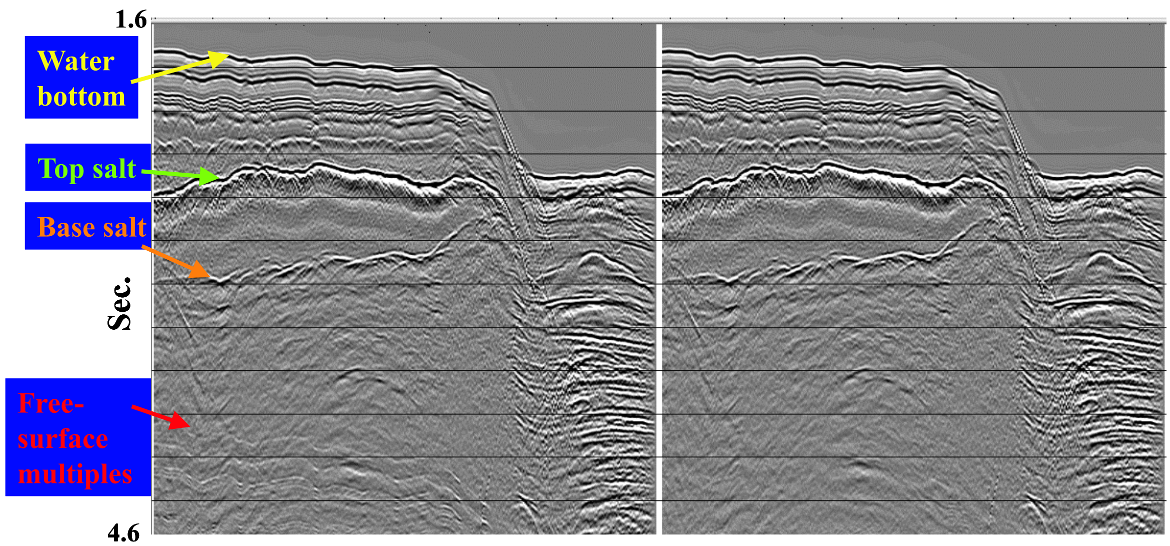


Figure 20: The left panel is a stack of a field dataset from the Gulf of Mexico. The right panel is the result of free surface demultiple. Data are courtesy of WesternGeco.

same Gulf of Mexico dataset. An internal multiple that has reverberated between the top of the salt body and the water bottom, is well attenuated through this method. The cartoons in Fig. 23 illsutrate the subsevents that are used by the algorithm to predict the internal multiples.

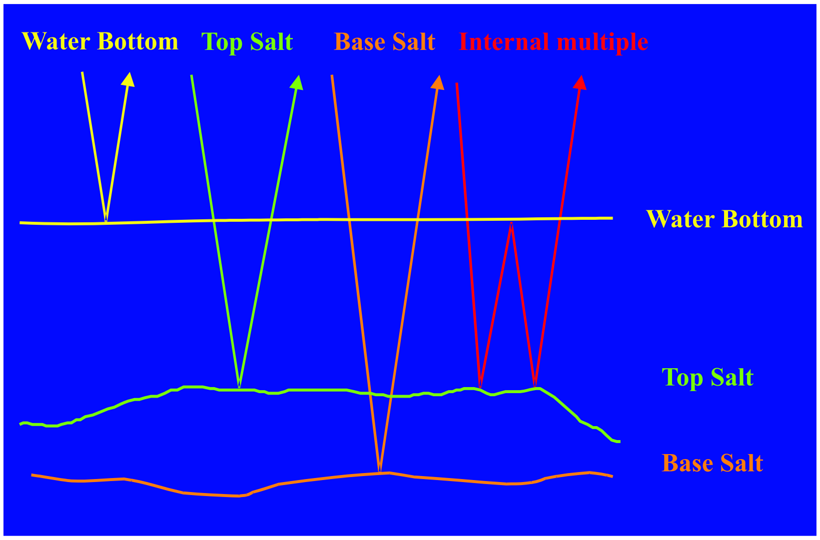


Figure 21: A cartoon illustrating the events that are used by the algorithm to predict free surface multiples.

10 Inverse subseries for imaging and inversion at depth without the velocity model for large contrast complex targets

Initial analysis for identifying the imaging and inversion tasks associated with primaries within the series have recently been reported by Weglein *et al.* (2002). Starting with the acoustic Eq. (6), and defining

$$\frac{1}{k} = \frac{1}{k_r}(1+\alpha)$$

$$\frac{1}{\rho} = \frac{1}{\rho_r}(1+\beta)$$

for a one dimensional variable velocity and density acoustic medium with point sources and receivers at depth ϵ_s and ϵ_g, respectively, Eq. (11′) becomes

$$\widetilde{D}(q_g,\theta,\epsilon_g,\epsilon_s) = -\frac{\rho_r}{4}e^{-iq_g(\epsilon_s+\epsilon_g)}[\frac{1}{\cos^2\theta}\widetilde{\alpha_1}(-2q_g) + (1-\tan^2\theta)\widetilde{\beta}_1(-2q_g)] \quad (89)$$

where the subscripts s and g denote source and receiver respectively, and q_g, θ and $k=\omega/c_0$ are shown in Fig. (24), and they have the following relations:

$$q_g = q_s = k\cos\theta$$

$$k_g = k_s = k \sin\theta \ .$$

Similarly, from Eq. (12′) we can get the solution for $\alpha_2(z)$ and $\beta_2(z)$ as a function of $\alpha_1(z)$ and $\beta_1(z)$

$$\begin{aligned}\frac{1}{\cos^2\theta}\alpha_2(z) + (1-\tan^2\theta)\beta_2(z) = &-\frac{1}{2\cos^4\theta}\alpha_1^2(z)\\ &+\frac{1}{\cos^4\theta}\alpha_1(z)\beta_1(z)\\ &-(\frac{3}{2}+\tan^2\theta+\frac{1}{2}\tan^4\theta)\beta_1^2(z)\\ &-\frac{1}{2\cos^4\theta}\alpha_1^{'}(z)\int_0^z dz^{'}[\alpha_1(z^{'})-\beta_1(z^{'})]\\ &+\frac{1}{2}(\tan^4\theta-1)\beta_1^{'}(z)\int_0^z dz^{'}[\alpha_1(z^{'})-\beta_1(z^{'})] \ . \end{aligned} \tag{90}$$

For a single reflection between two acoustic half-spaces where the upper half space corresponds to the reference medium the data consists of primaries and the inversion tasks they face are simply to locate the reflector and to invert for acoustic property charges across the reflector. When the primary data from this two half space model is substituted into Eq. (89) and (90), then the two terms involving integrals on the right hand side become zero. If the model would allow a second reflector, and a two primary wavefield, then those same terms involving the integrals are not zero. From an inversion point-of-view, the primary from the second reflector has more tasks to perform, (in comparison with the first primary) since the first event actually travelled through the reference medium. In addition to estimating changes in earth material properties, the second primary will be imaged where it is placed by the reference medium. From this type of observation and the detailed analysis formed in Weglein *et al.*, (2002) and Shaw and Weglein (2003), it is deduced that the last two terms in Eq. 90 assist in moving the second (deeper) primary to its correct location and the first three terms of Eq. 90 are associated with improving the linear inversion (Eq. 89).

The first three terms on the right-hand-side of equation (90) have two objectives. The first objective: for a primary off the shallower reflector, those first three terms start the nonlinear process of turning that events reflection coefficient into the earth property changes α and β. The reflection coefficient is a non-linear series in α and β; and, conversely, α and β are themselves nonlinear series in the reflection coefficient. For the second (deeper) primary, the first objective is more complicated, since the event amplitude is a function of both the reflection coefficient at the second reflector and the transmission coefficient down-through and up-past the first reflector. The communication between the two events allowed in e.g., α_1^2 can be shown to allow the reflection coefficient of the shallower reflector to work towards removing the transmission coefficients impeding the amplitude of the second event from inverting for local properties at the second reflector. Hence, specific communication between primaries from different reflectors work together to remove the extraneous transmission coefficients on deeper primaries that are suffering from being given the wrong velocity.

Similarly, the integral terms on the right-hand-side of equation (90) represent a recognition that the reference velocity will give an erroneous image, and asks for an integral of $\alpha_1 - \beta_1$, the linear approximation to the change in acoustic velocity, from the onset of $\alpha_1 - \beta_1$, down to the depth needing the imaging help. Two important observations: (1) When the actual velocity doesn't change across an interface, $R(\theta)$ is not a function of θ and from equation (89) it can be shown that

$$\alpha_1 - \beta_1 = (\frac{\Delta V}{V})_1 = 0 \ .$$

Therefore, when the actual velocity doesn't change then the linear approximation to the change in velocity is zero. Therefore, when the velocity is equal to the reference across all reflectors (e.g., when density changes but not velocity) then these equations understand and don't correct the location from where the reference velocity locates those events, which in that case is correct; and (2) the error in locating reflectors caused by an error in velocity depends on both the size of the error and the duration of the error. Hence, the integral of $\alpha_1 - \beta_1$ represents an amplitude and duration correction to the originally mislocated primary. This is a general principle, when an inversion task has a duration aspect to the problem being addressed, the response has an integral in the solution. The inverse series empowers the primary events in the data to speak to themselves for non linear inversion and to speak to each other to deal with the effect of erroneous velocity on amplitude analysis. The analogous "discussion between events" for multiple removal is described in the conclusions.

Figures (25) and (26) illustrate the imaging portion of the inverse series for a 1-D constant density, variable velocity acoustic medium. The depth that the reference velocity images the second reflector at is $z_{b'}$=136m. The band-limited singular functions of the imaging subseries act to extend the interface from $z_{b'}$ to z_b (Fig. (25)). The cumulative sum of these imaging subseries terms is illustrated in Fig. 26. After summing five terms, the imaging subseries has converged and the deeper reflector has moved towards its correct depth $z_b = 140$ m.

Figures (28)–(31) are a comparison of linear and non-linear prediction for a two-parameter acoustic medium and for a 1-D single interface example (Fig. 27). Figure (28) shows α_1 as a function of two different angles of incidence for a chosen set of acoustic parameters. Figure (29) shows the sum of α_1+α_2, and shows a clear improvement, for all precritical angles, as an estimate for α. Figure (31) illustrates similar improvements for the second parameter β over the linear estimate given in Fig. (30).

Early analysis and tests are encouraging demonstrating intrinsic potential for the of task specific subseries of the inverse series to perform imaging at the correct depth (Shaw *et al.*, 2003) and improving upon linear estimation of earth material properties (Zhang and Weglein, 2003), without an adequate velocity model. Furthermore, tests point to convergence for imaging for large error and duration of error in velocity and rapid improvement in estimates of earth material properties beyond the industry standard linear techniques.

11 Conclusions and Summary

The forward series begins with the reference propagator, $\mathbf{G}_0$, and the perturbation operator, $\mathbf{V}(\vec{r}, \omega)$, the difference between actual and reference medium properties as a function of space, $\vec{r}$. The inverse series inputs data, $D(t)$, in time, and the reference propagator, $\mathbf{G}_0$.

Since the forward series inputs the perturbation, $\mathbf{V}(\vec{r}, w)$, and rapid variation of $\mathbf{V}$ correspond to the exact spatial location of reflectors it follows that space is the domain of comfort of the forward series.

On the contrary, the computation of time of arrival of any (and every)seismic event for which the actual propagation path is not described by $\mathbf{G}_0$ requires an infinite series to get the correct time from the forward series. Time is the domain of discomfort for the forward series for seismic events.

For the inverse series the input is data, $D(t)$,in time, and processes that involve transforming, $D(t)$, to another function of time, e.g., the data without free-surface multiples, $D'(t)$,are simpler to achieve than tasks such as imaging primaries in space that require a map from $D(t)$ to $\mathbf{V}(\vec{r}, \omega)$.

In addition, if accurate a-priori information can be provided for the localization and separation of a given task (as in the case of a free-surface reflection coefficient or ${\mathbf{G}_0}^{FS}$) e.g., for the removal of ghosts and free-surface multiples, where the task is defined in terms of separating events that have a well defined experience (from those events that have not) then further efficiency can derive from subseries that involve time to time maps. In table (1), we summarize the amount of effort required to achieve a certain level of effectiveness for each of the four task-specific subseries.

The strategy is to accomplish one task at a time, in the order listed, and then restart the problem as though the already achieved task never existed. This avoids the coupled task terms in the series. Further more, the achievement of these tasks, in order, can enhance the ability of subsequent tasks to reach their objective. For example, the removal of free-surface and internal multiples significantly improves the ability to estimate the overburden velocity model and subsequently aids the efficacy and efficiency of the imaging and inversion subseries for primaries.

Since the rate of convergence of both multiple removal subseries doesn't benefit at all from anything closer to the earth than water speed, and the cost of the algorithms quickly increase with complexity of the reference medium, the idea is to perform tasks that prefer simple, cheap reference propagation with what they want. Then restart the problem with certain issues in the data addressed, i.e., with new data (e.g., primaries) that require proximal velocity information, with more complex and more costly subseries for tasks that appreciate that assistance for practical efficacy and efficiency. If you don't like the 'Isolated Task and Restart the Problem Strategy', and you wanted to be a purist and start and end with one inverse scattering series, you would need the single complex reference that would allow the toughest task to have an opportunity to succeed. Two issues with the latter approach: (1) where would you get the proximal velocity if troublesome multiples are in your input data; and (2) the one series, one time for all data is an "all or nothing at all" strategy that doesn't allow for stages to succeed and provide benefit, when the overall series or more ambitious goals are beyond reach.

Although both primaries and multiples have experienced the subsurface; and, hence, carry information encoded in their character, the indisputable attitude of the only multidimensional direct inversion method for acoustic and elastic media, the inverse scattering series, is to treat multiples as coherent noise to be removed and primaries as the provider of subsurface information. That doesn't mean that one could never use multiples in some inclusive method that seeks to exploit the information that both primaries and multiples contain. If simply means that an inclusive theory, starting with realistic a-priori information, doesn't now exist, and, further, that the inverse series definitely adopts the exclusive view.

While the ability to directly achieve seemingly impossible inversion objectives from data, $D(t)$, and only an estimated reference propagator, $\mathbf{G}_0$, (which can be inadequate) certainly follows from Eq. (11-14) (see also Weglein *et al.* (1997)) there is value in providing an understanding from an information content point of view, as well.

What basically happens in each task specific subseries is that specific conversations take place between events in the data as a whole that allows, e.g., multiple prediction or accurate depth imaging to take place without an accurate velocity model. "Non-linear in the data" is the key, and means that quadratic terms enter the picture (data times data, at least) and that allows different events to have multiplicative communication.

For example, if you provide the medium in detail you can readily determine whether any event in the data is a primary or multiple. However, if you provide only an isolated event, without the medium properties, then there is no way to determine if it is a primary or multiple, in fact it can be either for different models. So how does the inverse series figure out whether the event is a primary or multiple with out any subsurface information? Since it is a series there is a "conversation" set up with other events and then a yes or no to whether an event is a primary or multiple is completely achievable without any information about the medium. In Fig. (32) we show an internal multiple (dashed line), SABCR. Primaries SABE, DBCR and DBE have a phase relationship with the internal multiple SABCR such that.

$$(\text{SABE})_{\text{time}} + (\text{DBCR})_{\text{time}} - (\text{DBE})_{\text{time}} = (\text{SABCR})_{\text{time}}. \tag{91}$$

Hence, if the overall data contains three events such that two are longer time events and if the sum of the time of the two longer events *minus* one smaller time event corresponds to the time of the event under investigation, the event is an internal multiple and, if so, it is removed. This is the reason the third term in the inverse series, that involves three $D(t)$ data terms, starts the process of internal multiple removal and why the "W" diagram (see Fig (32)) is as the heart of the internal multiple prediction from the data procedure; and, finally, why the time prediction of all internal multiples is precise.

In the subseries for imaging at depth without the velocity the first term is current linear migration and places each events exactly where the input reference velocity dictates. The latter imaging process is linear in the data and events are not asked their individual view or opinion of the input velocity non are they allowed to discuss it amongst themselves.

The second term in the inverse series has integral terms (e.g., Eq. 90) that start to move the incorrectly imaged events (from using the reference velocity) towards their correct location. There is a quadratic dependence on the data, allowing multiplicative conversations between two events and they are empowered to have an opinion about the input velocity. If they decide together that (at least) one of the events has been provided with a velocity model not consistent with those two events, the troubled event (usually deeper) asks for assistance from a shallower event to help it use its amplitude and the degree of dissatisfaction to move the deeper primary towards its correct location.

Hence, the inverse series and the task-specific subseries concept, represent as fundamentally a new and potentially impactful way of thinking about imaging and inverting primaries as it represented for the earlier (and now mature and standard algorithms for) removal of multiples. There were serious conceptual and practical hurdles in the theoretical evolution, development and robust industrial application.

We anticipate that in bringing the subseries for imaging and inverting primaries through that same process, that higher hurdles and tougher prerequisites will be addressed. The potential benefits for exploration and production of hydrocarbons are great. We would be delighted if this paper would serve to encourage other fields of non-destructive evaluation e.g., medical imaging, environmental monitoring, and defense detection and identification, and earth quake deep earth definition, application to benefit from these efforts as well.

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References

Araújo, F. V., 1994, Linear and nonlinear methods derived from scattering theory: Backscattered tomography and internal multiple attenuation: Ph.D. thesis, Universidade Federal da Bahia, Brazil, (in Portuguese).

Araújo, F. V., Weglein, A. B., Carvalho, P. M., and Stolt, R. H., 1994a, Internal multiple attenuation: EAEG Abstracts.

Araújo, F. V., Weglein, A. B., Carvalho, P. M., and Stolt, R. H., 1994b, Inverse scattering series for multiple attenuation: An example with surface and internal multiples: 64th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded Abstracts, 1039-1041.

Boyse, W. E., Wave propagation and inversion in slightly inhomogeneous media, p.40. (1986)

Boyse, W.E. and Keller J.B., Inverse elastic scattering in three dimensions, 1986: J. Acoust. Soc. Am., **79**, 215–218.

Carvalho, P. M., 1992, Free-surface multiple reflection elimination method based on nonlinear inversion of seismic data: Ph.D. thesis, Universidade Federal da Bahia, Brazil, (in Portuguese).

Robert W. Clayton and Robert H. Stolt: Geophysics, **46**, 1559-1567.

Jack K. Cohen and Bleistein N.: SIAM Journal of Applied Math (1977), **32**, 784-799.

Devaney, A. J., and Weglein A. B., Inverse scattering using the Heitler equation, Inverse Problems (December, 1989) **5**, 3, p49–52.

Innanen, K.A. and Weglein, A.B., Viscoacoustic Born Series Continued: Toward Scattering-based Q Compensation /Estimation, 2003, (in preparation).

Keys, R. G., and Weglein, A. B., 1983, Generalized linear inversion and the first Born theory for acoustic media, J. Math. Phys., **24(6)**, 144-149.

ten Kroode, F., 2002, Prediction of internal multiples, Wave Motion 35, p.315-338.

Matson, K., 1996, The relationship between scattering theory and the primaries and multiples of reflection seismic data: J. Seis. Expl., **5**, 63-78.

Morley, L., and J. Claerbout: Geophysics, **48**, 515-531.

Moses, H. E., 1956, Calculation of scattering potential from reflection coefficients: Phys. Rev., **102**, 559-567.

Newton, R.G., 2002, Scattering Theory of Waves and Particles, Dover Publishing.

Nita, B., Matson, K. and Weglein, A. B., 2003, Forward scattering series seismic events: high frequency approximations, critical and postcritical reflections. Journal of Mathematical Physics.

Prosser, R. T., 1969, Formal solutions of inverse scattering problems: J. Math. Phys., **10**, 1819-1822.

Razavy, M., 1975, Determination of the wave velocity in an inhomogeneous medium from reflection data: J. Acoust. Soc. Am., **58**, 956-963.

Shaw, S.A. and Weglein, A.B., Imaging seismic reflection data at the correct depth without specifying an accurate velocity model: initial numerical examples of an inverse scattering subseries, 2003 in Frontiers of remote sensing information processing, C.H. Chen (ed.), World scientific publishing co. (in press).

Shaw, S.A., Weglein, A.B., Foster, D.J., Matson, K.H., and Keys, R.G., Isolation of a leading order depth imaging series and analysis of its convergence properties, 2003, (in preparation).

Stolt, R. H. and Weglein, A. B., 1985, Migration and inversion of seismic data: Geophysics, **50**, 2458-2472.

Stolt, R. H. and Jacobs, B., 1980, Inversion of seismic data in a laterally heterogeneous medium: SEP Report, **24**, 135-152.

Ware, J. A. and Aki, K., 1969, Continuous and discrete inverse-scattering problems in a stratified elastic medium. I. Plane waves at normal incidence: J. Acoust. Soc. Am., **45**, 911-921.

Weglein, A. B., Boyse, W. E. and Anderson, J. E., 1981, Obtaining three-dimensional velocity information directly from reflection seismic data: an inverse scattering formalism: Geophysics, **46**, 116-1120.

Weglein, A.B., D.J. Foster, K.H. Matson, S.A. Shaw, P.M. Carvalho and D. Corrigan, Predicting the correct spatial location of reflectors without knowing or determining the precise medium and wave velocity: initial concept, algorithm and analytic and numerical example, Journal of Seismic Exploration 10, 367-382 (2002).

Weglein, A.B., Gasparotto, F.A., Carvalho, P.M. and Stolt, R.H., 1997. An inverse scattering series method for attenuating multiples in seismic reflection data. Geophysics, 62: 1975-1989.

Weglein, A.B., Matson, K.H., Foster, D.J., Carvalho, P.M., Corrigan, D. and Shaw, S.A., 2000. Imaging and inversion at depth without a velocity model. 70th Ann. Internat. SEG Mtg., Calgary, Alberta.

Weglein, A.B. and Stolt, R.H., 1999. Migration-inversion revisited (1999). The Leading Edge, **18**, 8: 950-952.

Zhang, H. and Weglein, A.B., Target identification using the inverse scattering series; inversion of large-contrast, variable velocity and density acoustic media, 2003, (in preparation).

Further Reading

References

Carvalho, P. M., Weglein, A. B., and Stolt, R. H., 1991, Examples of a nonlinear inversion method based on the T matrix of scattering theory: application on multiple suppression: 61st Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded abstracts, 1319-1322.

Carvalho, P. M., Weglein, A. B., and Stolt, R. H., 1992, Nonlinear inverse scattering for multiple suppression: application to real data, Part I: 62nd Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded abstracts, 1093-1095.

Coates, R. T., and Weglein, A. B., 1996, Internal multiple attenuation using inverse scattering: Results from prestack 1 & 2D acoustic and elastic synthetics, 66th Ann. Internat. Mtg., Soc. Expl. Geophys., Expanded abstracts, 1522-1525.

Dragoset, W. H., 1992, Surface multiple attenuation - theory, practical issues, examples: EAEG Abstracts, BO27.

Fokkema, J. T. and van den Berg, P. M., 1993, Seismic applications of acoustic reciprocity: Elsevier Science Publ. Co., Inc.

Ikelle, L. T., Roberts, G., and Weglein, A. B., 1995, Source signature estimation based on the removal of first order multiples: 65th Ann. Internat. Mtg., Soc. Expl. Geohys., Expanded Abstracts, 1478-1481.

Kennett, B. L. N., 1979, The suppression of surface multiples on seismic records: Geophys. Prosp., **27**, 484-600.

Matson, K., and Weglein, A. B., 1996, Free-surface elastic multiple removal using inverse scattering: Canad. SEG Expanded Abstracts.

Newton, R. G., 1981, Inversion of reflection data for layered media: a review of exact methods: Geophys. J. Roy. Astr. Soc., **65**, 191-215.

Riley, D. C., and Claerbout, J. F., 1976, 2D multiple reflections: Geophysics, **41**, 592-620.

Stolt, R. H. and Jacobs, B., 1981, An approach to the inverse seismic problem: SEP Report, **25**, 121-132.

Stolt, R. H. and Weglein, A. B., 1985, Migration and inversion of seismic data: Geophysics, **50**, 2458-2472.

Taylor, J. R., Scattering Theory, John Wiley and Sons, Inc., New York.

Verschuur, D. J., Berkhout, A. J. and Wapenaar, C. P. A., 1992, Adaptative surface-related multiple elimination: Geophysics, **57**, 1166-1177.

Weglein, A. B., 1985, The inverse scattering concept and its seismic application, *in* Fitch, A. A., Developments in Geophysical Exploration Methods: Elsevier Science Publ. Co., Inc, **6**, 111-138.

Weglein, A. B., 1995, Multiple attenuation: Recent advances and the road ahead: 65th Ann. Internat. Mtg., Soc. Expl. Geohys., Expanded Abstracts, 1492-1495.

Carvalho, P.M. and Weglein, A.B., 1994. Wavelet estimation for surface multiple attenuation using a simulated annealing algorithm. Expanded Abstr., 64th Ann. Internat. SEG Mtg., Los Angeles.

Manin, M. and Spitz, S., 1995. 3-D attenuation of targeted multiples with a pattern recognition technique. Expandede Abstr., 57th EAGE Conf., Glasgow, BO46.

Matson, K.H., 2000. An overview of wavelet estimation using free-surface multiple removal. The Leading Edge, 19: 50-55. Matson, K.H. and Weglein, A.B., 1996. Removal of elastic interface multiples from land and ocean bottom data using inverse scattering. Expanded Abstr., 66th Ann. Internat. SEG Mtg., Denver: 1526-1529.

Matson, K.H., Paschal, D. and Weglein, A.B., 1999. A comparison of three multiple attenuation methods applied to a hard water-bottom data set. The Leading Edge, 18: 120-126.

Osen, A., Secrest, B.G., Amundsen, L. and Reitan, A., 1998. wavelet estimation from marine pressure measurements. Geophysics, 63: 2108-2119.

Tan, T.H., 1999. Wavelet spectrum estimation. Geophysics, 64: 1836-1846.

Snieder, R., The role of the Born approximation in nonlinear inversion, 1990: Inverse Problems, **6**, 247–266.

Snieder, R., An extension of Backus-Gilbert theory to nonlinear inverse problems, 1991: Inverse Problems, **7**, 409–433.

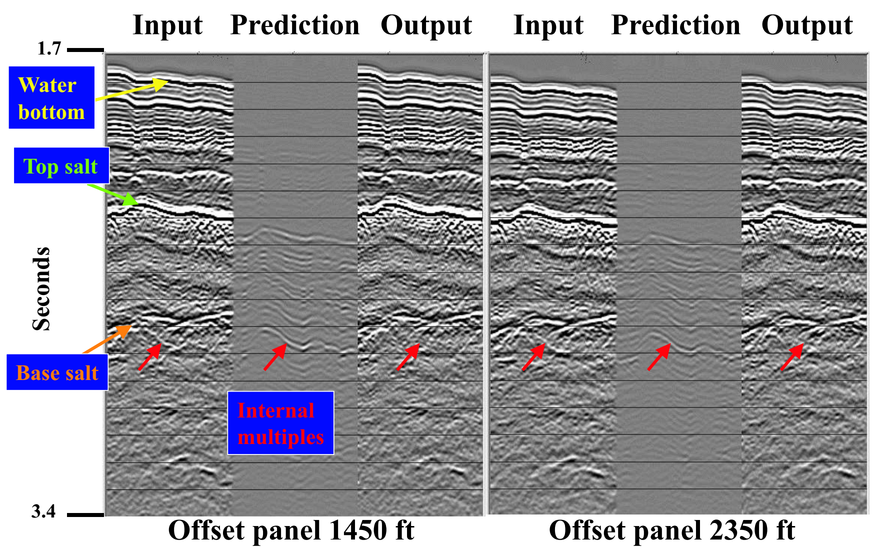

Figure 22: An example of internal multiple attenuation from the Gulf of Mexico. Data are courtesy of WesternGeco.

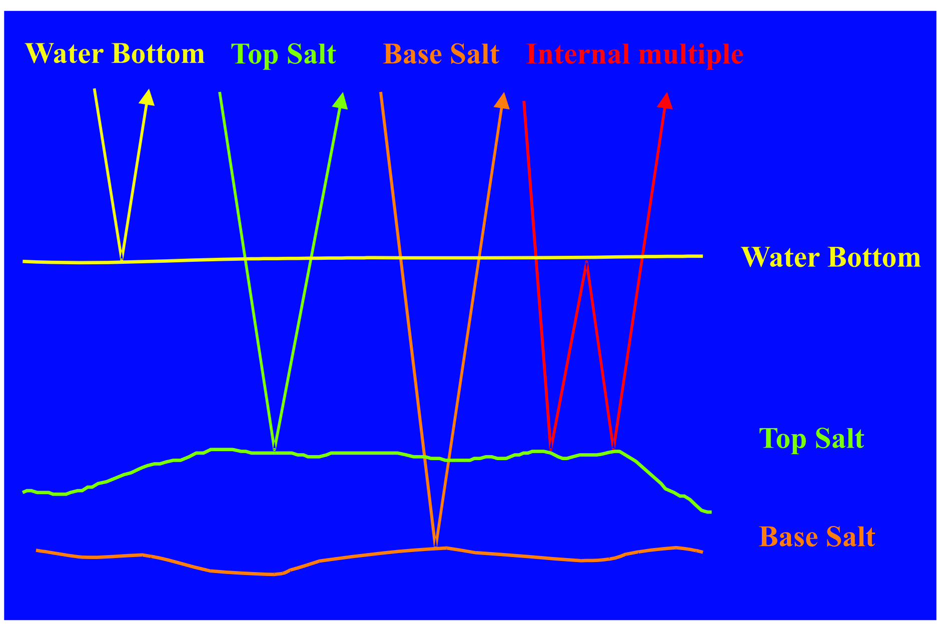

Figure 23: A cartoon illustrating the events that are used by the algorithm to predict a subsalt internal multiple.

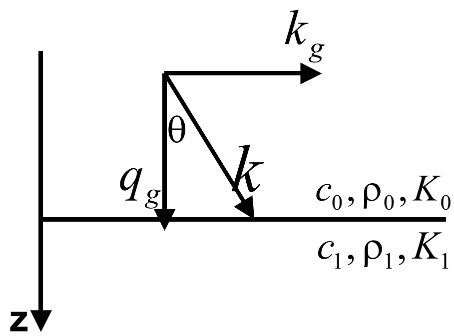

Figure 24: The relationship between q_g, k_g and θ

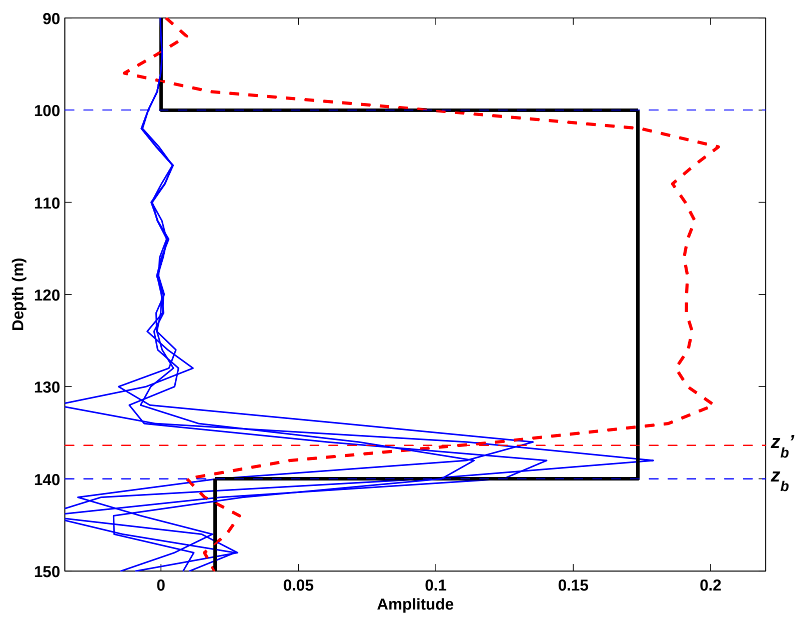

Figure 25: Five terms in the leading order imaging subseries. The solid black line is the actual perturbation α and the dashed red line is α_1, the first approximation to α. The blue lines are the leading order imaging subseries terms. The cumulative sum of these imaging terms is shown in Figure 26.

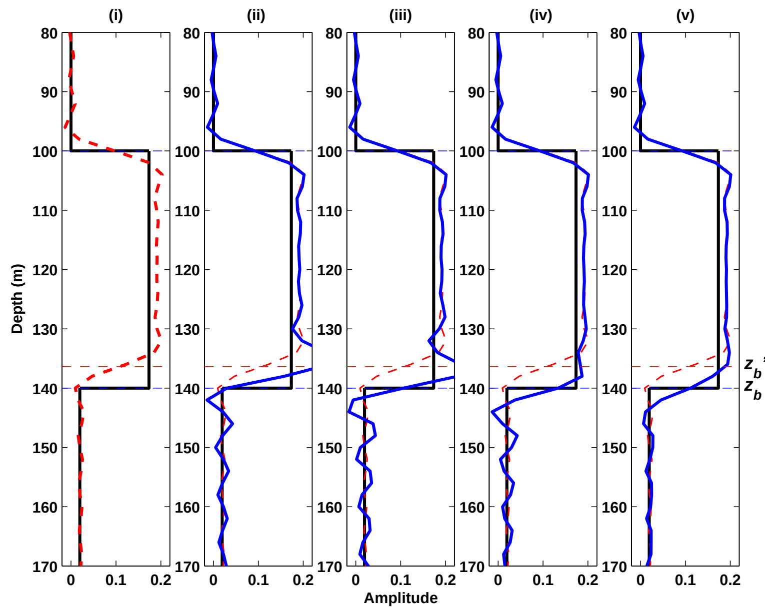

Figure 26: Cumulative sum of five terms in the leading order imaging subseries. The solid black line is the perturbation α and the red line is the first approximation to α or the first term in the inverse series, α_1. The blue line is the cumulative sum of the imaging subseries terms, e.g. in panel (ii) the sum of two terms in the subseries is shown, and in panel (v) the sum of five terms in the subseries is displayed.

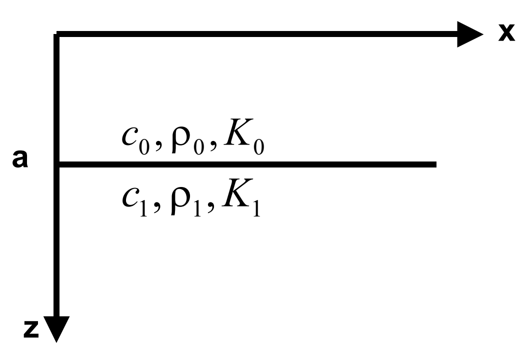

Figure 27: One-dimensional acoustic model.

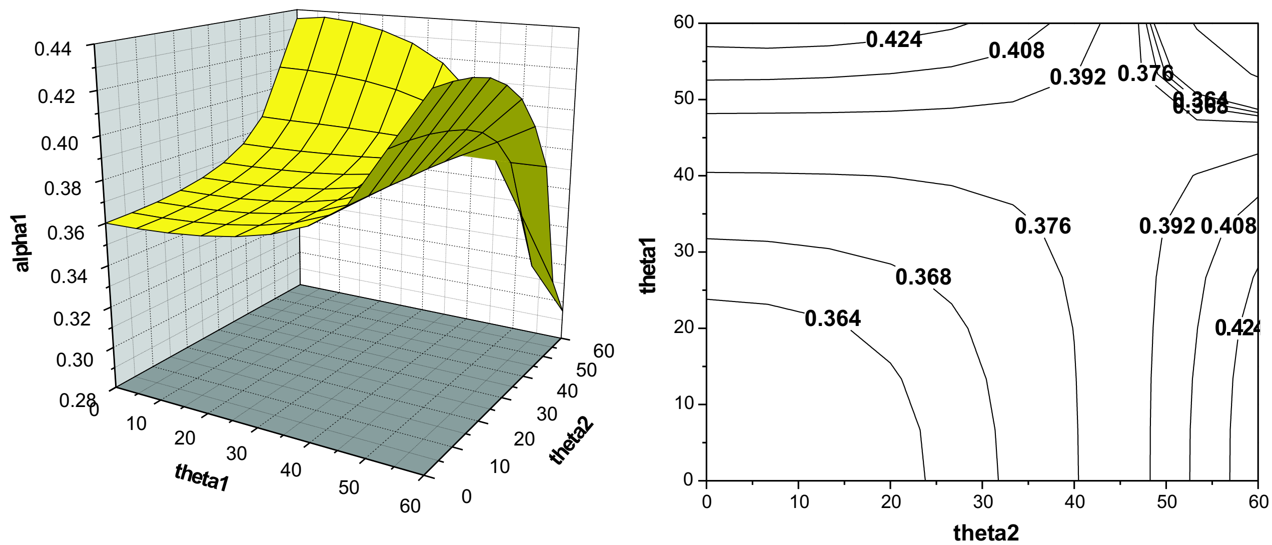

Figure 28: α_1 displayed as a function of two angles. The graph on the right is a contour plot of the graph on the left. In this example, the exact value of α is 0.292.

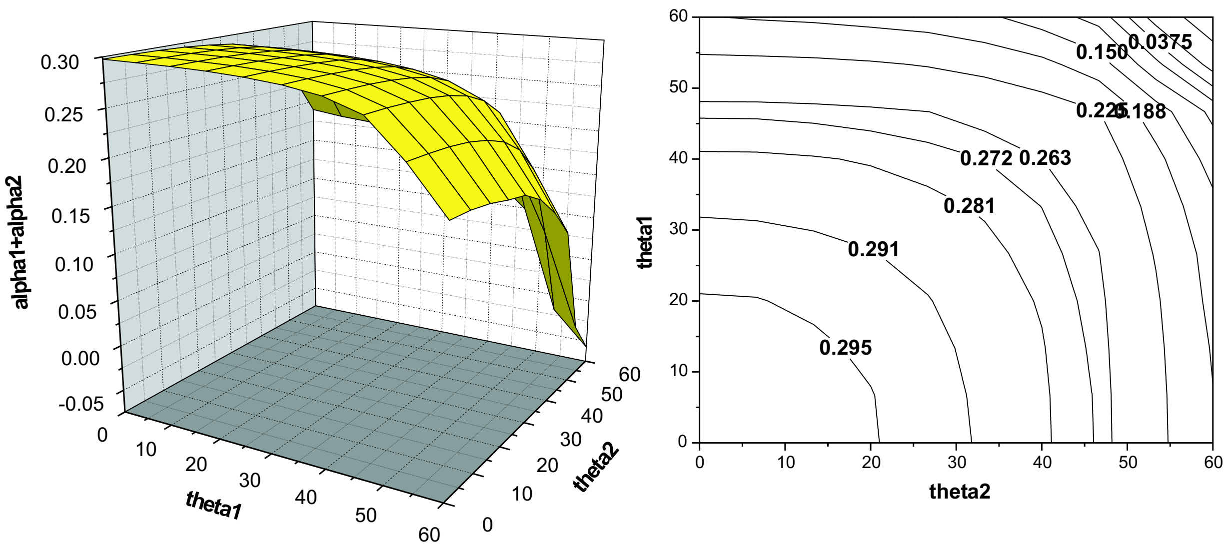

Figure 29: The sum of $\alpha_1 + \alpha_2$ displayed as a function of two angles for the same example as in Fig. 28.

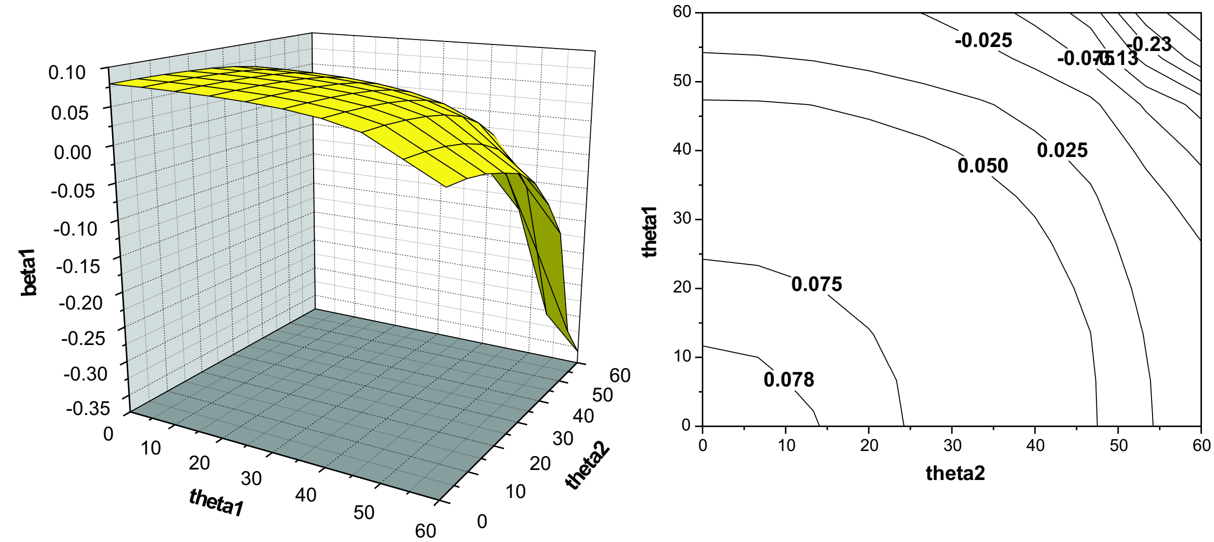

Figure 30: β_1 displayed as a function of two angles. The graph on the right is a contour plot of the graph on the left. In this example, the exact value of β is 0.09.

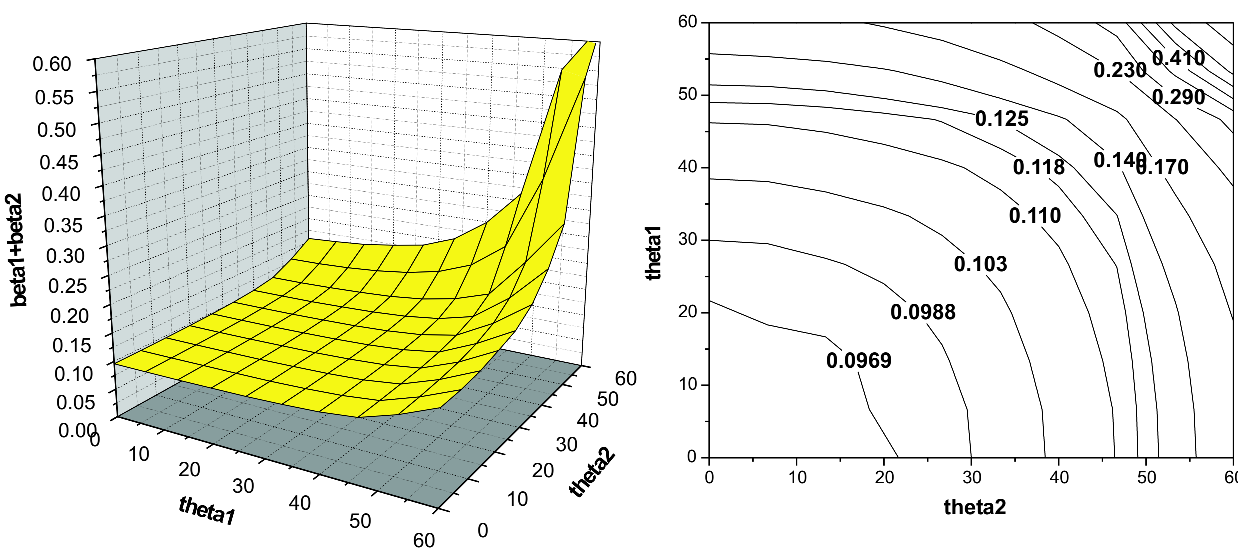

Figure 31: The sum of $\beta_1 + \beta_2$ displayed as a function of two angles for the same example as in Fig. 30.

TASK	PROPERTIES.
Free-surface multiple elimination	One term in the subseries predicts precisely the time and amplitude of all free-surface multiples of a given order independent of the rest of the history of the event. Order is defined as number of times the multiple has a downward reflection at the free-surface.
Internal multiple attenuation	One term in the inverse series predicts the precise time and approximate amplitude of all internal multiples of a given order. The order of an internal multiple is defined by the number of downward reflections from any subsurface reflector at any depth.
Imaging at depth without the precise velocity	First term in series corresponds to current migration or migration − inversion. To achieve a well-estimated depth-map requires an infinite series directly in terms of an inadequate velocity model. A priori velocity estimate will aid rate of convergence.
Inversion at depth without the precise overburden	First term in subseries corresponds to current linear amplitude analysis. For improvement to linear estimates of earth property changes and to account for inadequate overburden requires infinite series. Tests indicate rapid convergence for the first non-linear parameter estimation objective.

Table 1: Summary of task-specific subseries.

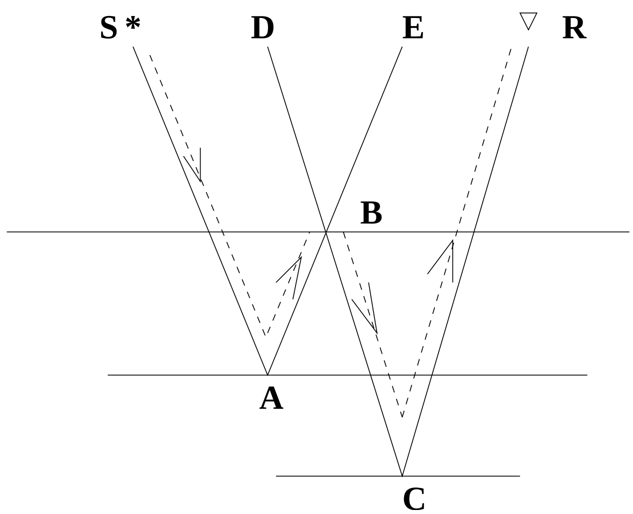

Figure 32: Subevents for an internal multiple.