A timely and necessary antidote to indirect methods and so-called P-wave FWI

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Editor's note: The following article brings to light a cautionary concern (and a set of fundamental and substantive issues, related to indirect methods, in general, that benefits from a broader and deeper understanding and perspective), regarding the validity of basic assumptions made in FWI. It was slated to appear in the special section on FWI in September. The article also describes and exemplifies a direct inverse method for the same FWI-type objectives. However as that issue was fully subscribed, given the popularity of FWI, it was decided, in conjunction with the section's guest editors (Antoine Guitton, Tariq Alkhalifah, and Chris Liner) that the article appear in the October TLE. *In the introduction to the FWI section, the guest editors pose some admonitory questions: "Are we heading in the right direction? Are we in the right valley? Or within a bigger context, is FWI the way to go?" In this context, Weglein's article is a timely and pertinent riposte that will be of significant interest and may elicit a degree of controversy to those working in the FWI field.*

A central purpose of this article is to bring an alternative voice, perspective, and understanding to the latest geophysical stampede, technical bubble, and self-proclaimed seismic cure-all, the so-called "full-waveform inversion" or FWI. If you think this is exaggerated, I refer to the advertisement/announcement of the 2013 SEG Workshop on FWI whose opening line is, "Full-waveform inversion has emerged as the final and ultimate solution to the Earth resolution and imaging objective."

Besides representing language, attitude, and a viewpoint that have no place anywhere in science, and, in particular, in exploration seismology, the fact is that the method, as put forth, is from a fundamental and basic-principle point of view (aside from, and well before, any practical considerations and track record of added-value are considered) hardly deserving of the label "inversion", let alone all the other extreme and unjustified claims and attributes, as being the "deliverance" and the last and final word on the subject.

From a direct-inversion point of view, and for the algorithms that are derived for solving the exact same problem of estimating, for example, the location of velocity anomalies and shallow hazards, and velocity changes at the top and base salt, all the current approaches to so-called full-waveform inversion are: (1) always using the wrong data, (2) always using the wrong algorithms, and (3) all too often, using the wrong Earth model, as well. Making this clear is one purpose of this article.

The issue being discussed in this article is not a matter of semantics and is not a labeling/mislabeling issue; it is the substantive issue of what data and what algorithms are called for by direct inversion to achieve certain seismic processing objectives. In particular, the focus here is on objectives that rely on the amplitude of reflection data as a function of incident angle to determine changes in, e.g., P-wave velocity, AVO parameters, or so-called FWI.

Another purpose of this article is to propose and exemplify an alternative and direct inverse solution that actually deserves the label "inversion" and could be useful for those goals and objectives, and perhaps can actually earn, deserve, and warrant a label of FWI, although never as the "ultimate and final solution." The direct-inversion approach provides not only a method but also a framework and platform for understanding when it will and will not work. All current so-called FWI methods are indirect model-matching methods, and indirect methods can never provide that capability and clarity. Model-matching run backward, or solving a forward problem in an inverse sense, resides behind all the current indirect P-wave-only so-called FWI and is never equivalent to a direct inverse solution for any nonlinear problem, nor does it even represent a fully and completely aligned goal and property of a direct inverse solution.

A third and perhaps the most important goal of this article is to provide a new, comprehensive overview and bridge for these two approaches for those who may be following, applying, and/or considering the current so-called indirect model-matching FWI approach and those proposing, interested in, or providing a road to a direct inverse methodology. It will be shown how these two approaches have the same starting point, and in fact, have the same exact generalized Taylor series expansion for modeling data and for expressing the actual data in terms of a reference model and reference data and the difference between actual and reference properties. The two approaches differ in how they view each of the same terms of that forward series. One view of those individual terms leads to a Taylor series form that does not allow a direct inverse series and that leaves as the only option the running of a forward (linear truncated) series in an inverse sense. That forward description viewed as only a generalized Taylor series results in, and provides no other choice other than, an indirect model-matching approach (e.g., as seen in AVO and the so-called FWI methods). This is the mainstream/conventional view of the forward description as a Taylor series, and, while easy to understand, that view precludes a direct inverse, and therefore explains the widespread use of indirect model-matching approaches. Another view of those individual terms in the forward Taylor series that derives from the fundamental equation of scattering theory (the Lippmann-Schwinger equation) recognizes that the forward Taylor series is a special class of generalized Taylor series—a generalized geometric series. Further, it is a geometric series for a forward problem, and it has a geometric series for a direct inverse solution. Without understanding and calling upon the scattering-theory equation, that recognition of the forward series as being geometric is not possible, and a direct inverse solution would not be achievable. All of the consequences and differences between the forward model-matching approach leading to methods such as so-called FWI and the direct inverse methods, derived from the inverse scattering series, have that simple, accessible, and understandable origin. The details, arguments, and examples behind these three objectives and goals are provided below.

Let's begin. Seismic processing is an inverse problem, in which measurements on or near the surface of the Earth are used to make inferences about the nature of the subsurface that are relevant to the exploration and production of hydrocarbons.

There was a time, not too long in the past, when a discussion of any method for solving inverse or data-processing problems always began with a definition of direct and indirect methods. The latter was deemed the less respectable and the lesser choice between the two, considered out of desperation and resignation and offered with hesitation and apology. It was associated among "inversionists" with searching and model matching rather than with seeking a direct, clear, and definitive solution through a math-physics analysis.

It appears that earlier, healthy understanding and respect for the framework and definitiveness of direct inverse methods have largely given way or have been pushed aside, with serious and substantive negative and injurious conceptual and practical consequences. Among the latter manifestations and consequences is the totally mislabeled and ubiquitous phenomenon of so-called "full-wave inversion" (FWI) methods. Among FWI references are Brossier et al. (2009), Crase et al. (1990), Gauthier et al. (1986), Nolan and Symes (1997), Pratt (1999), Pratt and Shipp (1999), Sirgue et al. (2010), Symes (2008), Tarantola (1984, 1986), Valenciano et al. (2006), Vigh and Starr (2008), and Zhou et al. (2012).

This note advocates (whenever possible) direct methods for solving processing problems and providing prerequisites. Direct methods offer many conceptual and practical benefits over indirect methods. Advantages of direct methods begin with actually knowing that you are solving the problem that you are interested in solving.

How can you recognize a direct versus an indirect method? Consider the quadratic equation

$$ax^2 + bx + c = 0\,, \tag{1}$$

and the solution

$$x = (-b \pm \sqrt{b^2 - 4ac})/2a\,. \tag{2}$$

Equation 2 is a direct solution for the roots of Equation 1. On the other hand, if you see a cost function involved in a solution, the solution is indirect. Also, if you see a modeling equation being solved in an inverse sense, or an iteratively linear updating, those are each direct indicators of an indirect solution and a model-matching approach, which too often can start with an incorrect or insufficient modeling equation and a matching of fundamentally inadequate data. The only time that a forward problem solved in an inverse sense can be equivalent to a direct inverse solution is when the direct inverse solution is linear. For example, locating reflectors at depth with a known velocity model is linear, and, hence, e.g., (asymptotic) RTM is a modeling run backward (i.e., in an inverse sense) to directly determine structure. Another transparent example is given by the forward geometric series

$$S = ar + ar^2 + ar^3 + \ldots = \frac{ar}{1-r} \qquad when\,|r|<1 \tag{3}$$

and the inverse

$$\begin{aligned} r &= \frac{S/a}{1+S/a} = S/a - (S/a)^2 + (S/a)^3 \ldots \\ &= r_1 + r_2 + r_3 + \ldots \text{ when } |S/a|. < 1 \end{aligned} \tag{4}$$

If, rather than these nonlinear relationships among S, a, and r, we instead imagine an exact linear relationship that S, a, and r might satisfy, e.g.,

$$S = ar, \tag{5}$$

then we have the forward problem of solving for S given a and r, and the inverse problem becomes solving for r in terms of S and a. The direct inverse solution $r = S/a$ is equivalent to the forward problem solved in an inverse sense, solving $S = ar$ for r in terms of S and a. However, if the forward relationship assumed among S, a, and r is a quadratic relationship (an approximate of the actual nonlinear forward problem given by Equation 3), we have

$$S = ar + ar^2. \tag{6}$$

Then, solving the forward problem, Equation 6, in an inverse sense is a quadratic solution with *two* roots that can be real or imaginary, whereas the solution to Equation 4 is a *single* real solution for r. In place of Equation 6, think of the linearized forward Zoeppritz equation for R_{pp} solved in an inverse sense, and the point is clear. This simple and transparent example demonstrates a pitfall of thinking that a direct inversion is equivalent to a forward problem solved in an inverse sense. Another example, pointed out in Weglein et al. (2009), is the direct inverse solution for predicting and removing free-surface and internal multiples, from the inverse-scattering series, where these two distinct algorithms are independent not only of subsurface information, they are also independent of whether we assume the Earth is acoustic, elastic, anelastic, heterogeneous, and anisotropic. The multiple-removal algorithms (which are direct and nonlinear) do not change one line of code when you change your mind about the Earth model type you want to consider. Can you imagine a model-matching and subtraction method or linear-updating method for predicting and removing multiples, with any cost function, L_1, L_2, L_p, that would be independent of subsurface properties and the type of Earth model you are using to generate the synthetic data? It is hard to overstate the significance of this point. The widely recognized benefit to industry from effectively removing free-surface and internal multiples using algorithms derived from the inverse scattering series, for offshore and onshore plays, never would have occurred if the indirect inversion, model-matching, and iterative updating, and FWI-like thinking, were the approaches pursued for removing multiples.

In general, we look at inversion as a set of tasks: free-surface and internal-multiple removal, depth imaging, and nonlinear AVO. For the purposes of this article and for discussing FWI, the focus is entirely on how the ISS addresses that parameter estimation task in isolation, and as if all other tasks (e.g., multiple removal) had been previously achieved.

Indirect methods such as flat common-image gathers (CIG) were developed as a response to the inability to directly solve for and adequately provide a velocity model for depth imaging, and those CIGs represent a necessary condition at the image that an accurate velocity would satisfy. References for CIGs are Anderson et al. (2012), Baumstein et al. (2009), Ben-Hadj-ali et al. (2008, 2009), Biondi and Sava (1999), Biondi and Symes (2004), Brandsberg-Dahl et al. (1999), Chavent and Jacewitz (1995), Fitchner (2011), Guasch et al. (2012), Kapoor et al. (2012), Rickett and Sava (2002), Sava et al. (2005), Sava and Fomel (2003), Sirgue et al. (2009, 2010, 2012), Symes and Carazzone (1991), Tarantola (1987), and Zhang and Biondi (2013). Many wrong velocity models can and will also satisfy a flat common-image-gather criterion, especially under complex imaging circumstances. Indeed, unquestioned faith in the power of satisfying the flat CIG criterion can and does contribute to dry-hole drilling. Mathematicians who work on the latter types of CIG problems would better spend their time describing the underlying lack of a necessary and sufficient condition, and the consequences, rather than dressing up and obfuscating the necessary but insufficient condition in fancy, rigorous, and abstract new clothes.

It seems that the recent surge of interest in estimating changes in velocity is fueled by: (1) the improved ability to produce low-frequency and low-vertical-wavenumber information from new acquisition and improved deghosting; (2) the implicit admission of serious problems with methods to estimate velocity models (e.g., with tomography, iterative flat CIG searching, and the like); and, of course, (3) the persistent and unacceptable dry-hole drilling rate. Today, for example, we basically remain fixed and without significant progress (at a one-in-ten success rate) in drilling successful exploration wells in the deep-water Gulf of Mexico (Hawthorn, 2009; Iledare and Kaiser, 2007).

Indirect methods should be considered only when direct methods are not available or are inadequate, or when you cannot figure out how to solve a problem directly. Indirect methods are often and reasonably employed to allow a channel or an adjustment (a dial) for phenomena and components of reality that are outside and external to the physics of the system you have chosen and defined. Of course, there always are, and always will be, phenomena outside your assumed and adopted physics and system that must be accommodated and that are ignored at your peril. That's the proper realm and role for indirect methods. Even then, however, they need to be applied judiciously and always with scrutiny of what resides behind cost-function-criteria assumptions. When a direct method to predict the amplitude and phase of free-surface multiples, such as inverse-scattering-series free-surface-multiple removal, includes the obliquity factor, and has the direct satisfaction of prerequisites such as source and receiver deghosting and wavelet estimation, then the better the direct method of providing the prerequisites performs, the better the free-surface demultiple provides the amplitude and phase of the free-surface multiples. If at any stage you decide you can "roll in" obliquity, source and receiver deghosting, and wavelet estimation into a catch-all energy-minimization adaptive subtraction, you run into the serious problem: No matter how much better you achieve a satisfaction of energy minimization, you still have no guarantee that that improved energy minimization aligns with and supports free-surface-multiple removal while preserving primaries. In fact, removal of multiples can increase "energy" (e.g., when you have destructive interference between a primary and a multiple), and it is widely understood that the energy-minimization criteria are among today's greatest impediments to effectively removing free-surface and internal multiples for complex onshore and marine plays. The criteria behind the indirect adaptive step matter. Within the area of free-surface and internal-multiple attenuation, the rush to and overreliance on energy-minimization adaptive subtraction contributes to the inability to effectively and surgically remove multiples at all offsets and without damaging primaries. That specific issue was discussed in a recent report to the M-OSRP consortium on seeking adaptive criteria (Weglein, 2012) that serve as an alternative and replacement for energy minimization for free-surface multiple removal. However, the trend of using indirect methods for phenomena and processing goals within the system, and for providing prerequisites within the system, is in general a conceptual and practical mistake. There has been a dangerous and growing tendency to solve everything inside and outside the system by using indirect methods and cost functions. Of course the need for ever-faster computers is universally recognized and supported. However, the growth in computational physics, often at the expense of mathematical physics, and the availability of ever-faster computers, encourages the rush to "cost functions" and to searching without thinking, and thus represents a ubiquitous, misguided, and unfortunate trend, with "solutions" that aren't. When we give up on physics and determinism, we look at statistics and searching, and indirect methods become a "natural" choice and are always readily available, along with their drawbacks and consequences.

A direct method provides a framework of precise data needs, and it delivers a straight-ahead formula that takes in your data and actually solves and explicitly and directly outputs the solution that you seek. Indirect methods can never provide that clarity or confidence. Model-matching and iterative updating by any fancy name, such as a new "Frechet derivative," and the so-called "full-wave inversion," are model-matching and are never, ever, equivalent to a direct inversion for the Earth's elastic mechanical property changes. The distinction is significant and has both conceptual and mercantile consequences.

Here is an example of the difference. Suppose someone said that you could take a single seismic trace that is a single function of time, and invert simultaneously for velocity and density, each as a function of depth in a 1D Earth.

Today, you might reasonably be cautious and concerned because the dimension of the data is less than the overall dimension of the quantities you seek to determine. We have learned as an industry to be dubious in the latter single-trace, solve-for-two-functions-of-depth case. We look skeptically at those who would model-match and pull all kinds of arcane cost functions and generalized inverses together, using different norms and constraints and full-wave predictions of that single trace that can be model-matched with amplitude and phase so that we can call that model-matching scheme "full-waveform inversion." Why can't we solve for density and velocity uniquely from a single trace, because we can certainly model the single trace from knowing the velocity and density as a function of depth? That's a beginning and an example of thinking that solving a forward problem in an inverse sense is in some way actually solving the inverse problem. What came along in that earlier time, as a response to this question, were direct acoustic inversion methods that said that inverting for velocity and density as functions of depth from a single trace is impossible, or at least that it is impossible to provide the unique and actual velocity and density as a function of depth. That direct-inversion framework convinced many (hopefully most) people that the one-trace-in, two-functions-out approach is not a question or an issue of which indirect algorithm or L_p cost function you are using. It is more basic and stands above algorithm; it's an inadequate-data issue. No algorithm with that single-trace data input should call itself "inversion," even if that single trace was model-matched and iteratively updated and computed with amplitude and phase and, with too much self-regard, labels itself as "full-wave inversion." We learned to stop running that single trace through search algorithms for velocity and density—and that lesson was absorbed within our collective psyches in our industry—for whatever the cost function and local or global minimum you employed. Using the wrong and fundamentally inadequate data closes the book and constitutes the end of the story. Thus, we learned to look for and respect dimension between the data and the sought-after parameters we want to identify. That is a good thing, but it turns out that it's not a good-enough thing. In fact, direct acoustic wavefield inversion for a 1D Earth requires all the traces for a given shot record in order to determine one or more parameters (e.g., V_P and density) as a function of depth.

This article will show (in a similar way) that the fact that you can solve the forward Zoeppritz equations (or a linear approximate) for a PP reflection coefficient as a function of incident angle and the changes in λ, μ, and ρ across the reflector does not imply that you can solve for changes in λ, μ, and ρ in terms of the PP reflection coefficient as a function of angle. A direct inverse for the changes in λ, μ, and ρ demands all multicomponent sources and receivers, or, equivalently, PP, PS, SP, and SS data.

These conditions on data requirements hold for any processing/inverse problem in which the reference or background medium is elastic—e.g., for all amplitude analysis, including AVO and so-called FWI and all ISS multiple removal and imaging with ocean-bottom or onshore acquisition. See Li et al. (2011), Liang et al. (2010), Matson (1997), Matson and Weglein (1998), Weglein et al. (2003), and H. Zhang (2006).

"Inadequate data" means something much more basic and fundamental than limitations due to sampling, aperture, and bandwidth. That is, indirect solutions can (and often do) input data that are fundamentally inadequate from a basic and direct inverse perspective and understanding. The indirect methods then search locally and globally around error surfaces with Frechet derivatives and conjugate gradients, and they keep hordes of math, physics, geophysics, and computer scientists busy using giant and super-fast computers looking at outputs and 3D color displays, and being convinced that with all the brainpower and resources that are invested, they are on track and are on their way to solving the problem. What's wrong with linear iterative updating? What's wrong begins with understanding the meaning of a linear inverse. Even in cases in which the data are adequate—e.g., cases with P-wave data and an acoustic inverse model—the algorithms that a direct inverse provides for explicit linear and each nonlinear estimate of changes in P-wave velocity and density, will differ at the first nonlinear step and at every subsequent step, with the nonlinear iterative linear estimate of these changes in physical properties. The linear, quadratic, cubic, … estimates of physical properties from a direct inverse method are explicit and unique (a generalized Taylor/geometric series) and order-by-order in the data and will not agree with an iterative linear update. Hence, although the iterative linear updating is nonlinear in the data, it does not represent a direct inverse solution. Further, the terms in the direct solution are analytically determined in terms of the first term, whereas iterative linear updating requires generalized inverses, SVD, cost functions, and numerical solutions. They could not be more different. If you had an alternative to the solution of the quadratic equation and it produced different roots from those produced by the direct quadratic formula, (Equation 2), would you call it "an inverse solution for the roots?" That's the issue, and it's that simple.

For the elastic inverse case, the difference is yet more serious. A direct inverse solution for the P-velocity, V_P, shear velocity, V_S, and density, ρ, and a linear iterative method, will already differ at the linear step, and that difference and resulting gap grow at each nonlinear step and estimate.

When it comes to directly inverting for changes in elastic properties and density, there are direct and explicit formulas for the linear and nonlinear estimates. The same single unchanged direct inverse ISS set of equations that derived the algorithms for free-surface and internal-multiple removal—and have demonstrated standalone capability (see, e.g., Ferreira, 2011; Luo et al., 2011; and Weglein et al., 2003, 2011)—have also provided the ISS depth imaging (Weglein et al. 2011, 2012) and direct inversion for Earth mechanical properties. In Zhang (2006), we find the first direct nonlinear equations for estimating the changes in elastic properties for a 1D Earth.

The mathematical origin of linear inverse theory (and linear iterative inversion) begins with a Taylor series of the recorded data, $D(m)$, from the actual Earth. Those data depend on the Earth properties characterized by the label m and the synthetic data $D(m_0)$ from an estimate or reference value of those properties that we label, m_0. To relate $D(m)$ and $D(m_0)$, we introduce a Taylor series

$$D(m)=D(m_0)+D'(m_0)\Delta m+\frac{D''(m_0)}{2}\Delta m^2+\ldots, \quad (7)$$

in which the derivatives are Frechet derivatives. A linearized form of Equation 7 is considered

$$D(m)=D(m_0)+D'(m_0)\Delta m_1^1, \quad (8)$$

where the Frechet derivative,

$$D'(m_0)=\frac{D(m_0+\varepsilon\Delta m)-D(m_0)}{\varepsilon\Delta m} \quad (9)$$

is approximated by a finite-difference approximation involving data at m_0 and data at a nearby model, $m_0+\varepsilon\Delta m$. Δm_1^1 means the first linear estimate of Δm, with the subscript standing for linear and the superscript for the first estimate. The matrix inversion of Equation 8 for Δm_1^1 leads to a new approximate $m_0+\Delta m_1^1$, and

$$D(m)-D(m_0+\Delta m_1^1)=D'(m_0+\Delta m_1^1)\Delta m_1^2. \quad (10)$$

The process is repeated and is the basis of iterative linear inversion. Properties of that process related to convergence to m are spelled out in Blum (1972), page 536, with issues where the constants such as M that appear in the convergence criteria are unknown.

Another starting point for this type of perturbative approach is from scattering theory, where $D(m)$ relates to the actual Green's function, G, and $D(m_0)$ relates to the reference Green's function, G_0, and $V = m-m_0$. The identity among G, G_0, and V is called the Lippmann-Schwinger or Scattering Equation (see, e.g., Taylor 1972)

$$G=G_0+G_0VG \quad (11)$$

and an expansion of Equation 11 for G in terms of G_0 and V produces

$$G=G_0+G_0VG_0+G_0VG_0VG_0+\ldots. \quad (12)$$

Keys and Weglein (1983) provide the formal association between $D'(m_0)\Delta m$ and G_0VG_0. Equation 7 is a Taylor series in Δm, and as such that series does not have an available inverse series. However, because Equation 12 (which follows from the scattering Equation 11) is a geometric series in $r = VG_0$ and $a = G_0$, then a geometric series for $S = G-G_0$ in terms of a and r—$S = ar/(1-r)$—has an inverse series $r = (S/a)/(1+S/a)$ with terms

$$r_1 = S/a$$

$$r_2 = -(S/a)^2$$

$$r_3 = (S/a)^3$$

$$r_4 = -(S/a)^4$$

$$\ldots .$$

A unique expansion of VG_0 in orders of measurement values of $(G-G_0)$ is

$$VG_0=(VG_0)_1+(VG_0)_2+\ldots \quad (13)$$

The scattering-theory equation allows that forward series form the opportunity to find a direct inverse solution. Substituting Equation 13 into Equation 12 and setting the terms of equal order in the data to be equal, we have $D = G_0 V_1 G_0$, where the higher order terms are $V_2, V_3, \ldots$, as given in Weglein et al. (2003) page R33 Equations 7—14.

For the elastic equation, V is a matrix and the relationship between the data and V_1 is

$$\begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix} = \begin{pmatrix} G_0^P & 0 \\ 0 & G_0^S \end{pmatrix} \begin{pmatrix} V_1^{PP} & V_1^{PS} \\ V_1^{SP} & V_1^{SS} \end{pmatrix} \begin{pmatrix} G_0^P & 0 \\ 0 & G_0^S \end{pmatrix}$$

$$V_1 = \begin{pmatrix} V_1^{PP} & V_1^{PS} \\ V_1^{SP} & V_1^{SS} \end{pmatrix}$$

$$V = \begin{pmatrix} V^{PP} & V^{PS} \\ V^{SP} & V^{SS} \end{pmatrix}$$

$$V = V_1 + V_2 + \ldots$$

where V_1, V_2 are linear, quadratic contributions to V in terms of the data,

$$D = \begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix}.$$

The changes in elastic properties and density are contained in $V = \begin{pmatrix} V^{PP} & V^{PS} \\ V^{SP} & V^{SS} \end{pmatrix}$, and that leads to direct and explicit solutions for the changes in mechanical properties in orders of the data, $D = \begin{pmatrix} D^{PP} & D^{PS} \\ D^{SP} & D^{SS} \end{pmatrix}$,

$$\frac{\Delta\lambda}{\lambda} = \left(\frac{\Delta\lambda}{\lambda}\right)_1 + \left(\frac{\Delta\lambda}{\lambda}\right)_2 + \ldots$$

$$\frac{\Delta\mu}{\mu} = \left(\frac{\Delta\mu}{\mu}\right)_1 + \left(\frac{\Delta\mu}{\mu}\right)_2 + \ldots$$

$$\frac{\Delta\rho}{\rho} = \left(\frac{\Delta\rho}{\rho}\right)_1 + \left(\frac{\Delta\rho}{\rho}\right)_2 + \ldots$$

The ability of the forward series to have a direct inverse series derives from (1) the identity among G, G_0, V provided by the scattering equation and then (2) the recognition that the forward solution can be viewed as a geometric series for the data, D, in terms of VG_0. The latter derives the direct inverse series for VG_0 in terms of the data.

Viewing the forward problem and series as the Taylor series (Equation 7) in terms of Δm does not offer a direct inverse series, and hence there is no choice but to solve the forward series in an inverse sense. It is that fact that results in all current AVO and FWI methods being modeling methods that are solved in an inverse sense. Among references that solve a forward problem in an inverse sense in P-wave AVO are Beylkin and Burridge (1990), Boyse and Keller (1986), Burridge et al. (1998), Castagna and Smith (1994), Clayton and Stolt (1981), Foster et al. (2010), Goodway (2010), Goodway et al. (1997), Shuey (1985), Smith and Gidlow (2000), Stolt (1992), and Stolt and Weglein (1985). The intervention of the explicit relationship among G, G_0, and V (the scattering equation) in a Taylor series-like form produces a geometric series and a direct inverse solution.

The linear equations are:

$$\begin{pmatrix} \hat{D}^{PP} & \hat{D}^{PS} \\ \hat{D}^{SP} & \hat{D}^{SS} \end{pmatrix} = \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \tag{14}$$

$$\hat{D}^{PP} = \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \tag{15}$$

$$\hat{D}^{PS} = \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \tag{16}$$

$$\hat{D}^{SP} = \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \tag{17}$$

$$\hat{D}^{SS} = \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \tag{18}$$

$$\begin{aligned} \tilde{D}^{PP}\left(k_g,0;-k_g,0;\omega\right) &= -\frac{1}{4}\left(1-\frac{k_g^2}{\nu_g^2}\right)\tilde{a}_\rho^{(1)}\left(-2\nu_g\right) \\ &-\frac{1}{4}\left(1+\frac{k_g^2}{\nu_g^2}\right)\tilde{a}_\gamma^{(1)}\left(-2\nu_g\right)+\frac{2k_g^2\beta_0^2}{\left(\nu_g^2+k_g^2\right)\alpha_0^2}\tilde{a}_\mu^{(1)}\left(-2\nu_g\right) \end{aligned} \tag{19}$$

$$\begin{aligned} \tilde{D}^{PS}\left(\nu_g,\eta_g\right) &= -\frac{1}{4}\left(\frac{k_g}{\nu_g}+\frac{k_g}{\eta_g}\right)\tilde{a}_\rho^{(1)}\left(-\nu_g-\eta_g\right) \\ &-\frac{\beta_0^2}{2\omega^2}k_g\left(\nu_g+\eta_g\right)\left(1-\frac{k_g^2}{\nu_g\eta_g}\right)\tilde{a}_\mu^{(1)}\left(-\nu_g-\eta_g\right) \end{aligned} \tag{20}$$

$$\begin{aligned} \tilde{D}^{PS}\left(\nu_g,\eta_g\right) &= \frac{1}{4}\left(\frac{k_g}{\nu_g}+\frac{k_g}{\eta_g}\right)\tilde{a}_\rho^{(1)}\left(-\nu_g-\eta_g\right) \\ &+\frac{\beta_0^2}{2\omega^2}k_g\left(\nu_g+\eta_g\right)\left(1-\frac{k_g^2}{\nu_g\eta_g}\right)\tilde{a}_\mu^{(1)}\left(-\nu_g-\eta_g\right) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \tilde{D}^{SS}\left(k_g,\eta_g\right) &= \frac{1}{4}\left(1-\frac{k_g^2}{\eta_g^2}\right)\tilde{a}_\rho^{(1)}\left(-2\eta_g\right) \\ &-\left[\frac{\eta_g^2+k_g^2}{4\eta_g^2}-\frac{2k_g^2}{\eta_g^2+k_g^2}\right]\tilde{a}_\mu^{(1)}\left(-2\eta_g\right), \end{aligned} \tag{22}$$

where $a_\gamma^{(1)}$, $a_\mu^{(1)}$, and $a_\rho^{(1)}$ are the linear estimates of the changes in bulk modulus, shear modulus, and density, respectively. The direct quadratic nonlinear equations are

$$\begin{aligned} &\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_2^{PP} & \hat{V}_2^{PS} \\ \hat{V}_2^{SP} & \hat{V}_2^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \\ &= -\begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^S \end{pmatrix} \begin{pmatrix} \hat{V}_1^{PP} & \hat{V}_1^{PS} \\ \hat{V}_1^{SP} & \hat{V}_1^{SS} \end{pmatrix} \begin{pmatrix} \hat{G}_0^P & 0 \\ 0 & \hat{G}_0^s \end{pmatrix}, \end{aligned} \tag{23}$$

$$\hat{G}_0^P \hat{V}_2^{PP} \hat{G}_0^P = -\hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P - \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \tag{24}$$

$$\hat{G}_0^P \hat{V}_2^{PS} \hat{G}_0^S = -\hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S - \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S, \tag{25}$$

$$\hat{G}_0^S \hat{V}_2^{SP} \hat{G}_0^P = -\hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \hat{V}_1^{PP} \hat{G}_0^P - \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P, \tag{26}$$

$$\hat{G}_0^S \hat{V}_2^{SS} \hat{G}_0^S = -\hat{G}_0^S \hat{V}_1^{SP} \hat{G}_0^P \hat{V}_1^{PS} \hat{G}_0^S - \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S \hat{V}_1^{SS} \hat{G}_0^S. \tag{27}$$

Because $\hat{V}_1^{PP}$ relates to $\hat{D}^{PP}$, $\hat{V}_1^{PS}$ relates to $\hat{D}^{PS}$, and so on, the four components of the data will be coupled in the nonlinear elastic inversion. We cannot perform the direct nonlinear inversion without knowing all components of the data. Thus, the direct nonlinear solution determines the data needed for a direct inverse. That, in turn, defines what a linear estimate means. That is, a linear estimate of a parameter is an estimate of a parameter that is linear in data that can directly invert for that parameter. Because D_{PP}, D_{PS}, D_{SP}, and D_{SS} are needed to determine a_γ, a_μ, and a_ρ directly, a linear estimate for any one of these quantities requires simultaneously solving Equations 19–22. See, e.g., Weglein et al. (2009) for further detail.

Those direct nonlinear formulas are like the direct solution for the quadratic equation mentioned above and solve directly and nonlinearly for changes in V_P, V_S, and density in a 1D elastic Earth. Stolt and Weglein (2012), present the linear equations for a 3D Earth that generalize Equations 19-22. Those formulas prescribe precisely what data you need as input, and they dictate how to compute those sought-after mechanical properties, given the necessary data. There is no search or cost function, and the unambiguous and unequivocal data needed are full multicomponent data—PP, PS, SP, and SS—for all traces in each of the P and S shot records. The direct algorithm determines first the data needed and then the appropriate algorithms for using those data to directly compute the sought-after changes in the Earth's mechanical properties. Hence, any method that calls itself inversion (let alone full-wave inversion) for determining changes in elastic properties, and in particular the P-wave velocity, V_P, and that inputs only P-data, is more off base, misguided, and lost than the methods that sought two or more functions of depth from a single trace. You can model-match P-data until the cows come home, and that takes a lot of computational effort and people with advanced degrees in math and physics computing Frechet derivatives, and requires sophisticated L_p norm cost functions and local or global search engines, so it must be reasonable, scientific, and worthwhile. Why can't we use just PP data to invert for changes in V_P, V_S, and density, because Zoeppritz says that we can model PP from those quantities, and because we have, using PP-data with angle variation, enough dimension? As stated above, data dimension is good, but not good enough for a direct inversion of those elastic properties.

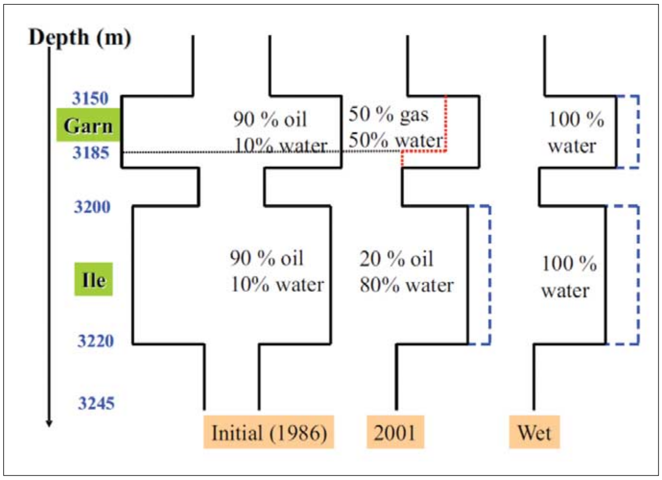

Figure 1. Synthetic well log A-52.

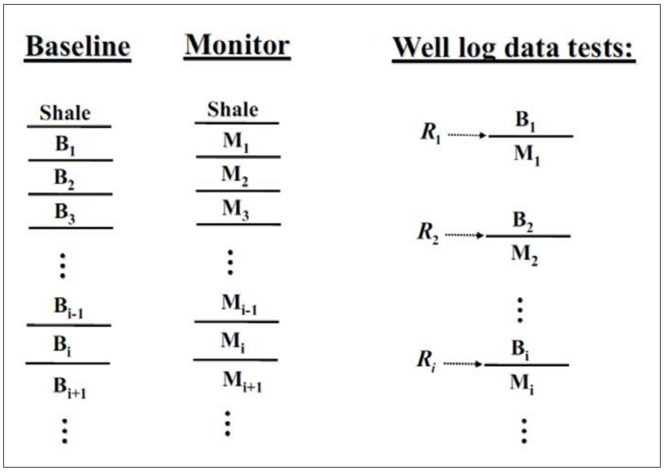

Figure 2. The baseline, monitor, and input reflection coefficients.

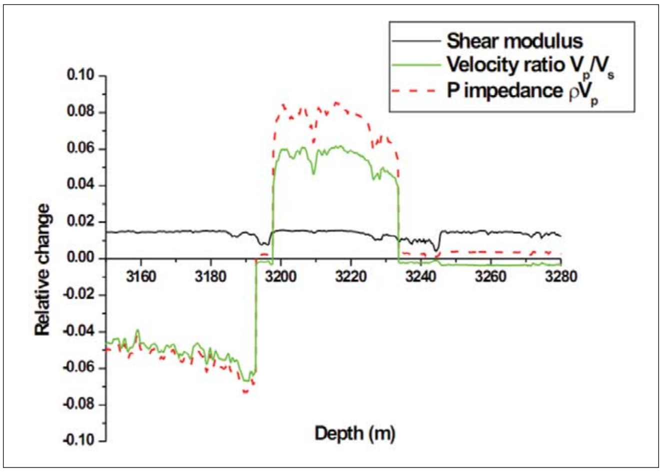

Figure 3. Comparison of actual changes in shear modulus, P-impedance, and velocity ratio V_p/V_S. The baseline is the log data in 1986 and the monitor is the log data in 2001.

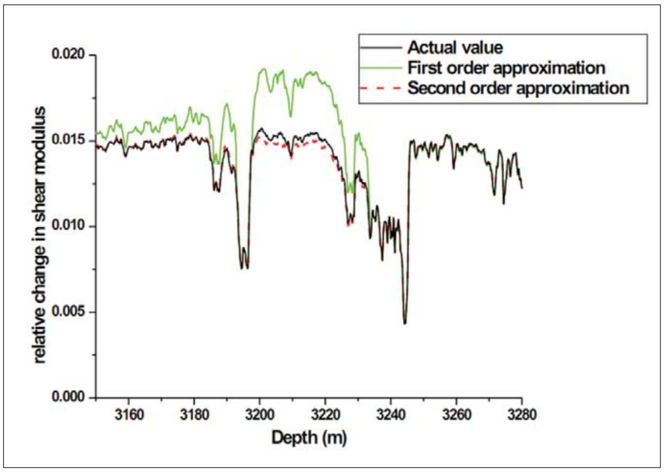

Figure 4. Comparison of first- and second-order approximations of relative change in shear modulus. The baseline is the log data in 1986 and the monitor is the log data in 2001.

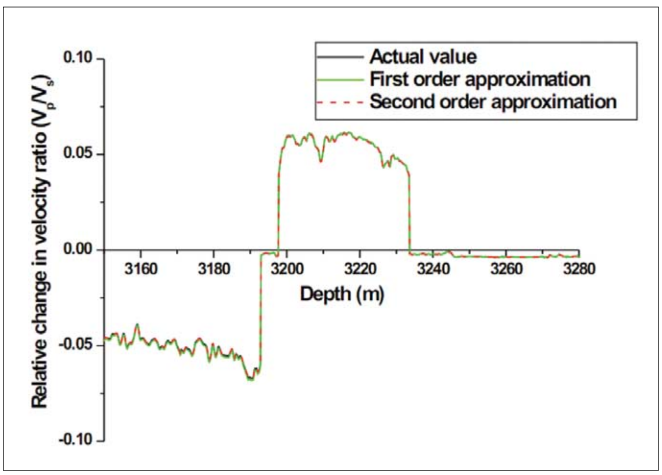

Figure 5. Comparison of first- and second-order approximations of relative change in V_p/V_S. The baseline is the log data in 1986 and the monitor is the log data in 2001.

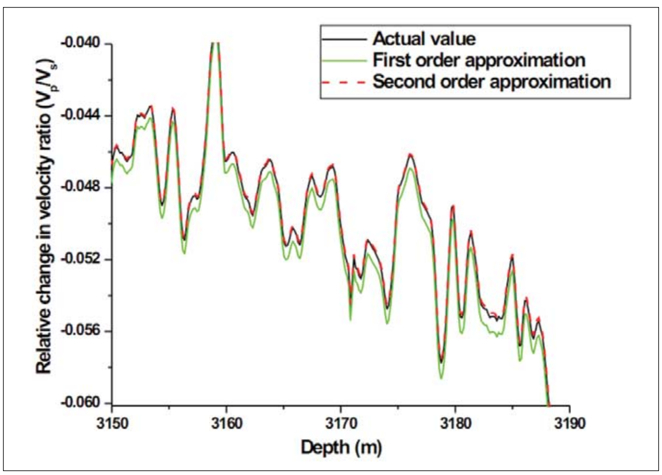

Figure 6. Zoomed-in comparison of first- and second-order approximations of relative change in V_p/V_S. The baseline is the log data in 1986 and the monitor is the log data in 2001.

The direct inverse is nonlinear. Iterative linear is nonlinear. But iterative linear inversion is not in any way equivalent to a direct nonlinear inversion. The further evidence that iterative linear inverse is not a direct elastic inverse solution, is that you can iteratively linear invert P-wave data. Hence, you can have the fundamentally inadequate data and perform iterative linear updating. That's not possible with a direct inverse method. The framework, data needs, and algorithms provided by direct inversion all matter. If you iteratively linear invert multicomponent data, you would not be performing a direct inversion, and your nonlinear estimates would not agree with the unique nonlinear terms provided by a direct solution. Multicomponent data are important, but the direct inverse algorithm of that data is essential. The framework of a direct method helps you understand what will allow things to work in principle, and, equally important, it helps you identify the issue or problem when things don't work. Indirect methods, on the other hand, can never match that definiteness, clarity, and value. When we use just P-wave data with an acoustic or elastic model-matching FWI for shallow-hazard detection or velocity estimation at top salt, and then issues arise, perhaps the framework and requirements described in this note might be among the issues behind a lack of predictive stability and usefulness.

In "Wave theory modeling of P-waves in a heterogeneous elastic medium" (Weglein 2012), a single-channel P-wave formalism is presented as a way to model P-waves in amplitude and phase without needing to model and predict shear waves. This P-only wave-modeling method is intractable as a parameter-estimation inverse procedure, blocked at the first and linear term. That supports the need for all multicomponent data in a direct inverse for estimating changes in the Earth's mechanical properties. If one somehow remained insistent that P-data were adequate for a direct elastic inverse, one would have to provide a response to that linear, intractable inverse step. Further, those direct and explicit nonlinear formulas are derivable only from the direct inverse machinery of the inverse scattering series (please see the References section).

Using P-wave data with amplitude and phase for an acoustic Earth model flies in the face of 40 years of AVO experience, which says that the elastic Earth is the minimum realistic Earth model for any amplitude-dependent algorithm or processing method. Using P-wave data for an elastic Earth model, with algorithms that utilize amplitude and phase, violates the necessary multicomponent data needs prescribed by direct inversion of V_P, V_S, and density. Having the adequate data (defined by a direct-inversion framework) is better than not having the necessary and sufficient data and is a good place to start. However, even when one is starting with the indicated multicomponent data, the train can still be taken off the track by indirect search and iterative linear-updating algorithms, when direct inverse algorithms are indicated and available. Iterative linear updating of multicomponent data is a model-matching indirect method and is never equivalent to a direct inversion of those data.

Some might say in response that P-wave FWI with either an acoustic or elastic medium, followed by use of some search algorithm, represents "an approximation," and what's wrong with approximations? The answer is precisely that "What IS wrong with the approximation?" If you purposefully or inadvertently ignore (or wish away) the framework and algorithms that a direct solution to the elastic parameter estimation provides, you will never know what you are ignoring and dropping and what your approximation is approximating, nor will you know what value your method actually represents and means, and how you could improve the reliability of your prediction.

In summary, so-called P-wave FWI is something less than advertised and is in general the wrong (acoustic) Earth model, the wrong data, and the wrong method—but besides that, it has a lot going for it.

In Zhang (2006), the direct elastic inverse was applied to a 4D application and the term beyond linear was able to help distinguish a pressure change from a fluid change. This line of research continued in Li (2011) and Liang (2010). This is comparatively illustrated with synthetic log data in Zhang's Figures 1 through 6 (which are included in this article).

Epilog

A direct method to find the route from where you are to where you want to go—e.g., for a scheduled meeting—would use MapQuest, while an indirect method would seek and search and stop at every possible location in the city until you arrive somewhere where someone seems to be happy to see you, and you have a toolbox of L_p cost functions to define "happy." A direct solution, in contrast to indirect methods, does not require or ever raise the issue of necessary but insufficient conditions or cost functions, and it's not a "condition" or property. It's a solution, a construction. Nothing beats that for clarity, efficiency, and effectiveness. The direct MapQuest inversion communication and message to the current indirect P-wave FWI methods is that the latter are searching for the meeting in the wrong city.

The message of this article is that direct inversion provides a framework, and a set of data requirements and algorithms, that not only have produced a standalone capability (with model-type independent algorithms) for removing free-surface and internal multiples, without subsurface information, but also for establishing the requirements for all seismic processing methods that depend on amplitude analysis, such as AVO and so-called FWI. Being frank, we wish these requirements were not the case, because it makes our lives more complicated and difficult—but the conclusions are inescapable. When the framework, data requirements, and direct methods are not satisfied, we have a clear and understandable reason for the resulting failure and for what we might do to provide more reliable and useful predictive capability. Direct and indirect methods both play an essential role in an effective seismic processing strategy: where the former accommodates the physics within the system, and the latter provides a channel for real-world phenomena beyond the assumed physics. TLE

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