Investigating the grouping of inverse scattering series terms: simultaneous imaging and inversion I

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Abstract

We consider those portions of the inverse scattering series (for the simple case of a 1D normal incidence acoustic problem) that are concerned with imaging and inversion of primaries in a measured wave field. We show that many of the terms involved in this effort can be captured with a subseries that involves the n'th derivative of the n'th power of the first integral of the Born approximation; it is referred to here as the simultaneous imaging and inversion subseries. The value of this subseries at present lies in what it can tell us about the functioning of the series as a whole; we investigate it for purposes of basic insight.

We begin by outlining the formulation of the simultaneous imaging and inversion subseries, initially using it to reproduce terms from the 1D normal incidence inverse series as a means to illustrate the extent of the approximations made, and note that an equivalent form for the coupled subseries may be developed which leads to a closed form. We lastly interpret the form of the simultaneous imaging and inversion algorithm from a signal-processing perspective, to provide insight into precisely what operations the inverse scattering series advocates as the general means for imaging and inverting primaries.

1 Introduction

The idea of task separation has been the critical conceptual leap in the success of the inverse scattering series to date (Weglein et al., 1997; Weglein et al., 2003). It hinges on the apparent willingness, or even predisposition, of the series to compartmentalize the entire inversion into portions that greatly resemble existing steps in seismic data processing. Task separation is twice beneficial, because (i) separately accomplishing any one task disengages the user from the other, often far more complex (and possibly divergent) problems of inversion, and (ii) the output of this task may be a valuable product in its own right. Amongst other advances, a leading-order imaging subseries has been identified (Shaw et al., 2003) in the 1D normal incidence case for which (i) a term of arbitrary order may be immediately written down, and (ii) a closed-form expression exists. This subseries has been shown to locate reflectors with a high-degree of accuracy without concerning itself with the issue of perturbation amplitude (i.e. the 1D version of the inversion task). Furthermore, the second term in the inversion subseries has been shown to improve the estimation of density and bulk modulus (and thereby

P-wave velocity etc.) beyond the linearized form for a 1D case with offset. This amounts to a method for nonlinear AVO (Zhang and Weglein, 2003). Hence, early evidence is strongly suggestive of the value of task separation.

We are, however, interested in the functioning of the inverse scattering series as a whole – we are interested in exposing the basic mechanisms of the series as a means to transform primaries into correctly-located Earth parameter distributions. That is the core motivation behind the work described in this paper, in which we purposefully couple the tasks of imaging and inversion and investigate the nature of the resulting algorithm. This paper constitutes part of the thesis research of one of us (Innanen, 2003), and has been partly reported in (Innanen and Weglein, 2003), and also includes a recent mathematical treatment.

We begin by outlining the formulation of the simultaneous imaging and inversion approach. We initially use the formulation to reproduce terms from the 1D normal incidence inverse series as a means to illustrate the extent of the approximations made. Then, retracing steps, we detail how, making these approximations, the form of the terms was deduced, and, in particular, how these approximations result in reducing the effect of entire classes of terms in the inverse series to simple alterations of the sign of the output. We then note that an equivalent form for the coupled subseries may be developed which leads to a closed form. Finally we interpret the form of the simultaneous imaging and inversion algorithm, to provide insight into precisely what operations the inverse scattering series advocates as the general means for imaging and inverting primaries. We show how a simultaneous imaging/inversion operator, whose form is dictated by the inverse scattering series, performs imaging via identification and correction of discontinuities in the measured data, and inversion via an alteration of the amplitudes that comes from the exponentiation of the integrals of the measured signal.

2 Background and a Useful Notation

This section follows closely the derivation and discussion of Weglein et al. (2003). To briefly review the theory of inverse scattering, it is useful to temporarily resort to an operator notation, whereby, for instance, a "true" wave field satisfies the equation

$$\mathbf{LG} = -\mathbf{I}, \tag{1}$$

and a "reference" wave field satisfies

$$\mathbf{L_0G_0} = -\mathbf{I}, \tag{2}$$

where $\mathbf{L}$ and $\mathbf{L_0}$ are the true and reference wave operators, and $\mathbf{G}$ and $\mathbf{G_0}$ are the true and reference Green's operators respectively. The operators are general in the sense of model, and fully 3D. Equations (1) and (2) are in the space/temporal frequency domain. Two important quantities are associated with the difference of these operators:

$$\mathbf{V} = \mathbf{L} - \mathbf{L_0}, \tag{3}$$

known as the perturbation operator, scattering potential, or scattering operator, and

$$\Psi_s = \mathbf{G} - \mathbf{G_0}, \tag{4}$$

known as the scattered wave field. The Lippmann-Schwinger equation is an operator identity in this framework:

$$\Psi_s = \mathbf{G} - \mathbf{G_0} = \mathbf{G_0 V G}, \tag{5}$$

and it begets the Born series through self-substitution:

$$\begin{aligned} \Psi_s &= \mathbf{G_0VG_0} + \mathbf{G_0VG_0VG_0} + \mathbf{G_0VG_0VG_0VG_0} + ... \\ &= (\Psi_s)_1 + (\Psi_s)_2 + (\Psi_s)_3 + ... \end{aligned} \tag{6}$$

In other words, the scattered field is represented as a series in increasing order in the scattering potential. This formalism constitutes a forward modelling of the wave field; it is a nonlinear mapping between the perturbation and the wave field, the latter being written in increasing orders of the former.

Inversion, or the solution for $\mathbf{V}$ from measurements of the scattered field *outside of* $\mathbf{V}$, has no closed form. The approach taken here is that of Jost and Kohn (1952) and Moses (1956); it was formulated for the inversion of wave velocity by Razavy (1975), and discussed in the framework of seismic data processing and inversion by Stolt and Jacobs (1981) and Weglein et al. (1981). It is to represent the solution (the perturbation operator) as an infinite series:

$$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2} + \mathbf{V_3} + ..., \tag{7}$$

where $\mathbf{V_j}$ is "j'th order in the data". This form is substituted into the terms of the Born series, and terms of like order in Ψ_s are equated (each term is considered to have been evaluated on the measurement surface m). This is the form of the inverse scattering series:

$$\begin{aligned} &(\Psi_s)_m = (\mathbf{G_0V_1G_0})_m, \\ &0 = (\mathbf{G_0V_2G_0})_m + (\mathbf{G_0V_1G_0V_1G_0})_m, \\ &0 = (\mathbf{G_0V_3G_0})_m + (\mathbf{G_0V_1G_0V_1G_0V_1G_0})_m + (\mathbf{G_0V_1G_0V_2G_0})_m + (\mathbf{G_0V_2G_0V_1G_0})_m, \\ &... \end{aligned} \tag{8}$$

The idea is that $\mathbf{V_1}$, the component of $\mathbf{V}$ that is linear in the data, is solved for with the first equation. This result is substituted into the second equation, leaving $\mathbf{V_2}$ as the only unknown, which may then also be solved for. This continues until a sufficient set of $\mathbf{V_j}$ are known to accurately approximate the desired result $\mathbf{V}$.

The form of the operators $\mathbf{L}$, $\mathbf{G}$, $\mathbf{L_0}$, $\mathbf{G_0}$, and $\mathbf{V}$ obviously vary depending on the desired form for the wave propagation (i.e. acoustic constant density, elastic, viscoacoustic, etc.), and, in the case of $\mathbf{V}$, are particularly dependent on how propagation in the reference medium differs from that of the true medium. We focus on the simplest possible cases: here that of 1D constant density acoustic media. Reference media are kept homogeneous, and the scattering potential is considered to be confined to a finite region on one side of the source and receiver locations.

This choice amounts to defining

$$\begin{aligned} \mathbf{L} &= \frac{d^2}{dz^2} + \left(\frac{\omega}{c(z)}\right)^2, \\ \mathbf{L_0} &= \frac{d^2}{dz^2} + \left(\frac{\omega}{c_0}\right)^2, \end{aligned} \tag{9}$$

in which case

$$\mathbf{V} = \left(\frac{\omega}{c(z)}\right)^2 - \left(\frac{\omega}{c_0}\right)^2 = k^2\alpha(z), \tag{10}$$

where $k = \omega/c_0$ and $\alpha(z) = 1 + c_0^2/c^2(z)$. This simple physical framework also permits the use of the Green's function

$$G_0(z|z_s;k) = \frac{e^{ik|z-z_s|}}{2ik} \tag{11}$$

(which becomes $\mathbf{G_0}$ when it is included as part of the kernel of the integrals of the series). In this framework the inverse scattering series terms of interest (equation (7)) can be reduced to

$$\alpha(z) = \alpha_1(z) + \alpha_2(z) + \alpha_3(z) + ... \tag{12}$$

Finally, following the conventional physical interpretation of the "rightmost" Green's operator in every term of equation (8) as being the incident wave field, these are replaced with incident plane-waves $\psi_0(z|z_s;k)$.

The terms of equation (8) become

$$
\begin{aligned}
\psi_s(z|z_s;k) &= \int_{-\infty}^{\infty} G_0(z|z';k)k^2\alpha_1(z')\psi_0(z'|z_s;k)dz', \\
0 &= \int_{-\infty}^{\infty} G_0(z|z';k)k^2\alpha_2(z')\psi_0(z'|z_s;k)dz' \\
&+ \int_{-\infty}^{\infty} G_0(z|z';k)k^2\alpha_1(z')\int_{-\infty}^{\infty} G_0(z'|z'';k)k^2\alpha_1(z'')\psi_0(z''|z_s;k)dz''dz', \\
&\dots
\end{aligned}
\tag{13}
$$

Consider these equations individually. Solving the first for $\alpha_1(z)$ is the 1D equivalent of Born inversion, and amounts, in essence, to trace integration. The resemblance of the equation to a Fourier transform results in (Weglein et al., 2003)

$$
\alpha_1(z) = 4\int_0^z \psi_s(z')dz', \tag{14}
$$

where the depth variable here is the so-called "pseudo-depth", determined by the natural time variable of the measurement of the wave field ψ_s and the reference wavespeed profile.

Weglein et al. (2003) approach the subsequent orders of $\alpha(z)$ by casting them all in terms of the Born approximation $\alpha_1(z)$. Also, choices for breaking the integrals up are made based on the resulting form of the terms in orders of α_1; these forms are by no means the only way to solve the integrals of equation (13); they are a reasoned choice, the basis for the separation of tasks into those of inversion and of imaging.

The formalism described above is based on the assumption that one measures the "scattered field" $\psi_s = \psi - \psi_0$. It is further assumed (as mentioned above) that all the scatterers occur "beneath" the source and receiver. These assumptions and others lead to a set of requirements on the data that characterize approaches, for the treatment of primary reflections, based on the inverse scattering series:

1. The source waveform has been compensated for.
2. The direct wave has been removed.
3. The source and receiver "ghosts" have been removed.
4. The free-surface multiples have been removed.
5. The internal (interbed) multiples have been removed.

We describe various manipulations of the mathematics of the (previously discussed) casting of the terms of the inverse scattering series. Here we present a notation for these integrals (Innanen, 2003) which speeds up some of the manipulations based on the chain rule and integration by parts. Subsequently, we use this notation to derive the 2nd order terms of the inverse series, and later, to derive and simplify 3rd and 4th order terms as well.

The terms in the inverse series are often profitably cast as an increasingly complex set of operations on $\alpha_1(z)$, the Born approximate solution for $\alpha(z)$, which is linear in the data. Of particular importance is the operation

$$\int_{-\infty}^{z} \alpha_1(z')dz', \tag{15}$$

and "nested" versions of the same, for instance

$$\int_{-\infty}^{z} \alpha_1(z') \int_{-\infty}^{z'} \alpha_1(z'')dz''dz', \tag{16}$$

and

$$\int_{-\infty}^{z} \alpha_1(z') \int_{-\infty}^{z'} \alpha_1(z'') \int_{-\infty}^{z''} \alpha_1(z''')dz'''dz''dz', \tag{17}$$

etc. Notice that equation (15) is a linear operator applied to $\alpha_1(z)$, namely the convolution of $\alpha_1(z)$ with a "left-opening" Heaviside function. Define this operator, i.e. convolution with a left-opening Heaviside, as $\mathcal{H}\{\cdot\}$, such that

$$\mathcal{H}\{\alpha_1(z)\} = \int_{-\infty}^{\infty} H(z - z')\alpha_1(z')dz' = \int_{-\infty}^{z} \alpha_1(z')dz'. \tag{18}$$

Although it will be used sparingly, we further define the convolution of $\alpha_1(z)$ with the time reverse of these Heaviside functions, that is, the "right-opening" kind. Let:

$$\mathcal{H}^{-}\{\alpha_1(z)\} = \int_{-\infty}^{\infty} H(z' - z)\alpha_1(z')dz' = \int_{z}^{\infty} \alpha_1(z')dz'. \tag{19}$$

The following relationship between $\mathcal{H}$ and $\mathcal{H}^{-}$ proves useful: for any $f(z)$ and $g(z)$,

$$\int_{-\infty}^{\infty} f(z')\mathcal{H}^{-}\{g(z')\}\,dz' = \int_{-\infty}^{\infty} g(z')\mathcal{H}\{f(z')\}\,dz'. \tag{20}$$

It can be derived by substituting $\mathcal{H}^{-}$ into the left-hand side of equation (20) and switching variables. An explicit form of this manipulation is done regularly in the derivation of the terms of the inverse series. We have developed this method/notation simply to speed up the derivations.

Since in this paper the operator $\mathcal{H}\{\cdot\}$ is often (but not exclusively) applied to $\alpha_1(z)$, for convenience call

$$H = \mathcal{H}\{\alpha_1(z)\}. \tag{21}$$

Since the operator $\mathcal{H}$ is essentially the antiderivative operator, it follows that for any $f(z)$ that is confined to a finite region,

$$\mathcal{H}\left\{\frac{df(z)}{dz}\right\} = f(z). \tag{22}$$

The nesting seen in equations (16) and (17) is incorporated into this operator framework straightforwardly. Define

$$\mathcal{H}_2\{\alpha_1(z)\} = \mathcal{H}\{\alpha_1(z)\mathcal{H}\{\alpha_1(z)\}\} = \int_{-\infty}^{z}\alpha_1(z')\int_{-\infty}^{z'}\alpha_1(z'')dz''dz', \tag{23}$$

and

$$\mathcal{H}_3\{\alpha_1(z)\} = \mathcal{H}\{\alpha_1(z)\mathcal{H}\{\alpha_1(z)\mathcal{H}\{\alpha_1(z)\}\}\} = \int_{-\infty}^{z}\alpha_1(z')\int_{-\infty}^{z'}\alpha_1(z'')\int_{-\infty}^{z''}\alpha_1(z''')dz'''dz''dz', \tag{24}$$

and so on. In general,

$$\mathcal{H}_n\{\alpha_1(z)\} = \mathcal{H}\{\alpha_1(z)\mathcal{H}_{n-1}\{\alpha_1(z)\}\}. \tag{25}$$

This nesting notation, i.e. $\mathcal{H}_n\{\alpha_1(z)\}$ will not be retained. Since

$$\frac{dH}{dz} = \alpha_1(z), \tag{26}$$

equation (23) can be written

$$\mathcal{H}_2\{\alpha_1(z)\} = \mathcal{H}\left\{\frac{dH}{dz}H\right\}. \tag{27}$$

Also, since

$$\frac{1}{2}\frac{dH^2}{dz} = \frac{dH}{dz}H, \tag{28}$$

equation (22) can be used to write

$$\mathcal{H}_2\{\alpha_1(z)\} = \frac{1}{2}H\left\{\frac{dH^2}{dz}\right\} = \frac{1}{2}H^2. \tag{29}$$

Using this result in the expression for $\mathcal{H}_3$, with similar arguments, and continuing on to $\mathcal{H}_n$, the general relationship

$$\mathcal{H}_n\left\{\alpha_1(z)\right\} = \frac{1}{n!}H^n \tag{30}$$

may be derived. The simplification implicit in equation (30) is not trivial – computing the right-hand side with a given α_1 is much simpler than computing the left.

This terminology is used in the next section, to assist in the derivation and exposition of the terms in the inverse scattering series, and again (and more extensively) in subsequent sections to develop and analyze simultaneous imaging and inversion.

Summing $\alpha_1 + \alpha_2 + ...$ implies the provision of the model α in terms of the measured wave field and the reference media; hence it is inversion, and imaging, without knowledge of, or determination of, the true wavespeed structure of the medium. Understanding the mechanisms of a series with this promise is made possible by study of the simplest possible models. We reproduce the 2nd order terms in the inverse series as cast by Weglein et al. (2003), using the operator notation defined above. The simplification achievable by the notation is not noticeable until later sections; here the idea is to show where in the process of deriving terms it becomes applicable.

The second order terms in the inverse series form the equation

$$\begin{aligned}\int_{-\infty}^{\infty} G_0(z|z';k)k^2\alpha_2(z')\psi_0(z'|z_s;k)dz' = \\ -\int_{-\infty}^{\infty} G_0(z|z';k)k^2\alpha_1(z')\int_{-\infty}^{\infty} G_0(z'|z'';k)k^2\alpha_1(z'')\psi_0(z''|z_s;k)dz''dz'.\end{aligned} \tag{31}$$

Upon substitution of the forms of the Green's functions etc. into equation (31), many like factors cancel. Furthermore, the left hand side of the equation, like in the Born approximate case, is a Fourier transform of the perturbation component (α_2 in this case). Hence equation (31) may be written

$$\alpha_2(-2k) = -\frac{1}{4}(i2k)\int_{-\infty}^{\infty} e^{ikz'}\alpha_1(z')\int_{-\infty}^{\infty} e^{ik|z'-z''|}\alpha_1(z'')e^{ikz''}dz''dz', \tag{32}$$

and, treating the two cases demanded by the absolute value bars, one has

$$\begin{aligned}\alpha_2(-2k) &= -\frac{1}{4}(i2k)\int_{-\infty}^{\infty} e^{ikz'}\alpha_1(z')\int_{-\infty}^{z} e^{ik(z'-z'')}\alpha_1(z'')e^{ikz''}dz''dz' \\ &\quad -\frac{1}{4}(i2k)\int_{-\infty}^{\infty} e^{ikz'}\alpha_1(z')\int_{z}^{\infty} e^{ik(z''-z')}\alpha_1(z'')e^{ikz''}dz''dz' \\ &= -\frac{1}{4}\int_{-\infty}^{\infty}(i2k)e^{i2kz'}\alpha_1(z')\int_{-\infty}^{\infty} H(z'-z'')\alpha_1(z'')e^{ikz''}dz''dz' \\ &\quad -\frac{1}{4}\int_{-\infty}^{\infty}(i2k)\alpha_1(z')\int_{-\infty}^{\infty} H(z''-z')\alpha_1(z'')e^{i2kz''}dz''dz'.\end{aligned} \tag{33}$$

The operators defined previously are applicable at this stage, i.e. with the appearance of the convolutions of various quantities with Heaviside functions. Equation (33) can be written

$$\begin{aligned}\alpha_2(-2k) = &-\frac{1}{4}\int_{-\infty}^{\infty}(i2k)e^{i2kz'}\left[\alpha_1 H\right](z')dz' \\ &-\frac{1}{4}\int_{-\infty}^{\infty}(i2k)\left[\alpha_1\mathcal{H}^-\left\{e^{i2kz'}\alpha_1\right\}\right](z')dz' \\ =&-\frac{1}{4}\int_{-\infty}^{\infty}(i2k)e^{i2kz'}\left[\alpha_1 H\right](z')dz' \\ &-\frac{1}{4}\int_{-\infty}^{\infty}(i2k)\left[e^{i2kz'}\alpha_1 H\right](z')dz'.\end{aligned} \tag{34}$$

Because the $\mathcal{H}$, $\mathcal{H}^-$ operators are convolution operators, the totality of their action in the integrands in equation (34) is an expression with z' dependence only. For convenience we therefore use the variable z' freely inside these operators even though strictly speaking it doesn't belong[1]. For instance,

$$\mathcal{H}^-\left\{e^{i2kz'}\alpha_1\right\} = \int_{z'}^{\infty} e^{i2kz''}\alpha_1(z'')dz''. \tag{35}$$

Both terms in equation (34) are Fourier transforms of derivatives with respect to z, so

$$\begin{aligned}\alpha_2(z) =& -\frac{1}{4}\frac{d}{dz}\left[2\alpha_1 H\right] \\ =& -\frac{1}{2}\left(\left[\frac{d\alpha_1}{dz}\right]H + \alpha_1^2\right).\end{aligned} \tag{36}$$

Finally, therefore,

$$\alpha_2(z) = \alpha_2^{(1)}(z) + \alpha_2^{(2)}(z) = -\frac{1}{2}\alpha_1^2(z) - \frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]\int_0^z \alpha_1(z')dz'. \tag{37}$$

Using similar manipulations, the third order terms may be derived as Weglein et al. (2003):

$$\alpha_3(z) = \alpha_3^{(1)}(z) + \alpha_3^{(2)}(z) + \alpha_3^{(3)}(z) + \alpha_3^{(4)}(z) + \alpha_3^{(5)}(z), \tag{38}$$

where

[1] It is reminiscent of the convention of discussing convolutions as $h(z') = f(z') * g(z')$, i.e. with seemingly careless use of the output variable z' in both input functions.

$$
\begin{aligned}
\alpha_3^{(1)}(z) &= \frac{1}{8}\left[\frac{d^2\alpha_1}{dz^2}\right]\left(\int_0^z \alpha_1(z')dz'\right)^2, \\
\alpha_3^{(2)}(z) &= \frac{3}{16}\alpha_1^3(z), \\
\alpha_3^{(3)}(z) &= \frac{3}{4}\alpha_1(z)\left[\frac{d\alpha_1}{dz}\right]\int_0^z \alpha_1(z')dz', \\
\alpha_3^{(4)}(z) &= -\frac{1}{8}\left[\frac{d\alpha_1}{dz}\right]\int_0^z \alpha_1^2(z')dz', \\
\alpha_3^{(5)}(z) &= -\frac{1}{16}\int_0^z\int_0^z \left[\frac{d\alpha_1(z')}{dz'}\right]\left[\frac{d\alpha_1(z'')}{dz''}\right]\alpha_1(z''+z'-z)dz''dz'.
\end{aligned}
\tag{39}
$$

This specific casting of the inverse series, as in equations (37) and (39), naturally separates the full inversion process into tasks (Weglein et al., 2003). For instance, at each order there is a term which is a weighted power of the Born approximation. Finding these in equations (37) and (39) and summing produces the subseries

$$
\alpha_{INV}(z) = \alpha_1(z) - \frac{1}{2}\alpha_1^2(z) + \frac{3}{16}\alpha_1^3(z) + ... \tag{40}
$$

This subseries has been identified (Weglein et al., 2003; Zhang and Weglein, 2003) and developed as the subseries that is concerned with target identification, or inversion proper.

Similarly, in each order are found terms that involve derivatives of the Born approximation $\alpha_1(z)$, weighted by integrals of the same; summing these produces

$$
\alpha_{LOISI}(z) = \alpha_1(z) - \frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]\int_0^z \alpha_1(z')dz' + \frac{1}{8}\left[\frac{d^2\alpha_1}{dz^2}\right]\left(\int_0^z \alpha_1(z')dz'\right)^2 + ... \tag{41}
$$

This subseries (hereafter referred to as $LOIS$) turns out (Shaw et al., 2003) to be the "leading order imaging subseries", that is, the leading order set of terms which are concerned with the correct location of reflectors in the subsurface. This identification continues; $\alpha_3^{(5)}(z)$ in equation (39) is the leading order internal multiple eliminator.

3 Simultaneous Imaging and Inversion

We begin this section with the statement of a formula for simultaneous imaging and inversion for this 1D normal incidence acoustic framework. We then back-track somewhat to give an idea of how the expression was derived. The motivation stems ultimately from the fact that in the terms of the inverse scattering series, formulated as they are above, certain combinations of operations on the Born approximation repeatedly arise. It appears that many of these terms might be produced by a core "generating expression", that for this reason would be equivalent to an engine for the imaging and inversion of the input.

Subsequently this formula is explored regarding (i) its ability to reproduce terms in the inverse series, and (ii) the simplifications inherent in its neglect of a class of series terms.

3.1 A Quantity Related to Imaging and Inversion

Consider the quantity

$$I_n(z) = K_n \frac{d^n H^n}{dz^n}, \tag{42}$$

where

$$K_n = \frac{(-1)^{n-1}}{n} \left(\frac{1}{2^{(n-1)}} \right)^2 \left[\sum_{k=0}^{n-1} \frac{1}{k!(n-k-1)!} \right]. \tag{43}$$

To compute this quantity, the Born approximation $\alpha_1(z)$ is integrated once to get $H = \mathcal{H}\{\alpha_1(z)\}$. The n'th power is taken, followed by the n'th derivative; finally it is weighted by K_n.

This quantity appears to be intimately connected with the terms of the inverse scattering series which relate to imaging and inversion. In and of itself, it is merely an expression that specifies a combination of derivative orders and (effective) numbers of nested integrals of the Born approximation.

3.2 Mapping Between $K_n d^n H^n/dz^n$ and α_n

One can best investigate equation (42) by carrying out the n'th derivative on the n'th power of H, without specifying $\alpha_1(z)$, and seeing what happens. In this section equation (42) is expanded in this way for $n = 1$, $n = 2$, $n = 3$, $n = 4$ and $n = 5$. The results are compared with existing derivations of the $\alpha_n(z)$ to clarify which aspects of the inverse problem are addressed by computing them. For convenience we sometimes suppress the z dependence of the Born approximation. At all times α_1 implies $\alpha_1(z)$.

Expansion for the $n = 1$ Term:

If we set $n = 1$ then $K_1 = 1$, and by equation (42),

$$I_1(z) = \frac{dH}{dz} = \alpha_1(z), \tag{44}$$

i.e. the Born approximation.

Expansion for the $n = 2$ Term:

For the $n = 2$ term we have

$$K_2 = -\frac{1}{4}\left[\frac{1}{2} + \frac{1}{2}\right] = -\frac{1}{4}. \tag{45}$$

Meanwhile,

$$\frac{d^2 H^2}{dz^2} = \frac{d}{dz}\left[2H\alpha_1\right] = 2\alpha_1^2 + 2\left[\frac{d\alpha_1}{dz}\right] H. \tag{46}$$

Recalling the definition of H in equation (21),

$$I_2(z) = K_2 \frac{d^2 H^2}{dz^2} = -\frac{1}{2}\alpha_1^2(z) - \frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]\int_{-\infty}^{z} \alpha_1(z')dz'. \tag{47}$$

Comparison of equations (37) and (47) demonstrates that, up to second order, the expression in equation (42) reproduces all of the expected inverse scattering series terms:

$$I_2(z) = \alpha_2(z). \tag{48}$$

Expansion for the $n = 3$ Term:

Proceeding as before, the third term is found by computing

$$K_3 = \frac{1}{16}\left[\frac{1}{6} + \frac{1}{3} + \frac{1}{6}\right] = \frac{1}{24}, \tag{49}$$

and then

$$\begin{aligned}
\frac{d^3 H^3}{dz^3} &= \frac{d^2}{dz^2}\left[3H^2\alpha_1\right] \\
&= 3\frac{d}{dz}\left[2H\alpha_1^2 + H^2\left[\frac{d\alpha_1}{dz}\right]\right] \\
&= 3\left[2\alpha_1^3 + 4\alpha_1\left[\frac{d\alpha_1}{dz}\right]H + 2\alpha_1 H\left[\frac{d\alpha_1}{dz}\right] + H^2\left[\frac{d^2\alpha_1}{dz^2}\right]\right] \\
&= 6\alpha_1^3 + 18\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + 3\left[\frac{d^2\alpha_1}{dz^2}\right]H^2.
\end{aligned} \tag{50}$$

All together,

$$I_3(z) = K_3 \frac{d^3 H^3}{dz^3} = \frac{1}{4}\alpha_1^3 + \frac{3}{4}\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + \frac{1}{8}\left[\frac{d^2\alpha_1}{dz^2}\right] H^2, \tag{51}$$

or, explicitly,

$$I_3(z) = \frac{1}{4}\alpha_1^3 + \frac{3}{4}\left[\frac{d\alpha_1}{dz}\right]\alpha_1\left(\int_{-\infty}^{z}\alpha_1(z')dz'\right) + \frac{1}{8}\left[\frac{d^2\alpha_1}{dz^2}\right]\left(\int_{-\infty}^{z}\alpha_1(z')dz'\right)^2. \tag{52}$$

These terms no longer match up one-to-one with the full set of inverse scattering series terms; the difference is due to approximations implied by equation (42), which are investigated in the next section. What is missing with this suppression is not merely the leading order internal multiple eliminator, but two other terms as well, including one which alters the coefficient of the inversion (α_1^3) term. This can be established by comparing equations (52) and (38).

On the other hand, equation (42) has correctly incorporated the other components, including those due to $-\mathbf{G_0 V_1 G_0 V_1 G_0 V_1}\psi_\mathbf{0}$, and $-(\mathbf{G_0 V_1 G_0 V_2}\psi_\mathbf{0} + \mathbf{G_0 V_2 G_0 V_1}\psi_\mathbf{0})$, in one fell swoop.

Expansion for the $n = 4$ Term:

For the $n = 4$ case, we have

$$K_4 = -\frac{1}{64}\left[\frac{1}{24} + \frac{1}{8} + \frac{1}{8} + \frac{1}{24}\right] = -\frac{1}{192}, \tag{53}$$

and

$$\begin{aligned}
\frac{d^4 H^4}{dz^4} &= \frac{d^3}{dz^3}\left[4H^3\alpha_1\right] \\
&= 4\frac{d^2}{dz^2}\left[3H^2\alpha_1^2 + H^3\left[\frac{d\alpha_1}{dz}\right]\right] \\
&= 4\frac{d}{dz}\left[6H\alpha_1^3 + 9\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H^2 + \left[\frac{d^2\alpha_1}{dz^2}\right]H^3\right] \\
&= 24\alpha_1^4 + 144\left[\frac{d\alpha_1}{dz}\right]\alpha_1^2 H + 36\left[\frac{d\alpha_1}{dz}\right]^2 H^2 + 48\left[\frac{d^2\alpha_1}{dz^2}\right]\alpha_1 H^2 + 4\left[\frac{d^3\alpha_1}{dz^3}\right]H^3
\end{aligned} \tag{54}$$

So in total,

$$I_4(z) = -\frac{1}{8}\alpha_1^4 - \frac{3}{4}\left[\frac{d\alpha_1}{dz}\right]\alpha_1^2 H - \frac{3}{16}\left[\frac{d\alpha_1}{dz}\right]^2 H^2 - \frac{1}{4}\left[\frac{d^2\alpha_1}{dz^2}\right]\alpha_1 H^2 - \frac{1}{48}\left[\frac{d^3\alpha_1}{dz^3}\right]H^3, \tag{55}$$

avoiding the replacement of H with its explicit form this time around.

Expansion for $n = 5$ Term:

The $n = 5$ expansion is

$$I_5(z) = \frac{1}{16}\alpha_1^5 + \frac{5}{8}\left[\frac{d\alpha_1}{dz}\right]\alpha_1^3 H + \frac{15}{32}\left[\frac{d\alpha_1}{dz}\right]^2 \alpha_1 H^2 + \frac{5}{16}\left[\frac{d^2\alpha_1}{dz^2}\right]\alpha_1^2 H^2 + \frac{1}{24}\left[\frac{d^3\alpha_1}{dz^3}\right]\alpha_1 H^3 + \frac{5}{48}\left[\frac{d^2\alpha_1}{dz^2}\right]\left[\frac{d\alpha_1}{dz}\right] H^3 + \frac{1}{384}\left[\frac{d^4\alpha_1}{dz^4}\right] H^4, \tag{56}$$

again omitting the replacement of H with its explicit form for convenience.

4 Inherent Simplicities and Approximations

Comparing the expansion of equation (42) with the derived terms of the inverse series, it is clear that some of the terms are missing, and others have the wrong coefficients. It is important to be very clear regarding what has been "kept" of the full inverse series, and what has been "rejected", in utilizing equation (42). This section is concerned with developing both a clear sense of the approximations made in this simultaneous imaging and inversion formulation, and explicitly demonstrating the simplified role of certain classes of terms in the series for the 3rd and 4th order (a role that is assumed to continue at all orders).

4.1 Deriving $K_n d^n H^n/dz^n$: Drop a Term, Find a Pattern

Equation (42) was deduced by noticing patterns in some components of the terms involving $\mathbf{V_1}$ *only*; for instance, by paying attention to the $\mathbf{G_0 V_1 G_0 V_1 G_0 V_1} \psi_\mathbf{0}$ term in the 3rd order equation, and ignoring terms like $\mathbf{G_0 V_1 G_0 V_2} \psi_\mathbf{0}$. In fact, it was designed initially to compute these terms only. We will demonstrate by considering $\alpha_3(z)$.

Set

$$\alpha_3(z) = I_p(z) + I_s(z), \tag{57}$$

where

$$
\begin{aligned}
I_p(-2k) &= \frac{1}{4}k^2 \int_{-\infty}^{\infty} e^{i2kz'} \alpha_1(z') \int_{-\infty}^{z'} \alpha_1(z'') \int_{-\infty}^{z''} \alpha_1(z''') dz''' dz'' dz' \\
&+ \frac{1}{4}k^2 \int_{-\infty}^{\infty} \alpha_1(z') \int_{z'}^{\infty} e^{i2kz''} \alpha_1(z'') \int_{-\infty}^{z''} \alpha_1(z''') dz''' dz'' dz' \\
&+ \frac{1}{4}k^2 \int_{-\infty}^{\infty} e^{i2kz'} \alpha_1(z') \int_{-\infty}^{z'} e^{-i2kz''} \alpha_1(z'') \int_{z''}^{\infty} e^{i2kz'''} \alpha_1(z''') dz''' dz'' dz' \\
&+ \frac{1}{4}k^2 \int_{-\infty}^{\infty} \alpha_1(z') \int_{z'}^{\infty} \alpha_1(z'') \int_{z''}^{\infty} e^{i2kz'''} \alpha_1(z''') dz''' dz'' dz',
\end{aligned}
\tag{58}
$$

i.e. I_p is due to $\mathbf{G_0 V_1 G_0 V_1 G_0 V_1} \psi_\mathbf{0}$; I_s is due to $\mathbf{G_0 V_1 G_0 V_2} \psi_\mathbf{0}$ and $\mathbf{G_0 V_2 G_0 V_1} \psi_\mathbf{0}$, and is considered later. These terms are broken up based on the geometry of the scattering interactions. The third term in equation (58) involves a "down-up-down" scattering event, or a change in the directions of propagation; such expressions differ from those involving only upward scattering events, and don't immediately simplify in the same way. For the present analysis, we set this term to zero. Implementing the operator notation, the expression

$$
\begin{aligned}
I_p(-2k) &= \frac{1}{4}k^2 \int_{-\infty}^{\infty} e^{i2kz'} \alpha_1(z') \mathcal{H}\left\{\alpha_1 H\left\{\alpha_1\right\}\right\} dz' \\
&+ \frac{1}{4}k^2 \int_{-\infty}^{\infty} \alpha_1(z') \mathcal{H}^{-}\left\{e^{i2kz'} \alpha_1 \mathcal{H}\left\{\alpha_1\right\}\right\} dz' \\
&+ \frac{1}{4}k^2 \int_{-\infty}^{\infty} \alpha_1(z') \mathcal{H}^{-}\left\{\alpha_1 \mathcal{H}^{-}\left\{e^{i2kz'} \alpha_1\right\}\right\} dz'
\end{aligned}
\tag{59}
$$

is produced. We proceed by making repeated use of the relationship in equation (20) in the definitions section, to produce from equation (59)

$$
\begin{aligned}
I_p(-2k) &= -\frac{1}{16}(i2k)^2 \int_{-\infty}^{\infty} e^{i2kz'} \alpha_1(z') \mathcal{H}\left\{\alpha_1 \mathcal{H}\left\{\alpha_1\right\}\right\} dz' \\
&- \frac{1}{16}(i2k)^2 \int_{-\infty}^{\infty} e^{i2kz'} \alpha_1(z') \mathcal{H}^2\left\{\alpha_1\right\} dz' \\
&- \frac{1}{16}(i2k)^2 \int_{-\infty}^{\infty} e^{i2kz'} \alpha_1(z') \mathcal{H}\left\{\alpha_1 \mathcal{H}\left\{\alpha_1\right\}\right\} dz',
\end{aligned}
\tag{60}
$$

noting that $-(1/16)(i2k)^2 = (1/4)k^2$. Identifying these terms as Fourier transforms of second derivatives, equation (60) may be replaced with

$$
\begin{aligned}
I_p(z) = &-\frac{1}{16}\frac{d^2}{dz^2}\left(\alpha_1(z)\mathcal{H}\left\{\alpha_1\mathcal{H}\left\{\alpha_1\right\}\right\}\right) \\
&-\frac{1}{16}\frac{d^2}{dz^2}\left(\alpha_1(z)\mathcal{H}^2\left\{\alpha_1\right\}\right) \\
&-\frac{1}{16}\frac{d^2}{dz^2}\left(\alpha_1(z)\mathcal{H}\left\{\alpha_1\mathcal{H}\left\{\alpha_1\right\}\right\}\right),
\end{aligned} \tag{61}
$$

and finally recalling that $\mathcal{H}_2\{\alpha_1\} = (1/2)H^2$, this becomes

$$
I_p(z) = -\frac{1}{8}\frac{d^2}{dz^2}\left(\alpha_1(z)\mathcal{H}^2\left\{\alpha_1\right\}\right) = -\frac{1}{24}\frac{d^3}{dz^3}\left(H^3\right). \tag{62}
$$

The preceding gives one a sense of how equation (42) was developed – doing this for a number of orders, and watching how the coefficient was generated. Interestingly, equation (62) is the negative of the associated $I_3(z)$:

$$
I_p(z) = -K_3\frac{d^3H^3}{dz^3} = -I_3(z), \tag{63}
$$

and hence, using equation (51), is

$$
I_p(z) = -\frac{1}{4}\alpha_1^3 - \frac{3}{4}\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H - \frac{1}{8}\left[\frac{d^2\alpha_1}{dz^2}\right]H^2. \tag{64}
$$

The difference between I_3 and the third order series terms of equation (38) indicates the effect of dropping the term from equation (58).

4.2 Simplifying Groups of Terms

Any reader who has taken the trouble to write down the inverse scattering series terms (even in general operator form), beyond the third or fourth order, say, would be excused if they looked at equation (42) with mounting skepticism. Its implied simplicity seems to contradict the growing complexity of the interactions of the series with increasing order. In this part of the paper, we attempt to reconcile this apparent contradiction by explicitly considering terms in third and fourth order. We use the results of this investigation to deduce a pattern of behaviour for the series, a pattern that we assume holds at all orders.

Shaw et al. (2003) point out that the *LOIS* terms are contributed-to from both terms that are in $\mathbf{V_1}$ only (e.g. $\mathbf{G_0V_1G_0V_1G_0V_1}\psi_\mathbf{0}$), and terms that are in $\mathbf{V_1}$, $\mathbf{V_2}$,... etc. (e.g. $\mathbf{G_0V_1G_0V_2}\psi_\mathbf{0}$). Here we call these "primary" terms and "secondary" terms respectively[2].

[2] We will use the terms *primary* and *secondary* a lot in this section. The former shouldn't be confused with *primaries* as in "the primaries and multiples of seismic data". In this paper, we use the word to mean those specific portions of the inverse scattering series defined here.

The fact that the *LOIS* is reproduced in the expansion of equation (42) for the third order (this can be seen by comparing equations (41) and (52)) suggests that simultaneous imaging and inversion correctly incorporates both primary and secondary terms. But we have just shown that equation (42) is almost exactly the same as the terms due to $\mathbf{G_0V_1G_0V_1G_0V_1}\psi_\mathbf{0}$ only – primary terms. The only way all these statements make sense is if the secondary terms are bringing about simple, predictable, alterations to their associated primary terms.

Secondary terms for 3rd Order

Consider again the 3rd order term in which

$$\alpha_3(z) = I_p(z) + I_s(z), \tag{65}$$

for primary and secondary terms respectively (that is, $I_p(z)$ is due to $-\mathbf{G}_0\mathbf{V_1G_0V_1G_0V_1}\psi_\mathbf{0}$ and $I_s(z)$ is due to $-\mathbf{G_0V_1G_0V_2}\psi_\mathbf{0} - \mathbf{G_0V_2G_0V_1}\psi_\mathbf{0}$). In the third order, the difference between the primary term due to $\mathbf{G_0V_1G_0V_1G_0V_1}\psi_\mathbf{0}$ and the full expression of equation (42) is a minus sign – see equation (63). However, since the secondary terms only have the power to change the primary term *by adding something to it*, the simple process of changing a sign can only be achieved by constructing twice the negative of $I_3(z)$ – a fair amount of work. We can see this by computing $I_s(z)$ from equation (57). Writing down equation (47), generalized to accommodate α_1 and α_2, and substituting the expression for α_2 therein, one has

$$\begin{aligned}
I_s(z) &= -\frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]\mathcal{H}\left\{\alpha_2\right\} - \alpha_1\alpha_2 - \frac{1}{2}\left[\frac{d\alpha_2}{dz}\right]\mathcal{H}\left\{\alpha_1\right\} \\
&= -\frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]\mathcal{H}\left\{K_2\frac{d^2H^2}{dz^2}\right\} - \alpha_1\left[K_2\frac{d^2H^2}{dz^2}\right] - \frac{1}{2}\left[K_2\frac{d^3H^2}{dz^3}\right]\mathcal{H}\left\{\alpha_1\right\} \\
&= -K_2\left(\frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]\left[\frac{dH^2}{dz}\right] + \alpha_1\frac{d}{dz}\left[2H\alpha_1\right] + \frac{1}{2}\frac{d^2}{dz^2}\left[2H\alpha_1\right]H\right) \\
&= -K_2\left(\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + 2\alpha_1\left[\alpha_1^2 + \left[\frac{d\alpha_1}{dz}\right]H\right] + \frac{d}{dz}\left[\alpha_1^2 + \left[\frac{d\alpha_1}{dz}\right]H\right]H\right) \\
&= -K_2\left(3\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + 2\alpha_1^3 + 2\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + \left[\frac{d^2\alpha_1}{dz^2}\right]H^2 + \left[\frac{d\alpha_1}{dz}\right]\alpha_1 H\right) \\
&= -K_2\left(6\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + 2\alpha_1^3 + \left[\frac{d^2\alpha_1}{dz^2}\right]H^2\right) \\
&= \frac{1}{2}\alpha_1^3 + \frac{3}{2}\left[\frac{d\alpha_1}{dz}\right]\alpha_1 H + \frac{1}{4}\left[\frac{d^2\alpha_1}{dz^2}\right]H^2,
\end{aligned} \tag{66}$$

which is twice $-I_p(z)$. (Note that the constant K_2 may be brought out of the $\mathcal{H}\left\{\cdot\right\}$ operator.) This result supports the postulate that the secondary terms effect some simple alteration to the primary terms. It seems reasonable to look for this behaviour in the other, higher order, terms as well.

Secondary terms for 4th Order

We proceed by considering the secondary components of the fourth order term. Since $\mathbf{V_4}$ is solved-for through the equation

$$\begin{aligned}\mathbf{G}_0\mathbf{V_4}\psi_\mathbf{0} = &- \mathbf{G}_0\mathbf{V_1}\mathbf{G_0}\mathbf{V_1}\mathbf{G_0}\mathbf{V_1}\mathbf{G_0}\mathbf{V_1}\psi_\mathbf{0} \\ &- (\mathbf{G}_0\mathbf{V_1}\mathbf{G_0}\mathbf{V_1}\mathbf{G_0}\mathbf{V_2}\psi_\mathbf{0} + \mathbf{G}_0\mathbf{V_1}\mathbf{G_0}\mathbf{V_2}\mathbf{G_0}\mathbf{V_1}\psi_\mathbf{0} + \mathbf{G}_0\mathbf{V_2}\mathbf{G_0}\mathbf{V_1}\mathbf{G_0}\mathbf{V_1}\psi_\mathbf{0}) \\ &- (\mathbf{G}_0\mathbf{V_1}\mathbf{G_0}\mathbf{V_3}\psi_\mathbf{0} + \mathbf{G}_0\mathbf{V_3}\mathbf{G_0}\mathbf{V_1}\psi_\mathbf{0} + \mathbf{G}_0\mathbf{V_2}\mathbf{G_0}\mathbf{V_2}\psi_\mathbf{0}),\end{aligned} \quad (67)$$

there are now six secondary components that need attention. Computation of the secondary terms in this fourth order set is different from that of the third order, because in this case *incomplete* terms are substituted in, namely the approximation $I_3 \approx \alpha_3$. Previously, i.e. in equation (66), only substitution for α_2, which is fully expressed by equation (42), was needed. Not so this time.

Calling the inversion terms counterpart to those of equation (67) respectively

$$\begin{aligned}\alpha_4 = &\ \text{PRIMARY} \\ &+ I_{112} + I_{121} + I_{211} \\ &+ I_{13} + I_{31} + I_{22},\end{aligned} \quad (68)$$

We use the same approach as that taken in equations (58) – (61) to find expressions for the six secondary terms above. At the risk of repeating ourselves, terms like the third of four in equation (58) are neglected in equation (68); this means that not only are they not included in the primary terms, but also they are not ever substituted into the secondary terms where they'll be expected, such as in a $\mathbf{G_0V_3G_0V_1G_0V_1}\psi_\mathbf{0}$-type expression.

In appendix A, we show that, after neglecting such terms,

$$I_{112} + I_{121} + I_{211} + I_{13} + I_{31} + I_{22} = 0. \quad (69)$$

That is, through a careful balancing, the fourth order secondary terms, none of which are zero on their own, sum to nil.

To summarize, in spite of considerable effort, these "secondary" components of the inverse series with one quarter of the scattering interactions neglected (the neglecting occurs in going from equation (58) to (59)) only either change the sign of their associated primary terms, as in the third order, or do nothing, as in the fourth. This is why equation (42), although designed to compute only primary terms, produces the output of both primary and secondary terms with the simple inclusion of a factor which alters the sign of the output. At present we assume that the patterns seen explicitly, thus far, hold for all the orders of this portion of the inverse series. The correct reproduction of the *LOIS* subseries found by Shaw et al. (2003), from equation (42), would be a good check of this assumption.

5 Associated Subseries

In this section we look more carefully at some of the subseries which arise when equation (42) is expanded for n orders. Some of these subseries have the form of the pure imaging and pure inversion type tasks, which have been developed and discussed elsewhere.

5.1 Leading Order Imaging Subseries

Previously, a set of $I_n(z)$'s were explicitly computed and/or listed. Notice that if we collect and sum the *last* term in each:

$$\alpha_1 - \frac{1}{2}\left[\frac{d\alpha_1}{dz}\right]H + \frac{1}{8}\left[\frac{d^2\alpha_1}{dz^2}\right]H^2 - \frac{1}{48}\left[\frac{d^3\alpha_1}{dz^3}\right]H^3 + ..., \tag{70}$$

the *LOIS* series of Shaw et al. (2004) is reproduced:

$$\alpha_{LOIS} = \sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1}\frac{1}{(k-1)!}\left[\frac{d^k\alpha_1}{dz^k}\right]H^k. \tag{71}$$

Equation (42) is therefore computing terms at all orders which are combinations of primary and secondary components. This supports the idea that (i) equation (42) is capturing much of the behaviour of the series, and (ii) therefore the net effect of secondary terms on this portion of the series is the aforementioned change (or not change) of sign.

5.2 Inversion Subseries

Next, notice that if the first terms in each of the expansions above are summed, one gets

$$\alpha_1 - \frac{1}{2}\alpha_1^2 + \frac{1}{4}\alpha_1^3 - \frac{1}{8}\alpha_1^4 + \frac{1}{16}\alpha_1^5 + ..., \tag{72}$$

a series which, because of the form of its constituents, must be devoted to inversion tasks. This is partial inversion only, since as discussed, some terms were neglected in the derivation of equation (42). Nevertheless there is a pattern:

$$\alpha_{PINV}(z) = \sum_{k=1}^{\infty} \left(-\frac{1}{2}\right)^{k-1} \alpha_1^k(z). \tag{73}$$

The question is, what does this series do? The full inversion subseries is

$$\alpha_{INV} = \alpha_1 - \frac{1}{2}\alpha_1^2 + \frac{3}{16}\alpha_1^3 - ..., \tag{74}$$

such that, for 1D reflection over a single interface with a reflection coefficient of R_1, the inverted perturbation amplitude becomes, in terms of this R_1,

$$\alpha_{INV} = 4R_1 \left(1 - 2R_1 + 3R_1^2 - 4R_1^3 + ...\right) = \frac{4R_1}{(1+R_1)^2}. \tag{75}$$

Clearly equation (73) is not this same series, therefore it will not produce the correct results. To find out what results are produced, we set a similar problem up, i.e. a 1D reflection experiment over a single interface. No imaging will be required, so omitting all z dependence, a Born approximate amplitude for the contrast of

$$\alpha_1 = 4R_1 \tag{76}$$

is produced. Substituting this into the partial inversion subseries, for α_{PINV} in equation (73) produces

$$\alpha_{PINV} = 4R_1 \left(1 - 2R_1 + 4R_1^2 - 8R_1^3 + 16R_1^4 - ...\right), \tag{77}$$

so up to second order in R_1 the partial inversion subseries is the same as the full; compare equation (75) with (77). Beyond that they begin to differ, more-so with higher order in R_1. This begins to suggest that the partial inversion subseries (α_{PINV}) is a low R_1 approximation. Of course, without infinite terms, so is the full! This means that at low R_1 both the full and partial inversion series are equivalent, but at higher R_1 they differ, with the full inversion series performing better than the partial. This is demonstrated in a series of plots of α_{INV} and α_{PINV} against R_1, in Figure 1. The partial subseries seems to capture the true reflection coefficient very well up until approximately $R_1 = 0.4$ over 8 terms. It seems that α_{PINV} converges to something very close to α_{INV}, but more slowly as R_1 increases.

The point is that the portion of the full inverse scattering series that come from the neglected terms appear to supply – in part – the inversion subseries with higher-order components. At low order, the partial inversion subseries within equation (42) and the full inversion subseries are almost equivalent.

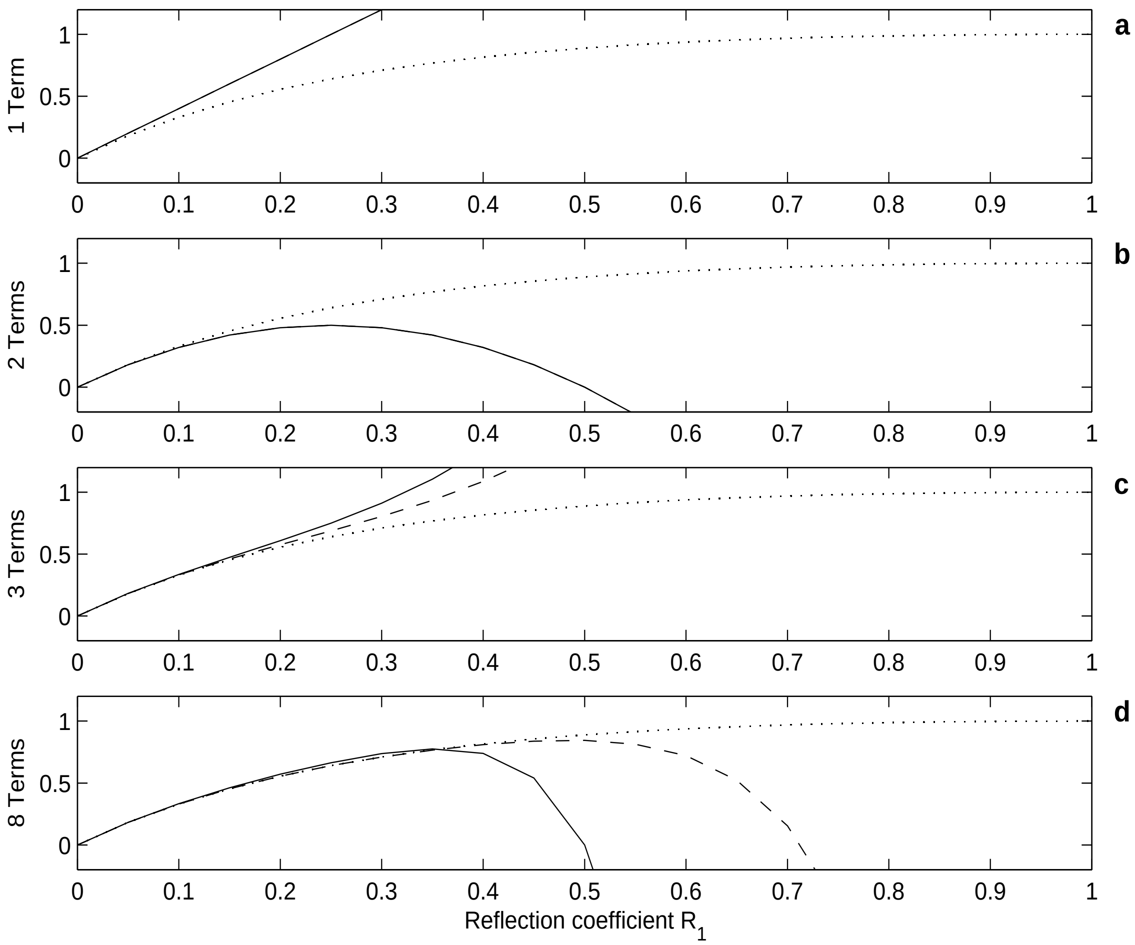

Figure 1: *Perturbations $\alpha_{INV}/\alpha_{PINV}$ vs. R_1. Comparison of full (dashed) and partial (solid) inversion subseries for single interface experiment using progressively more terms (dotted line is the true model). Both are low R_1 approximations, but we know that in its totality the full inversion converges to the true model. (a) 1 term, (b) 2 terms, (c) 3 terms, (d) 8 terms. The partial inversion subseries follows the full subseries well for R_1 values under 0.4 after 8 terms.*

6 Alternative Mathematical Forms

The simultaneous imaging and inversion form of equation (42) may equivalently be expressed

$$\alpha_{SII}(z) = \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} \frac{d^n}{dz^n} \left[\alpha_1(z) H^n\right]. \tag{78}$$

Using the forward and inverse Fourier transform,

$$
\begin{aligned}
\alpha_{SII}(z) &= \frac{1}{2\pi}\sum_{n=0}^{\infty}\frac{(-1/2)^n}{n!}\int_{-\infty}^{\infty}e^{ikz}\left[(-ik)^n\int_{-\infty}^{\infty}e^{-ikz'}[\alpha_1(z')H^n](z')dz'\right]dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{ik(z-z')}\alpha_1(z')\left[\sum_{n=0}^{\infty}\frac{\left[-\frac{ik}{2}H(z')\right]^n}{n!}\right]dz'dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}e^{ik(z-z')}\alpha_1(z')e^{\frac{ik}{2}H(z')}dz'dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikz}\left[\int_{\infty}^{\infty}e^{-ik[z'-\frac{1}{2}H(z')]}\alpha_1(z')dz'\right]dk
\end{aligned}
\tag{79}
$$

having made use of

$$
e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}. \tag{80}
$$

In other words we may interpret $\alpha_{SII}(z)$ as being the inverse Fourier transform of α_1 *after* $\alpha_1(z)$ has undergone a Fourier-like transformation where the kernel,

$$
\exp\left\{-ik\left[z' - \frac{1}{2}\int_0^{z'}\alpha_1(z'')dz''\right]\right\}, \tag{81}
$$

depends on the integral of $\alpha_1(z)$.

As an aside, in Figure 2 we demonstrate the numerical computation of this closed form solution for three numerical models which will appear again in Innanen et al. (2004) in this report (see that paper for model details and further numerical details of such reconstructions). In solid black is the reconstruction, overlying the true perturbation $\alpha(z)$; contrasting these, the Born approximation $\alpha_1(z)$ is displayed as a dotted line. The reconstruction has captured the expected location and amplitude of the true contrasts for all three examples (not to mention a smoothness that is probably the result of the crude quadrature routine used to compute the Fourier-like integral; the authors continue to investigate).

7 Coupled Imaging-Inversion: Intuitive Interpretation

One of the keys to pursuing (eventual) practical implementation of algorithms based on inverse scattering is to have a clear understanding of the operations being visited upon the data as a result of the theory. Such a working knowledge advances our general understanding of how amplitude and timing information is extracted and synthesized in the theoretical milieu of the inverse scattering series. In this section we explore the numerical action of

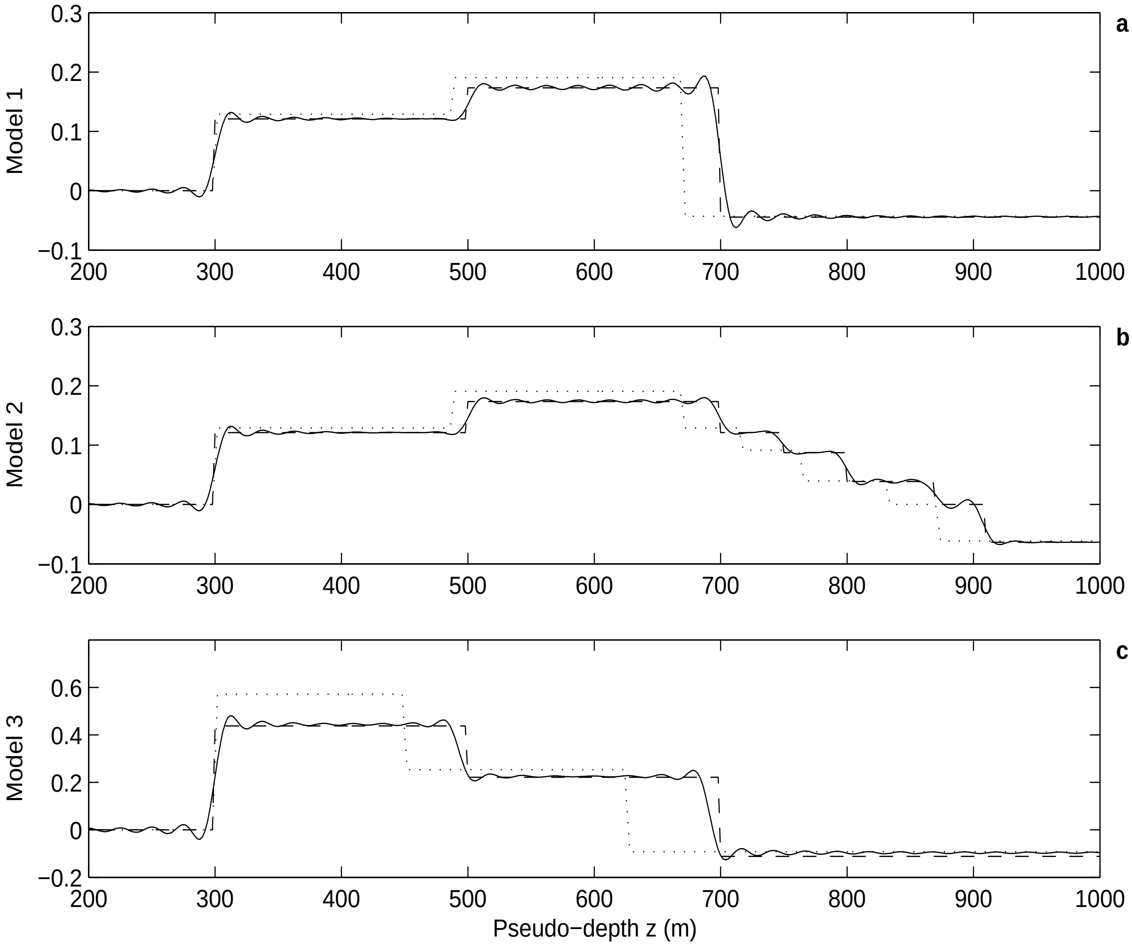

Figure 2*: Closed form for the simultaneous imaging and inversion algorithm, illustrated numerically on three input models (see Innanen et al. (2004) for more detail on these input models). In solid black is the reconstruction, overlying the true perturbation $\alpha(z)$; contrasting these, the Born approximation $\alpha_1(z)$ is displayed as a dotted line.*

equation (42) on a general input, with the aim of gaining a clearer signal processing-based view of what simultaneous imaging and inversion does.

The engine of simultaneous imaging and inversion is

$$\frac{d^n(\cdot)^n}{dz^n}, \tag{82}$$

where the quantity in brackets $(\cdot)$ is a discontinuous input akin to the second integral of a seismic trace (H in this paper). In words, the quantity $(\cdot)$ is taken to the n'th power, and the n'th derivative with respect to z is carried out upon the result. This is done for a (large) range of values of n, and the various outputs are weighted and summed.

The two cascaded operations, (1) *take the n'th power* and (2) *take the n'th derivative*, have a variable impact upon the integral of the input Born approximation. This helps explain how the processes of imaging and inversion can proceed simultaneously through a simple computation.

We consider synthetic data due to a 1D normal incidence model. In pseudo-depth the resulting trace is a series of discrete impulses. This is integrated once to produce $\alpha_1(z)$, and again to produce H. In general for piecewise constant impedance Earth models, the form of H tends, therefore, to be a piecewise linear signal. The key is to follow what the cascaded operation d^nH^n/dz^n does to such an input H.

H has two distinct types of structure, each of which reacts very differently to this operator. First consider an element of H away from all discontinuities. In general elements of H, called, say, $H_{lin}(z)$, on such an interval may be described, as a general linear function, by

$$H_{lin}(z) = az + b, \tag{83}$$

where a and b are some constants determined by the data at and above z_1. (Notice that if z_1 is the location of the shallowest interface, with a reflection coefficient of R_1, then $a = 4R_1$.) Computing the first four orders of the operator $K_n d^n H^n_{lin}/dz^n$ gives

$$\begin{aligned} K_1 \frac{dH_{lin}(z)}{dz} &= a \\ &(= 4R_1)\,, \end{aligned} \tag{84}$$

$$\begin{aligned} K_2 \frac{d^2H^2_{lin}(z)}{dz^2} &= -\frac{1}{4}\frac{d^2}{dz^2}\left[a^2z^2 + 2azb + b^2\right] \\ &= -\frac{1}{2}a^2 \\ &\left(= -8R_1^2\right), \end{aligned} \tag{85}$$

$$\begin{aligned} K_3 \frac{d^3H^3_{lin}(z)}{dz^3} &= \frac{1}{24}\frac{d^3}{dz^3}\left[a^3z^3 + ...\right] \\ &= \frac{1}{4}a^3 \\ &\left(= 16R_1^3\right), \end{aligned} \tag{86}$$

$$\begin{aligned} K_4 \frac{d^4H^4_{lin}(z)}{dz^4} &= -\frac{1}{192}\frac{d^4}{dz^4}\left[a^4z^4 + ...\right] \\ &= -\frac{1}{8}a^4 \\ &\left(= -32R_1^4\right), \end{aligned} \tag{87}$$

where in brackets the particular value $a = 4R_1$ is used. It is clear that, at all orders, the operator $d^n(\cdot)^n/dz^n$ does not take these linear features H_{lin} and create an output with any complicated spatial structure. In fact, the n'th power and the n'th derivative counteract each other almost completely for an input with a linear dependence. $K_n d^n(H_{lin})^n/dz^n$ is the transformation from a linear function to a constant function, in which the only important task of n is in determining the weight – or the eventual value – of the constant output.

So equation (42) in fact operates quite gently on the second integral of the data away from its discontinuities. It acts to alter its amplitudes in a way that corresponds exactly to the partial inversion subseries discussed earlier in this paper. To see this, compare the bracketed results of equations (84) – (87), for the single interface case, with the terms of this partial inversion subseries for the same case in equation (77).

But the genteel act of transforming a linear function into a constant function belies the volatility of the operator that performs it. When the operator encounters the second structural type found in H-like inputs, at and near its discontinuities, things change drastically.

$d^n(\cdot)^n/dz^n$ actually behaves like an edge-detector – it tends to do little to portions of a function that are well approximated by low-order polynomials, while strongly reacting to portions that resemble high-order polynomials, and especially signal edges and discontinuities. In the case of H-like functions, these discontinuities are at points where piecewise linear functions conjoin. In Figure 3 a test discontinuity (a) of this kind is operated on by $d^n(\cdot)^n/dz^n$, again for $n = 1$ – 4. Each of Figures 3a – d has 2 panels; the top is the function of Figure 3a taken to the n'th power, and the bottom is the numerical n'th derivative thereof.

Interestingly, what is returned is a signal with the characteristics of weighted derivatives of the original function. This is not necessarily an intuitive result, since by eye, the discontinuities associated with Figures 3a – d, top panels, appear to be of changing order (or regularity). However, numerically, the output continues reflect a "triangle"-like discontinuity, i.e. of order 1. What changes is that the differences in the slope on either side of the discontinuity become much larger, and this produces the effective weights on the output derivatives.

So, near the discontinuities that typify the integral of the Born approximation, $d^n(\cdot)^n/dz^n$ is outputting the sum of weighted derivatives of a piecewise constant function, 0'th (Figure 3a, bottom panel) through 3rd (Figure 3d, bottom panel) in these examples. This is in agreement with the mechanics of the leading order imaging subseries as investigated in Shaw et al. (2003), and seen in equation (71) in this paper.

In summary, the simultaneous imaging and inversion formula can be seen to act as a flexible operator that mimics *both* the inversion subseries and the leading order imaging subseries. One dominates over the other based on the proximity of the operator to discontinuities in the integral of the Born approximation.

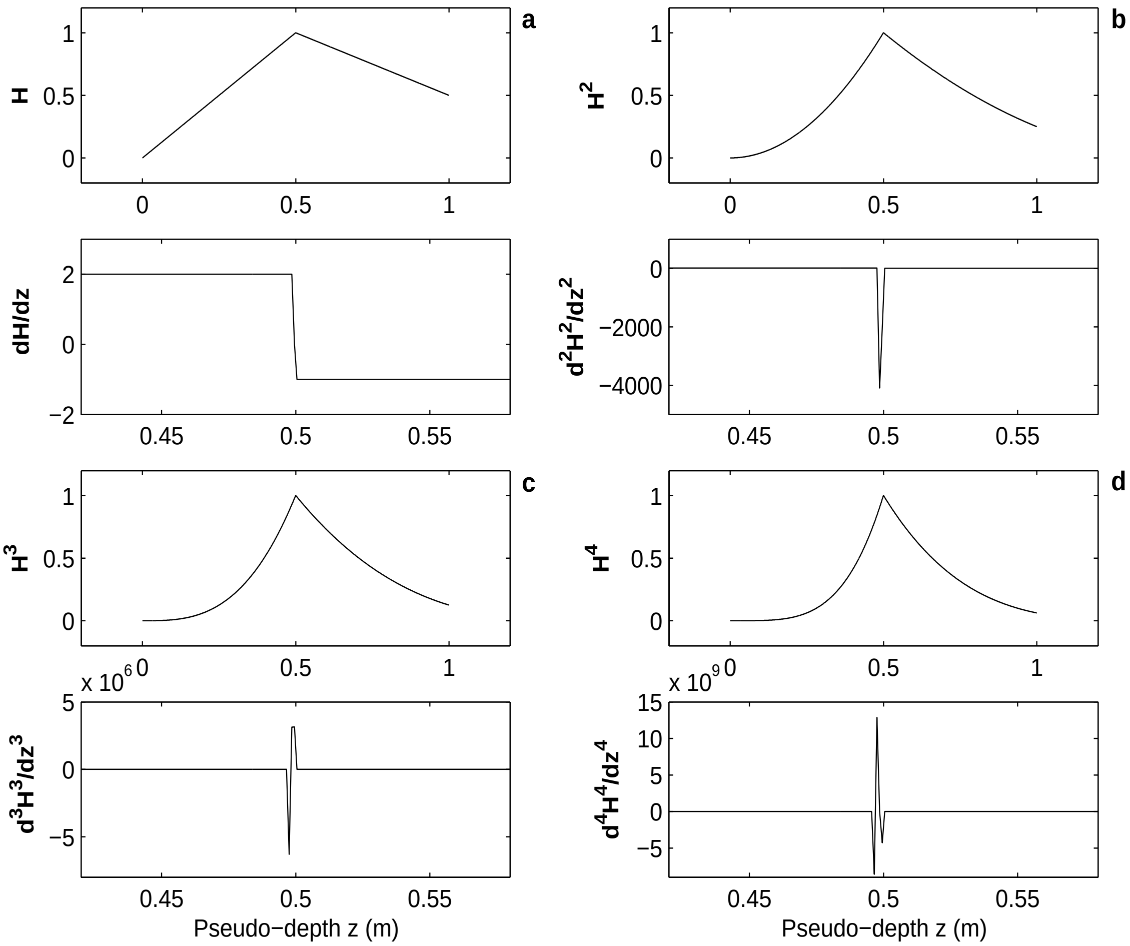

Figure 3: *Illustration of the effect of the operator $d^n(\cdot)^n/dz^n$ on the n'th power of the characteristic discontinuities of the input H (or second integral of the data). Each of the four examples (a)–(d) has two panels. Top panel: n'th power of input H; bottom panel: n'th numerical derivative of the top panel. This is done for four example orders: (a) 1, (b) 2, (c), 3, and (d) 4. In spite of the increasing curvature on either side of the discontinuity, numerically the results consistently resemble weighted n'th derivatives of order* 1 *discontinuities.*

8 Conclusions

We present an analysis of an algorithm that results from a purposeful coupling of terms in the inverse scattering series that concern the processing and inversion of seismic primaries. The interest lies principally in the fact that the coupling represents such a large fraction of the terms in the series that "remain" after multiples have been eliminated: the consequent algorithm and its properties speak volumes about the overall behaviour of the series.

The simplicity of form that results from a particular choice of inverse scattering series terms is developed, and the extent of the approximation of the full series is investigated – it is noted that what is missing effectively involves higher order imaging and inversion terms.

As such this “simultaneous imaging and inversion” procedure is very accurate for low and intermediate contrast levels, with some error accumulating at very high contrast.

We then consider a second analytic form for the 1D normal incidence acoustic algorithm, which permits a collapse to closed form (again for this simplified instance).

Finally, we attempt to describe the behaviour of this algorithm from a signal processing point of view, in particular with an aim to understand how a simple operator can perform such varied tasks (reflector location and target identification) at the same time. The answer is that the operator responds flexibly to the structure it encounters, behaving strongly at discontinuities and weakly away from them, to respectively move them in depth or merely change their amplitude.

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Appendix A: Inverse Scattering Terms

We show that the chosen terms from the fourth order imaging/inversion subseries (for 1D constant density acoustic imaging and wavespeed inversion) sum to nil. These components are

$$I_{112} + I_{121} + I_{211} + I_{13} + I_{31} + I_{22}. \tag{88}$$

As in the derivations of this paper, the operators $\mathcal{H}$ and the derivatives, that appear here arise after substitution of the form of the Green's functions into the terms of the inverse scattering series. They are due, respectively, to the alterations of the integration limits and the appearance of powers of the wavenumber. To start,

$$\begin{aligned}
I_{112} &= -\frac{1}{16}\frac{d^2}{dz^2}\left[\alpha_2\mathcal{H}\left\{\alpha_1\mathcal{H}\left\{\alpha_1\right\}\right\} + \alpha_1\mathcal{H}\left\{\alpha_1\right\}\mathcal{H}\left\{\alpha_2\right\} + \alpha_1\mathcal{H}\left\{\alpha_1\mathcal{H}\left\{\alpha_2\right\}\right\}\right], \\
I_{121} &= -\frac{1}{16}\frac{d^2}{dz^2}\left[\alpha_1\mathcal{H}\left\{\alpha_2\mathcal{H}\left\{\alpha_1\right\}\right\} + \alpha_2\mathcal{H}\left\{\alpha_1\right\}\mathcal{H}\left\{\alpha_1\right\} + \alpha_1\mathcal{H}\left\{\alpha_2\mathcal{H}\left\{\alpha_1\right\}\right\}\right], \\
I_{211} &= -\frac{1}{16}\frac{d^2}{dz^2}\left[\alpha_1\mathcal{H}\left\{\alpha_1\mathcal{H}\left\{\alpha_2\right\}\right\} + \alpha_1\mathcal{H}\left\{\alpha_1\right\}\mathcal{H}\left\{\alpha_2\right\} + \alpha_2\mathcal{H}\left\{\alpha_1\mathcal{H}\left\{\alpha_1\right\}\right\}\right],
\end{aligned} \tag{89}$$

so

$$I_{112} + I_{121} + I_{211} = I_{3V} = -\frac{1}{16}\frac{d^2}{dz^2}[2\alpha_2\mathcal{H}\{\alpha_1\mathcal{H}\{\alpha_1\}\} + 2\alpha_1\mathcal{H}\{\alpha_1\}\mathcal{H}\{\alpha_2\} + 2\alpha_1\mathcal{H}\{\alpha_1\mathcal{H}\{\alpha_2\}\} + 2\alpha_1\mathcal{H}\{\alpha_2\mathcal{H}\{\alpha_1\}\} + \alpha_2\mathcal{H}\{\alpha_1\}\mathcal{H}\{\alpha_1\}]. \tag{90}$$

Substituting appropriate versions of equation (42) into the above produces

$$\begin{aligned}
I_{3V} =& -\frac{1}{16}\frac{d^2}{dz^2}\left[2\left(K_2\frac{d^2H^2}{dz^2}\right)\mathcal{H}\{\alpha_1\mathcal{H}\{\alpha_1\}\} + 2\frac{dH}{dz}\mathcal{H}\{\alpha_1\}\mathcal{H}\left\{K_2\frac{d^2H^2}{dz^2}\right\}\right.\\
&\left.+2\frac{dH}{dz}\mathcal{H}\left\{\alpha_1\mathcal{H}\left(K_2\frac{d^2H^2}{dz^2}\right)\right\} + 2\frac{dH}{dz}\mathcal{H}\left\{K_2\frac{d^2H^2}{dz^2}H\right\} + K_2\frac{d^2H^2}{dz^2}H^2\right]\\
=& -\frac{K_2}{16}\frac{d^2}{dz^2}\left[\frac{d^2H^2}{dz^2}H^2 + 2\frac{dH}{dz}\frac{dH^2}{dz}H + 2\frac{dH}{dz}\mathcal{H}\left\{\frac{dH}{dz}\frac{dH^2}{dz}\right\}\right.\\
&\left.+2\frac{dH}{dz}\mathcal{H}\left\{\frac{d}{dz}\left(2\frac{dH}{dz}H\right)\right\} + \frac{d}{dz}\left(2\frac{dH}{dz}H\right)H^2\right]\\
=& -\frac{K_2}{16}\frac{d^2}{dz^2}\left[4H^3\frac{d^2H}{dz^2} + 8\left(\frac{dH}{dz}H\right)^2 + 8\frac{dH}{dz}\mathcal{H}\left\{\left(\frac{dH}{dz}\right)^2H\right\}\right.\\
&\left.+4\frac{dH}{dz}\mathcal{H}\left\{\frac{d^2H}{dz^2}H^2\right\}\right]\\
=& -\frac{K_2}{16}\frac{d^2}{dz^2}\left[4H^3\frac{d^2H}{dz^2} + 8\left(\frac{dH}{dz}H\right)^2 + 8\frac{dH}{dz}\left(\frac{1}{2}\frac{dH}{dz}H^2\right)\right.\\
&\left.-8\frac{dH}{dz}\mathcal{H}\left\{\frac{1}{2}\frac{d^2H}{dz^2}H^2\right\} + 4\frac{dH}{dz}\mathcal{H}\left\{\frac{d^2H}{dz^2}H^2\right\}\right]\\
=& -\frac{K_2}{16}\frac{d^2}{dz^2}\left[4\frac{d^2H}{dz^2}H^3 + 12\left(\frac{dH}{dz}H\right)^2\right]\\
=& -\frac{K_2}{4}\frac{d^2}{dz^2}\left[\frac{d}{dz}\left(\frac{dH}{dz}H^3\right) - 3H^2\left(\frac{dH}{dz}\right)^2 + 3H^2\left(\frac{dH}{dz}\right)^2\right]\\
=& -\frac{K_2}{4}\frac{d^3}{dz^3}\left[\frac{dH}{dz}H^3\right]\\
=& -\frac{K_2}{16}\frac{d^4H^4}{dz^4}.
\end{aligned} \tag{91}$$

The terms $I_{31} + I_{13}$ are of the same form as the secondary terms of the third order case, i.e. equation (66):

$$
\begin{aligned}
I_{31} + I_{13} &= -\frac{1}{2}\frac{d}{dz}\left[\frac{dH}{dz}H\left\{\alpha_3\right\} + \alpha_3 H\right] \\
&= -\frac{K_3}{2}\frac{d^2}{dz^2}\left[\frac{d^2H^3}{dz^2}H\right] \\
&= -\frac{3K_3}{2}\frac{d^2}{dz^2}\left[2\left(\frac{dH}{dz}H\right)^2 + H^3\frac{d^2H}{dz^2}\right] \\
&= -\frac{3K_3}{2}\frac{d^2}{dz^2}\left[2\left(\frac{dH}{dz}H\right)^2 + \frac{d}{dz}\left(\frac{dH}{dz}H^3\right) - 3\left(\frac{dH}{dz}H\right)^2\right] \\
&= \frac{3K_3}{2}\frac{d^2}{dz^2}\left[\left(\frac{dH}{dz}H\right)^2 - \frac{d}{dz}\left(\frac{dH}{dz}H^3\right)\right] \\
&= -\frac{3K_3}{8}\frac{d^4H^4}{dz^4} + \frac{3K_3}{2}\frac{d^2}{dz^2}\left[\left(\frac{dH}{dz}H\right)^2\right].
\end{aligned} \tag{92}
$$

Similarly compute I_{22}:

$$
\begin{aligned}
I_{22} &= -\frac{1}{2}\frac{d}{dz}\left[\alpha_2\mathcal{H}\left\{\alpha_2\right\}\right] \\
&= -\frac{1}{2}\frac{d}{dz}\left[K_2\frac{d^2H^2}{dz^2}\mathcal{H}\left\{K_2\frac{d^2H^2}{dz^2}\right\}\right] \\
&= -\frac{K_2^2}{2}\frac{d}{dz}\left[\frac{d^2H^2}{dz^2}\frac{dH^2}{dz}\right] \\
&= -K_2^2\frac{d^2}{dz^2}\left[\left(\frac{dH}{dz}H\right)^2\right].
\end{aligned} \tag{93}
$$

Summing equations (92) and (93), we have

$$
\begin{aligned}
I_{31} + I_{13} + I_{22} &= -\frac{3K_3}{8}\frac{d^4H^4}{dz^4} + \frac{3K_3}{2}\frac{d^2}{dz^2}\left[\left(\frac{dH}{dz}H\right)^2\right] - K_2^2\frac{d^2}{dz^2}\left[\left(\frac{dH}{dz}H\right)^2\right] \\
&= -\frac{3K_3}{8}\frac{d^4H^4}{dz^4} + \left(\frac{3}{2}K_3 - K_2^2\right)\frac{d^2}{dz^2}\left[\left(\frac{dH}{dz}H\right)^2\right].
\end{aligned} \tag{94}
$$

However,

$$
\frac{3}{2}K_3 - K_2^2 = \frac{3}{2}\left(\frac{1}{24}\right) - \left(-\frac{1}{4}\right)^2 = \frac{1}{16} - \frac{1}{16} = 0, \tag{95}
$$

so

$$I_{31} + I_{13} + I_{22} = -\frac{3K_3}{8}\frac{d^4H^4}{dz^4}. \tag{96}$$

Equations (91) and (96) now contain the totality of the fourth order secondary terms; these are summed, and the result is added to the primary term (due to $\mathbf{G_0V_1G_0V_1G_0V_1G_0V_1}\psi_\mathbf{0}$) to produce α_4. But summing these secondary terms produces

$$I_{3V} + I_{31} + I_{13} + I_{22} = -\frac{d^4H^4}{dz^4}\left(\frac{K_2}{16} + \frac{3K_3}{8}\right) = -\frac{d^4H^4}{dz^4}\left(-\frac{1}{64} + \frac{1}{64}\right) = 0. \tag{97}$$