Target identification using the inverse scattering series: data requirements for the direct inversion of large-contrast, inhomogeneous elastic media

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Abstract

In this paper we extend the earlier work (Zhang and Weglein, 2003) on direct non-linear inversion (parameter identification) of 1D acoustic media to a 1D isotropic inhomogeneous three parameter elastic medium. A formalism that generalizes the L-S scalar acoustic equation to the elastic matrix case is presented and the associated inverse series provides a framework for direct elastic inverse processing. The first step in that process corresponds to direct linear inversion. An important new conclusion derived from this framework is that direct non-linear inversion for material properties of the simplest 1D elastic earth using line sources and receivers requires four components of data, $\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}$, where the left subscripts of the matrices represent the type of measurement and the right ones are the source type. For example, $\hat{D}^{PS}$ is the P component of the measured scattered field corresponding to an incident S wave field in the reference medium. This provides an explanation of the often reported ambiguous results from indirect global search inversion of the R_{PP} from the exact Zoeppritz's equations.

1 Introduction

In the context of exploration seismology, the promise of the inverse scattering series is to produce a direct non-linear inversion procedure in terms of data and reference information only. A strategy that has proven to be useful is to seek task isolated subseries contained within the series (Weglein et al., 2003) where these tasks are typically solved in sequence, from easiest to most difficult. These tasks typically associated with direct inversion are: (1) free surface multiple removal; (2) internal multiple removal; (3) imaging reflectors to their correct spatial location and (4) earth property identification. Determining subsurface material properties is the last and most difficult task to achieve. Weglein and Stolt (1992) introduce an elastic L-S equation and provide a specific linear inverse formalism for parameter estimation. Matson (1997) pioneered the development and application of methods for attenuating ocean bottom and on-shore multi component data.

The evolution of series-based methods from those associated with multiple removal to those associated with primaries (imaging at depth and parameter identification) began with 1D normal incidence acoustic analysis (Weglein et al., 2000, 2002; Shaw et al., 2003) to prestack 1D (Shaw and Weglein, 2004) and to prestack 1D two parameter acoustic (Zhang and Weglein, 2003) and 2D one parameter acoustic (Liu et al., 2004). Those analyses were able to isolate the imaging-only and inversion-only tasks within the series using explicit expressions for the inverse series in terms of their respective model parameters.

The objective of this paper is to begin similar analysis, i.e. towards a task (4) isolated subseries for a 1D elastic isotropic medium. The willingness of the inverse series to cooperate with our interests in task isolation is a non-trivial matter and significant effort (and luck) often is required in choosing favorable parameters and degrees of freedom to realize a map between our interests and the ways the series operates. We illustrate that issue for the acoustic two parameter model, where a slight change in parameters can make a significant difference in the ability to isolate tasks and to provide an interpretation to the terms of a given order in the inverse series for a given choice of parameters. Those lessons are invaluable in our ongoing efforts to choose parameters for the elastic inverse problem, that are most agreeable to physical interpretation in terms of imaging and inversion tasks.

However, overarching all considerations of physical interpretation and task specific identification is an unambiguous message from direct elastic inversion: for any choice of material property parameters, and choice of free parameter, the first step towards improvement beyond linear estimation of parameters requires the full data matrix $\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}$. Strategies when only $\hat{D}^{PP}$ is available will be the subject of future reports.

In the following sections, at first, we give the background of elastic inversion in the displacement space[1], then, we transform those operators and equations from the displacement space to the PS space. This is followed by the linear solution in PS space. For non-linear inversion, two results corresponding to two different sets of parameters are compared.

2 2D elastic inversion

In this section we consider the inversion problem in two dimensions for an elastic medium.

2.1 In the displacement space

We begin with some basic equations in the displacement space (Matson, 1997):

$$L\mathbf{u} = \mathbf{f}, \tag{1}$$

[1]Operators without hats in this paper are in the displacement space, those with hats are in PS space.

$$L_0\mathbf{u} = \mathbf{f}, \tag{2}$$

$$LG = \mathbf{I}, \tag{3}$$

$$L_0 G_0 = \mathbf{I}, \tag{4}$$

where L and L_0 are the differential operators that describe the wave propagation in the actual and reference medium, respectively, $\mathbf{u}$ and $\mathbf{f}$ are the corresponding displacement and source terms, respectively, and G and G_0 are the corresponding Green operators for the actual and reference medium.

Defining the perturbation $V = L_0 - L$, the Lippmann- Schwinger equation for the elastic media in the displacement space is

$$G = G_0 + G_0 V G. \tag{5}$$

Iterating this equation back into itself generates the Born series

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \cdots . \tag{6}$$

We define the data D as the measured values of the scattered wave field. Then, on the measurement surface, we have

$$D = G_0 V G_0 + G_0 V G_0 V G_0 + \cdots . \tag{7}$$

Expanding V as a series in orders of D (Weglein et al., 1997), we have

$$V = V_1 + V_2 + V_3 + \cdots . \tag{8}$$

Substituting Eq. (8) into Eq. (7), evaluating Eq. (7), and setting terms of equal order in the data equal, we get the equations that determine V_1, V_2, ... from D and G_0.

$$D = G_0 V_1 G_0, \tag{9}$$

$$0 = G_0 V_2 G_0 + G_0 V_1 G_0 V_1 G_0, \tag{10}$$

$$\vdots$$

In the actual medium, the 2-D elastic wave equation is (Weglein and Stolt, 1992)

$$L\mathbf{u} \equiv \left[\rho\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \partial_1\gamma\partial_1 + \partial_2\mu\partial_2 & \partial_1(\gamma - 2\mu)\partial_2 + \partial_2\mu\partial_1 \\ \partial_2(\gamma - 2\mu)\partial_1 + \partial_1\mu\partial_2 & \partial_2\gamma\partial_2 + \partial_1\mu\partial_1 \end{pmatrix}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{f}, \quad (11)$$

where

$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ = displacement,

ρ = density,

γ = bulk modulus ($\equiv \rho\alpha^2$ where α = P velocity),

μ = shear modulus ($\equiv \rho\beta^2$ where β = S velocity),

ω = temporal frequency (angular), and

$\mathbf{f}$ is the source term.

For constant $(\rho, \gamma, \mu) = (\rho_0, \gamma_0, \mu_0)$, $(\alpha, \beta) = (\alpha_0, \beta_0)$, the operator L becomes

$$L_0 \equiv \left[\rho_0\omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma_0\partial_1^2 + \mu_0\partial_2^2 & (\gamma_0 - \mu_0)\partial_1\partial_2 \\ (\gamma_0 - \mu_0)\partial_1\partial_2 & \mu_0\partial_1^2 + \gamma_0\partial_2^2 \end{pmatrix}\right]. \quad (12)$$

Then,

$$\begin{aligned} V \equiv& L_0 - L \\ =& -\rho_0 \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2\partial_1 a_\gamma\partial_1 + \beta_0^2\partial_2 a_\mu\partial_2 & \partial_1(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_2 + \beta_0^2\partial_2 a_\mu\partial_1 \\ \partial_2(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_1 + \beta_0^2\partial_1 a_\mu\partial_2 & a_\rho\omega^2 + \alpha_0^2\partial_2 a_\gamma\partial_2 + \beta_0^2\partial_1 a_\mu\partial_1 \end{bmatrix}, \end{aligned} \quad (13)$$

where $a_\rho \equiv \frac{\rho}{\rho_0} - 1$, $a_\gamma \equiv \frac{\gamma}{\gamma_0} - 1$ and $a_\mu \equiv \frac{\mu}{\mu_0} - 1$. For a 1D earth (i.e. a_ρ, a_γ and a_μ are only functions of depth z), we have

$$\begin{bmatrix} V^{11} & V^{12} \\ V^{21} & V^{22} \end{bmatrix} = -\rho_0 \begin{bmatrix} a_\rho\omega^2 + \alpha_0^2 a_\gamma\partial_1^2 + \beta_0^2\partial_2 a_\mu\partial_2 & (\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_1\partial_2 + \beta_0^2\partial_2 a_\mu\partial_1 \\ \partial_2(\alpha_0^2 a_\gamma - 2\beta_0^2 a_\mu)\partial_1 + \beta_0^2 a_\mu\partial_1\partial_2 & a_\rho\omega^2 + \alpha_0^2\partial_2 a_\gamma\partial_2 + \beta_0^2 a_\mu\partial_1^2 \end{bmatrix}. \quad (14)$$

2.2 Transform to PS space

In the reference medium, we diagonalize the operator L_0. Consider the transform matrix $\Pi = \begin{pmatrix} \partial_1 & \partial_2 \\ -\partial_2 & \partial_1 \end{pmatrix}$ and a constant matrix $\Gamma_0 = \begin{pmatrix} \gamma_0 & 0 \\ 0 & \mu_0 \end{pmatrix}$, which satisfy

$$\hat{L}_0 \equiv \Pi L_0 \Pi^{-1} \Gamma_0^{-1} = \begin{pmatrix} \hat{L}_0^P & 0 \\ 0 & \hat{L}_0^S \end{pmatrix},$$

where $\hat{L}_0$ is L_0 transformed to PS space, $\Pi^{-1} = \nabla^{-2} \begin{pmatrix} \partial_1 & -\partial_2 \\ \partial_2 & \partial_1 \end{pmatrix}$ is the inverse matrix of Π, $\hat{L}_0^P = \omega^2/\alpha_0^2 + \nabla^2$, $\hat{L}_0^S = \omega^2/\beta_0^2 + \nabla^2$. Multiplying Eq. (2) from the left by the operator Π, we find

$$\Pi L_0 \Pi^{-1} \Gamma_0^{-1} \Gamma_0 \Pi \mathbf{u} = \Pi \mathbf{f}. \tag{15}$$

Introduce as new independent variables the pressure ϕ_P and the shear stress ϕ_S defined as

$$\mathbf{\Phi} = \begin{pmatrix} \phi_P \\ \phi_S \end{pmatrix} = \Gamma_0 \Pi \mathbf{u} = \begin{bmatrix} \gamma_0(\partial_1 u_1 + \partial_2 u_2) \\ \mu_0(\partial_1 u_2 - \partial_2 u_1) \end{bmatrix}, \tag{16}$$

and define

$$\mathbf{F} = \Pi \mathbf{f} = \begin{pmatrix} F^P \\ F^S \end{pmatrix}. \tag{17}$$

Then, in PS domain, Eq. (2) becomes,

$$\begin{pmatrix} \hat{L}_0^P & 0 \\ 0 & \hat{L}_0^S \end{pmatrix} \begin{pmatrix} \phi_P \\ \phi_S \end{pmatrix} = \begin{pmatrix} F^P \\ F^S \end{pmatrix}. \tag{18}$$

Since $G_0 \equiv L_0^{-1}$, let $\hat{G}_0^P = \left(\hat{L}_0^P\right)^{-1}$ and $\hat{G}_0^S = \left(\hat{L}_0^S\right)^{-1}$, then, displacement G_0 in PS domain becomes

$$\hat{G}_0 = \Gamma_0 \Pi G_0 \Pi^{-1} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}. \tag{19}$$

So, in the reference medium, after transforming from the displacement domain to PS domain, both L_0 and G_0 become diagonal.

Multiplying Eq. (5) from the left by the operator $\Gamma_0 \Pi$ and from the right by the operator Π^{-1}, and using Eq. (19),

$$\begin{aligned}\Gamma_0\Pi G\Pi^{-1} &= \hat{G}_0 + \hat{G}_0\left(\Pi V\Pi^{-1}\Gamma_0^{-1}\right)\Gamma_0\Pi G\Pi^{-1}\\ &= \hat{G}_0 + \hat{G}_0\hat{V}\hat{G}, \end{aligned} \tag{20}$$

where the displacement Green's operator G is transformed to the PS domain as

$$\hat{G} = \Gamma_0\Pi G\Pi^{-1} = \begin{pmatrix} \hat{G}^{PP} & \hat{G}^{PS} \\ \hat{G}^{SP} & \hat{G}^{SS} \end{pmatrix}. \tag{21}$$

The perturbation V in the PS domain becomes

$$\hat{V} = \Pi V\Pi^{-1}\Gamma_0^{-1} = \begin{pmatrix} \hat{V}^{PP} & \hat{V}^{PS} \\ \hat{V}^{SP} & \hat{V}^{SS} \end{pmatrix}, \tag{22}$$

where, as before, the left subscripts of the matrices represent the type of measurement and the right ones are the source type.

Similarly, applying the PS transformation to the entire inverse series gives

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + \hat{V}_3 + \cdots. \tag{23}$$

It follows, from Eqs. (20) and (23) that

$$\hat{D} = \hat{G}_0\hat{V}_1\hat{G}_0, \tag{24}$$

$$\hat{G}_0\hat{V}_2\hat{G}_0 = -\hat{G}_0\hat{V}_1\hat{G}_0\hat{V}_1\hat{G}_0, \tag{25}$$

$$\vdots$$

where $\hat{D} = \begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix}$ are the data in the PS domain.

In the displacement space we have, for Eq. (1),

$$\mathbf{u} = G\mathbf{f}. \tag{26}$$

Then, in the PS domain, Eq. (26) becomes

$$\mathbf{\Phi} = \hat{G}\mathbf{F}. \tag{27}$$

On the measurement surface, we have

$$\hat{G} = \hat{G}_0 + \hat{G}_0\hat{V}_1\hat{G}_0. \tag{28}$$

We substitute Eq. (28) into Eq. (27), and rewrite Eq. (27) in matrix form:

$$\begin{pmatrix}\phi_P\\ \phi_S\end{pmatrix}=\begin{pmatrix}\hat{G}_0^P & 0\\ 0 & \hat{G}_0^S\end{pmatrix}\begin{pmatrix}F^P\\ F^S\end{pmatrix}+\begin{pmatrix}\hat{G}_0^P & 0\\ 0 & \hat{G}_0^S\end{pmatrix}\begin{pmatrix}\hat{V}_1^{PP} & \hat{V}_1^{PS}\\ \hat{V}_1^{SP} & \hat{V}_1^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^P & 0\\ 0 & \hat{G}_0^S\end{pmatrix}\begin{pmatrix}F^P\\ F^S\end{pmatrix}. \tag{29}$$

This can be written as the following two equations

$$\phi_P=\hat{G}_0^P F^P+\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P F^P+\hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S F^S, \tag{30}$$

$$\phi_S=\hat{G}_0^S F^S+\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P F^P+\hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S F^S, \tag{31}$$

We can see from the two equations above that for homogeneous media, (no perturbation, $\hat{V}_1=0$), there are only direct P and S waves and that the two kinds of waves are separated. However, for inhomogeneous media, these two kinds of waves will be mixed together. If only the P wave is incident, $F^P=1$, $F^S=0$, then the above two equations (30) and (31) are respectively reduced to

$$\phi_P=\hat{G}_0^P+\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P, \tag{32}$$

$$\phi_S=\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P, \tag{33}$$

Hence, in this case, there is only the direct P wave $\hat{G}_0^P$, and no direct wave S. But there are two kinds of scattered waves: one is the P-to-P wave $\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P$, and the other is the P-to-S wave $\hat{G}_0^S\hat{V}_1^{SP}\hat{G}_0^P$. For the acoustic case, only the P wave exists, and hence we only have one equation $\phi_P=\hat{G}_0^P+\hat{G}_0^P\hat{V}_1^{PP}\hat{G}_0^P$.

Similarly, if only the S wave is incident, $F^P=0$, $F^S=1$, and the two equations (30) and (31) are respectively reduced to

$$\phi_P=\hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S, \tag{34}$$

$$\phi_S=\hat{G}_0^S+\hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S, \tag{35}$$

In this case, there is only the direct S wave $\hat{G}_0^S$, and no direct wave P. There are also two kinds of scattered waves: one is the S-to-P wave $\hat{G}_0^P\hat{V}_1^{PS}\hat{G}_0^S$, the other is the S-to-S wave $\hat{G}_0^S\hat{V}_1^{SS}\hat{G}_0^S$.

3 Linear inversion

Writing Eq. (24) in matrix form

$$\begin{pmatrix}\hat{D}^{PP} & \hat{D}^{PS}\\ \hat{D}^{SP} & \hat{D}^{SS}\end{pmatrix}=\begin{pmatrix}\hat{G}_0^P & 0\\ 0 & \hat{G}_0^S\end{pmatrix}\begin{pmatrix}\hat{V}_1^{PP} & \hat{V}_1^{PS}\\ \hat{V}_1^{SP} & \hat{V}_1^{SS}\end{pmatrix}\begin{pmatrix}\hat{G}_0^P & 0\\ 0 & \hat{G}_0^S\end{pmatrix}, \tag{36}$$

leads to four equations

$$\hat{D}^{PP} = \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P, \quad (37)$$
$$\hat{D}^{PS} = \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S, \quad (38)$$
$$\hat{D}^{SP} = \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \quad (39)$$
$$\hat{D}^{SS} = \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S. \quad (40)$$

For $z_s = z_g = 0$, in the $(k_s, z_s; k_g, z_g; \omega)$ domain, we get

$$\widetilde{D}^{PP}(k_g, 0; -k_g, 0; \omega) = -\frac{1}{4}\left(1 - \frac{k_g^2}{\nu_g^2}\right)\widetilde{a}_\rho(-2\nu_g) - \frac{1}{4}\left(1 + \frac{k_g^2}{\nu_g^2}\right)\widetilde{a}_\gamma(-2\nu_g) + \frac{2k_g^2\beta_0^2}{(\nu_g^2 + k_g^2)\alpha_0^2}\widetilde{a}_\mu(-2\nu_g), \quad (41)$$

$$\widetilde{D}^{PS}(\nu_g, \eta_g) = -\frac{1}{4}\left(\frac{k_g}{\nu_g} + \frac{k_g}{\eta_g}\right)\widetilde{a}_\rho(-\nu_g - \eta_g) - \frac{\beta_0^2}{2\omega^2}k_g(\nu_g + \eta_g)\left(1 - \frac{k_g^2}{\nu_g\eta_g}\right)\widetilde{a}_\mu(-\nu_g - \eta_g), \quad (42)$$

$$\widetilde{D}^{SP}(\nu_g, \eta_g) = \frac{1}{4}\left(\frac{k_g}{\nu_g} + \frac{k_g}{\eta_g}\right)\widetilde{a}_\rho(-\nu_g - \eta_g) + \frac{\beta_0^2}{2\omega^2}k_g(\nu_g + \eta_g)\left(1 - \frac{k_g^2}{\nu_g\eta_g}\right)\widetilde{a}_\mu(-\nu_g - \eta_g), \quad (43)$$

$$\widetilde{D}^{SS}(k_g, \eta_g) = -\frac{1}{4}\left(1 - \frac{k_g^2}{\eta_g^2}\right)\widetilde{a}_\rho(-2\eta_g) - \left[\frac{\eta_g^2 + k_g^2}{4\eta_g^2} - \frac{2k_g^2}{\eta_g^2 + k_g^2}\right]\widetilde{a}_\mu(-2\eta_g), \quad (44)$$

where

$$\nu_g^2 + k_g^2 = \frac{\omega^2}{\alpha_0^2},$$
$$\eta_g^2 + k_g^2 = \frac{\omega^2}{\beta_0^2}.$$

4 Non-linear inversion

Writing Eq. (25) in matrix form:

$$\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}$$
$$= -\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}\begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix}\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix}, \quad (45)$$

leads to four equations

$$\hat{G}_0^P \hat{V}_2^{PP} \hat{G}_0^P = -\hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P - \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \tag{46}$$

$$\hat{G}_0^P \hat{V}_2^{PS} \hat{G}_0^S = -\hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S - \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S, \tag{47}$$

$$\hat{G}_0^S \hat{V}_2^{SP} \hat{G}_0^P = -\hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P - \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \tag{48}$$

$$\hat{G}_0^S \hat{V}_2^{SS} \hat{G}_0^S = -\hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S - \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S. \tag{49}$$

From the equations above, we can see that we cannot perform the direct non-linear inversion without knowing all components of the data.

In the derivation for the second order inversion, some difficulties were encountered which made us think about the parametrization in inversion. The choice of the parameters and the definition of the perturbation for inversion turns out to be a crucial step.

In the acoustic case, if we start directly with the pressure wave equation and choose θ as the free parameter, α and β as the two material property parameters, (Weglein et al., 2003; Zhang and Weglein, 2003) we arrive at the following equation for the second order (first term beyond linear):

$$\begin{aligned}
&\frac{1}{\cos^2\theta}\alpha_2(z) + (1-\tan^2\theta)\beta_2(z) \\
=& -\frac{1}{2\cos^4\theta}\alpha_1^2(z) \\
& -\frac{1}{2}(1+\tan^4\theta)\beta_1^2(z) \\
& +\frac{\tan^2\theta}{\cos^2\theta}\alpha_1(z)\beta_1(z) \\
& -\frac{1}{2\cos^4\theta}\alpha_1'(z)\int_0^z dz'[\alpha_1(z')-\beta_1(z')] \\
& +\frac{1}{2}(\tan^4\theta-1)\beta_1'(z)\int_0^z dz'[\alpha_1(z')-\beta_1(z')].
\end{aligned} \tag{50}$$

If we start with the displacement domain, as discussed, letting μ_0, β_0, μ, and $\beta = 0$ and choose θ as the free parameter, a_γ and a_ρ as the two material property parameters, the following solution for second order is produced:

$$
\begin{aligned}
&\frac{1}{\cos^2\theta} a_\gamma^{(2)}(z) + (1-\tan^2\theta) a_\rho^{(2)}(z) \\
= &-\frac{1}{2}(\tan^4\theta - 1) a_\gamma^{(1)^2}(z) \\
&-\frac{1}{2}\left(\frac{1}{\cos^4\theta} - 2\right) a_\rho^{(1)^2}(z) \\
&+\frac{\tan^2\theta}{\cos^2\theta} a_\gamma^{(1)}(z) a_\rho^{(1)}(z) \\
&-\frac{1}{2\cos^4\theta} a_\gamma^{(1)'}(z) \int_0^z dz' [a_\gamma^{(1)}(z') - a_\rho^{(1)}(z')] \\
&+\frac{1}{2}(\tan^4\theta - 1) a_\rho^{(1)'}(z) \int_0^z dz' [a_\gamma^{(1)}(z') - a_\rho^{(1)}(z')] \\
&+ A(z),
\end{aligned} \tag{51}
$$

where the definition of θ is the same as that of Eq. (50), $a_\gamma^{(1)'} = \frac{da_\gamma^{(1)}}{dz}$, $a_\rho^{(1)'} = \frac{da_\rho^{(1)}}{dz}$, and $A(z)$ in (ν_g) domain, i.e., before the Fourier transform over ν_g, is

$$
\begin{aligned}
\widetilde{A}(\nu_g) = &\frac{1}{i\pi\nu_g} \int_{-\infty}^{+\infty} dz' a_\rho^{(1)'}(z') \, a_\gamma^{(1)}(z') \, e^{2i\nu_g z'} \\
&+ \frac{\tan\theta}{\pi} \frac{1}{\nu_g} \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dz'' a_\rho^{(1)'}(z') \, a_\rho^{(1)'}(z'') \, H(z'-z'') \, e^{i\nu_g(1+i\tan\theta)z'} e^{i\nu_g(1-i\tan\theta)z''} \\
&+ \frac{1}{2\pi\cos^2\theta} \frac{1}{i\nu_g(1-i\tan\theta)} \cdot \\
&\cdot \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dz'' a_\rho^{(1)'}(z') \, a_\gamma^{(1)'}(z'') \, H(z'-z'') \, e^{i\nu_g(1+i\tan\theta)z'} e^{i\nu_g(1-i\tan\theta)z''} \\
&- \frac{1}{2\pi\cos^2\theta} \frac{1}{i\nu_g(1+i\tan\theta)} \cdot \\
&\cdot \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dz'' a_\rho^{(1)'}(z') \, a_\gamma^{(1)'}(z'') \, H(z''-z') \, e^{i\nu_g(1-i\tan\theta)z'} e^{i\nu_g(1+i\tan\theta)z''} \\
&+ \frac{1}{2\pi} \frac{1}{\nu_g^2} \int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dz'' \left\{ e^{i\nu_g z'} a_\rho^{(1)'}(z') \, sgn(z'-z'') \left[e^{i\nu_g|z'-z''|} - e^{-k_g|z'-z''|} \right] \cdot \right. \\
&\left. \cdot \left[a_\gamma^{(1)''}(z'') + 2i\nu_g a_\gamma^{(1)'}(z'') \right] e^{i\nu_g z''} \right\}.
\end{aligned}
$$

Obviously, the second method or solution is much more complex than the first one. In addition, if another choice of free parameter other than θ (e.g., ω or k_h) was selected, then the functional form between the data and the first order perturbation changes. Furthermore, the relationship between the first and second order perturbation is then also different, and new analysis will then be required for the purposes of identifying specific task separated subseries. In our experience, the choice of θ as free parameter (for a 1D medium) is particularly well suited for allowing a task separated identification of terms in the inverse series.

For acoustic media with fixed density, from Eq. (50), we have

$$\alpha_2(z) = -\frac{1}{2\cos^2\theta}\left[\alpha_1^2(z) + \alpha_1'(z)\int_0^z dz'\alpha_1(z')\right], \tag{52}$$

while from Eq. (51) we have

$$a_\gamma^{(2)}(z) = -\frac{1}{2}(\tan^2\theta - 1)a_\gamma^{(1)^2}(z) - \frac{1}{2\cos^2\theta}a_\gamma^{(1)'}(z)\int_0^z dz' a_\gamma^{(1)}(z'). \tag{53}$$

Since $\alpha = 1 - \frac{\gamma_0}{\gamma}$, then

$$a_\gamma = \frac{\gamma}{\gamma_0} - 1 = \frac{\alpha}{1-\alpha} = \alpha + \alpha^2 + \alpha^3 + \cdots,$$

where the series expansion is valid for $|\alpha| < 1$. And then we have

$$a_\gamma^{(1)} = \alpha_1,$$

$$a_\gamma^{(2)} = \alpha_2 + \alpha_1^2,$$

$$\vdots$$

Then,

$$a_\gamma^{(2)} = -\frac{1}{2\cos^2\theta}\left[\alpha_1^2(z) + \alpha_1'(z)\int_0^z dz'\alpha_1(z')\right] + \alpha_1^2(z),$$

from the equation above we can get Eq. (53):

$$a_\gamma^{(2)}(z) = -\frac{1}{2}(\tan^2\theta - 1)a_\gamma^{(1)^2}(z) - \frac{1}{2\cos^2\theta}a_\gamma^{(1)'}(z)\int_0^z dz' a_\gamma^{(1)}(z').$$

Eq. (52) and Eq. (53) agree!

Next, for acoustic media with variable density, from Eq. (50) we have

$$(1-\tan^2\theta)\beta_2(z) = -\frac{1}{2}(1+\tan^4\theta)\beta_1^2(z) - \frac{1}{2}(\tan^4\theta - 1)\beta_1'(z)\int_0^z dz'\beta_1(z'), \tag{54}$$

while from Eq. (51) we can get

$$(1-\tan^2\theta)a_\rho^{(2)}(z) = -\frac{1}{2}\left(\frac{1}{\cos^4\theta}-2\right)a_\rho^{(1)^2}(z) - \frac{1}{2}(\tan^4\theta-1)a_\rho^{(1)'}(z)\int_0^z dz' a_\rho^{(1)}(z') + B(z), \tag{55}$$

where $B(z)$ in (ν_g) domain, i.e., before the Fourier transform over ν_g, is

$$\widetilde{B}(\nu_g) = \frac{\tan\theta}{\pi}\frac{1}{\nu_g}\int_{-\infty}^{+\infty} dz' \int_{-\infty}^{+\infty} dz'' a_\rho^{(1)'}(z')\, a_\rho^{(1)'}(z'')\, H(z'-z'')\, e^{i\nu_g(1+i\tan\theta)z'} e^{i\nu_g(1-i\tan\theta)z''}.$$

Since $\beta = 1 - \frac{\rho_0}{\rho}$, then

$$a_\rho = \frac{\rho}{\rho_0} - 1 = \frac{\beta}{1-\beta} = \beta + \beta^2 + \beta^3 + \cdots,$$

where the series expansion is valid for $|\beta| < 1$. And then we have

$$a_\rho^{(1)} = \beta_1,$$

$$a_\rho^{(2)} = \beta_2 + \beta_1^2,$$

$$\vdots$$

Then,

$$(1-\tan^2\theta)a_\rho^{(2)}(z) = \left[-\frac{1}{2}(1+\tan^4\theta)\beta_1^2(z) - \frac{1}{2}(\tan^4\theta-1)\beta_1'(z)\int_0^z dz'\beta_1(z')\right] + (1-\tan^2\theta)\beta_1^2(z),$$

from the equation above we can get

$$\left(1-\tan^2\theta\right)a_\rho^{(2)}(z) = -\frac{1}{2}\left(\frac{1}{\cos^4\theta}-2\right)a_\rho^{(1)^2}(z) - \frac{1}{2}(\tan^4\theta-1)a_\rho^{(1)'}(z)\int_0^z dz' a_\rho^{(1)}(z').$$

The amplitude term(s) and the imaging term(s) in Eq. (54) and Eq. (55) agree!

5 Conclusion

This paper provides a framework and analysis of issues involved in data requirements, computation and interpretation of the non-linear direct elastic inverse problem. Specifically, a detailed analysis of how different choice of acoustic parameters (and free parameters) have a marked difference on the ability of task separated interpretation. That analysis provides a guide and lesson for ongoing effort at parameter inversion and structural location specific subseries for the elastic world.

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