Isolation of an elimination subseries for the surgical removal of first-order internal multiples with downward reflection at the shallowest reflector

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Abstract

In this work a subseries of the ISS is isolated, with the specific task of removing internal multiples of first-order, with downward reflection at the shallowest reflector. The algorithm predicts both the phase and exact amplitude of the internal multiples and does not modify any primary; therefore the internal multiples are removed surgically. This algorithm may be relevant and provide added value when one of the internal multiples under discussion is interfering destructively with (or is proximal to) a primary, and the attenuation of the internal multiple provided by previous algorithms is not adequate for the clean removal of the multiple and not touching the primary. To show how the elimination subseries proposed in this work deals with this challenging situation, an analytic example with three interfaces is included, with one of the relevant first-order internal multiples interfering destructively with the primary generated at the third reflector. We show in particular how the interfering internal multiple is eliminated with no damage to the amplitude or the phase of the primary, as is expected from a method for surgical removal of internal multiples.

1 Introduction

Today, there are a number of methodologies in the oil industry that are designed to predict internal multiples. These methods are followed by energy-minimization adaptive subtraction to try to accommodate all shortcomings in the prediction, as it addresses contributions left outside of the system by the prediction method. In other words, the energy-minimization adaptive subtraction deals with issues not included in the physical framework behind the prediction method.

In particular, by using the ISS and the concept of specific-task subseries, a multidimensional algorithm was derived in Araújo (1994), Araújo et al. (1994) and Weglein et al. (1997) to predict and attenuate internal multiples present in the data. However, there are situations in which the energy-minimization adaptive-subtraction technique is not suitable anymore, and the attenuation of internal multiples is not enough for a correct interpretation of the seismic data. An example of this challenging situation for the oil industry can arise when an internal multiple is interfering destructively with (or is proximal to) a primary associated to a target e.g. subsalt targets. This situation is often present in onshore exploration, but it can also happen offshore. While the energy-minimization adaptive-subtraction technique is of value for isolated multiples, in this case it might also affect the primary that is experiencing interference from the internal multiple.

Therefore, it is important to develop new algorithms with enhanced capabilities. In response to this need, Ramírez and Weglein (2005) and Ramírez (2007) discuss early ideas for moving attenuation of internal multiples towards elimination through higher order terms in the ISS. Those ideas and concepts are here progressed and developed leading to a subseries which surgically removes at the same time all internal multiples of first-order having their single downward reflection generated at the shallowest reflector. We refer to this subseries as the leading-order internal multiple elimination subseries (LOIMES). We also illustrate how to use this subseries in a three-interface analytic model, to surgically remove the first-order internal multiple with its downward reflection at the shallowest interface and upward reflections at the second reflector. To highlight the importance of this work, the parameters of the model are chosen to mimic the situation described in the paragraphs above; i.e., to allow the internal multiple to interfere destructively with a primary. In particular, the primary that is experiencing interference corresponds to the third reflector.

The report's organization is as follows: Section 2 provides a review of the leading-order attenuation of internal multiples of first order, which is the initial step toward their complete elimination. In Section 3 we explain how to isolate the LOIMES, with emphasis on the first contribution beyond the leading-order attenuator; i.e., with full details of the derivation of the second term of the subseries provided. Section 4 is devoted to application of the LOIMES to the analytic model mentioned in the paragraph above. Finally, in Section 5 we present final comments and conclusions. There are two appendices, in which we show the details of the calculations needed to follow the main body of this paper.

2 Review of the internal multiple attenuation subseries

2.1 The inverse scattering series and seismic physics

The inverse scattering series (ISS) is a direct inversion method which can in principle determine, in seismic applications, subsurface properties of the earth using only the measured data D in a seismic experiment, and a Green's function for a chosen reference medium. The information about the earth is contained in the perturbation operator V, which is the difference between the actual medium (the earth) and the reference medium. Also, the data are the value of the scattered field[1] at the measurement surface. The ISS starts with the expansion of the perturbation operator (at the measurement surface) as

$$V = V_1 + V_2 + V_3 + ... \tag{1}$$

where V_i is the portion of V that is ith order in the measured data. Then, at the measurement surface the ISS takes the form (Weglein et al. 2003)

$$\begin{aligned} G_0V_1G_0 &= D \\ G_0V_2G_0 &= -G_0V_1G_0V_1G_0 \\ G_0V_3G_0 &= -G_0V_1G_0V_1G_0V_1G_0 - G_0V_1G_0V_2G_0 - G_0V_2G_0V_1G_0 \\ &\vdots \end{aligned} \tag{2}$$

As D is provided by the seismic experiment, we can solve for V_1 in the first equation of (2). Then, we can substitute V_1 into the second equation and solve for V_2. Now we can substitute V_1 and V_2 into the third equation and solve for V_3. Following this procedure we can determine all the components in the right hand side of (1). However, empirical tests performed in Carvalho (1992) suggest that with no *a priori* information, convergence is restricted to small contrasts and short duration of the perturbation.

A solution for the issue of convergence explained in Weglein et al. (2003) is to split the inversion into specific tasks:

1. Removal of free-surface multiples.
2. Removal of internal multiples.
3. Location and imaging of reflectors in space.
4. Inversion for earth material properties.

A free-surface multiple is by definition a seismic event with at least one downward reflection at the air-water interface; the number of downward reflections at the air-water interface is the order of

[1]The scattered field is defined as $\psi_s \equiv G - G_0$, where G and G_0 are Green's functions for the actual and reference medium respectively.

the free-surface multiple. On the other hand, an internal multiple is by definition a seismic event with at least one downward reflection, and with all of its downward reflections created at the earth (Figure 1). The order an the internal multiple is defined as the number of downward reflections it experiences anywhere during its travel time.

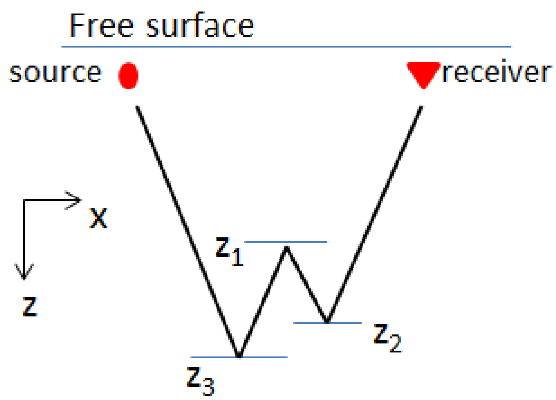

Figure 1: First-order internal multiple.

In Figure 1 the direction of increasing Z is downwards, hence $Z_2 > Z_1$ and $Z_3 > Z_1$. We also say that, on the basis of the locations where reflections occur, the interfaces generate an internal multiple of first order are in a "lower-higher-lower" configuration.

The recipe is to isolate distinct subseries from the ISS, with each subseries having as its goal only one of the specific tasks just listed. It turns out that those specific-task subseries have better convergence properties than the entire ISS. A fundamental part of this approach, mentioned in Weglein et al. (2003), is that the four tasks listed above are accomplished sequentially in the order in which they are mentioned. Each time a task is achieved, the problem is restarted, as if the task(s) accomplished had not existed before.

With regard to internal multiples, a subseries was isolated in Araújo (1994) and Weglein et al. (1997). Its task is attenuation of internal multiples of all orders. In particular, first-order internal multiples are attenuated by the leading-order contribution[2] of this subseries, conveniently named the *leading-order attenuator*.

[2]The leading-order contribution in a specific-task subseries refers here to the first term of that subseries that provides the initial contribution towards the achievement of the specific task.

2.2 The leading-order attenuator

As we will see in later sections, the LOIMES isolated in this work shares the same leading-order contribution that the internal multiple attenuation subseries (IMAS) has. Hence, it is important to first understand how the leading-order attenuator works, and then to move to higher-order contributions to the LOIMES. In this subsection we will provide a review of the leading-order attenuator.

A detailed study of the isolation of the IMAS, and in particular of the leading-order attenuator, is beyond the scope of this work. The interested reader can consult Araújo (1994), Ramírez (2007), and Weglein et al. (2003) for more details. For this work it is enough to say that the leading-order attenuator is contained in the third equation of the ISS. This is because first-order internal multiples experience three reflections and therefore they are of third order in data. The leading-order attenuator is isolated from $V_1G_0V_1G_0V_1$ in the references just mentioned.

For the 1D and normal-incidence case, the analytic expression for the leading-order attenuator is

$$b_3(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{3}$$

where ϵ is a small and positive parameter introduced to ensure the characteristic "lower-higher-lower" configuration for first-order internal multiples, which was mentioned in Section 2.1, and to avoid the configurations that include the contributions of the self-interactions $z'' = z'$ and $z' = z$. In the general case, ϵ is chosen to match the width of the source wavelet, and the consequence is that thin-bed multiples will not be attenuated (Weglein et al. 2003). However, we will consider 1D models and spike waves with normal incidence and therefore there is no wavelet to worry about, that is, there is no restriction on the value of ϵ other that it must be small and positive[3]. Also, $k = \frac{2\omega}{c_0}$ is the vertical wavenumber, and $b_1(z)$ is the result of performing Stolt's migration on the data of the model using the water speed, denoted c_0.

We will consider the 1D model shown in Figure 2, where Z_i denotes the depth of the ith reflector for $i = 1, 2, 3$.

[3]In practice, the computational implementation requires a discretization of time. In this case $\epsilon = \frac{c_0 \Delta t}{2}$, where Δt is a time sample interval and usually it has assigned the value of $1ms$. Also, c_0 is the water speed.

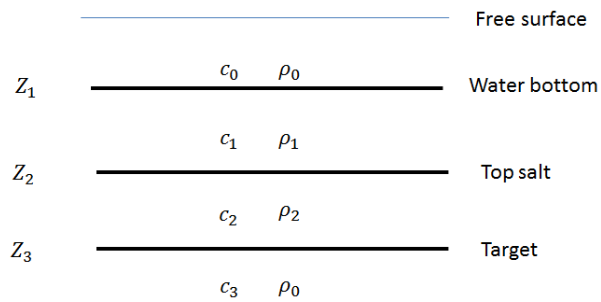

Figure 2: A 1D earth model, with three interfaces. The first interface, with depth Z_1, is the water bottom. The second interface, with depth Z_2, can be identified as the top salt, and the third interface, with depth Z_3, can be identified as the target.

We also consider data composed of primaries and internal multiples, generated by spike waves at normal incidence:

$$D(t) = R_1\delta(t - t_1) + R_2'\delta(t - t_2) + R_3'\delta(t - t_3) + IM, \tag{4}$$

where $R_2' = T_{01}R_2T_{10}$, $R_3' = T_{01}T_{12}R_2T_{21}T_{10}$, and t_i is the travel time of the primary associated with the interface at depth Z_i. Also, R_i is the reflection coefficient experienced by a wave that is reflected upward at the interface at depth Z_i. T_{ij} represents the transmission coefficient experienced by a wave traveling from the acoustic medium that has parameters (c_i, ρ_i) to the acoustic medium that has parameters (c_j, ρ_j).

In this case, the input of the leading-order attenuator, eq. (3), becomes (Appendix A.1):

$$b_1(z) = R_1\delta(z - z_1) + R_2'\delta(z - z_2) + R_3'\delta(z - z_3) + \cdots, \tag{5}$$

where $z_i = \frac{c_0 t_i}{2}$ represents the position of the reflector at depth Z_i, after Stolt's migration. The z_i are usually referred to as *pseudodepths*, and we say that eq. (5) is in the *pseudodepth* domain. Although the input data of the leading-order attenuator, eq. (5), includes primaries and internal multiples, we only consider the effect of the primaries. Initial steps towards the inclusion of internal multiples are addressed in Ma and Weglein (2012) and Liang and Weglein (2012).

According to Appendix A.2, in the time domain the result for the evaluation of eq. (3), using eq. (5), is

$$b_3(t) = -T_{01}T_{10} * (IM)_{j=1} - (T_{01}T_{10})^2 * T_{12}T_{21} * (IM)_{j=2} + \cdots , \quad (6)$$

where $(IM)_{j=1}$ is the sum of all first-order internal multiples with their downward reflection at the first (shallowest) reflector of the model, and $(IM)_{j=2}$ is the first-order internal multiple with its downward reflection at the second interface of the model. The analytic expressions are

$$(IM)_{j=1} = -T_{01}R_2R_1R_2T_{10}\delta(t-(2t_2-t_1))$$

$$-2T_{01}R_2R_1T_{21}R_3T_{12}T_{10}\delta(t-(t_2+t_3-t_1)) - T_{01}T_{12}^2R_3R_1R_3T_{21}^2\delta(t-(2t_3-t_1)). \quad (7)$$

$$(IM)_{j=2} = -T_{01}T_{12}R_3R_2R_3T_{10}T_{21}\delta(t-(2t_3-t_2)). \quad (8)$$

In order to see why $b_3(t)$ is an attenuator of internal multiples, let's add it to the data of the model:

$$b_1(t) + b_3(t) = primaries + [1 - T_{01}T_{10}](IM)_{j=1} + [1 - (T_{01}T_{10})^2 * T_{12}T_{21}] * (IM)_{j=2} + \cdots . \quad (9)$$

As $0 < T_{01}T_{10} < 1$, it becomes evident from (9) that the amplitude contribution of $(IM)_{j=1}$ i.e., the amplitude contribution of the internal multiples generated at the shallowest reflector is reduced by an amount $T_{01}T_{10}$ with respect to the contribution of those multiples prior to the addition of $b_3(z)$. $T_{01}T_{10}$ is referred to as *attenuation factor*.

An analogous situation is present for the internal multiple with its downward reflection at the second reflector. In this case, the amplitude contribution is reduced by an amount of $(T_{01}T_{10})^2 * T_{12}T_{21}$.

Finally, it is convenient to summarize some features of the leading-order attenuator:

- It is completely data-driven, and no subsurface information is required.
- It predicts the exact time and well understood amplitude of all first-order internal multiples.
- It also predicts the exact time and approximate amplitude for internal multiples with converted waves.

3 The leading-order internal multiple eliminator subseries (LOIMES)

In Section 2 we illustrated, using the specific model of Figure 2, how the leading-order attenuator decreases the amplitude contribution for first-order internal multiples with their downward reflection at the shallowest interface, by an amount of $T_{01}T_{10}$. This means that to promote this attenuation to an elimination, the contribution of higher-order terms from the elimination subseries need to move this attenuator factor to the unity: when those higher-order contributions are added to the initial attenuation provided by $b_3(t)$, the predicted amplitude will exactly match $(IM)_{j=1}$. Hence, the collective contribution of the terms in the elimination subseries will remove $(IM)_{j=1}$ from the data.

As the input of the ISS is water-speed migrated data, in order to isolate the terms within the ISS giving the right contributions, we need to express 1 in terms of reflection coefficients, and in particular in terms of R_1. This can be done by the following geometric series expansion:

$$1 = T_{01}T_{10} * \left(\frac{1}{T_{01}T_{10}}\right) = T_{01}T_{10} * \frac{1}{(1-R_1^2)} = T_{01}T_{10} * (1 + R_1^2 + R_1^4 + R_1^6 + R_1^8 + \ldots). \tag{10}$$

Notice that, upon distribution of the product, the first term on the right-hand side of eq. (10) is the initial attenuation provided by the leading-order attenuator. Therefore, the remaining terms are the required amplitude contributions from the higher-order terms, in any subseries claiming to promote the attenuation to elimination. For simplicity, we will focus on isolation of the term within the ISS that provides the next contribution following the leading-order attenuation; i.e., on the isolation of the term whose contribution is $T_{01}T_{10} * R_1^2$ on the right-hand side of eq. (10).

The first step towards the isolation of the second term of the LOIMES from the ISS is to notice that $T_{01}T_{10} * R_1^2$ is the attenuation provided by the leading-order attenuator, $T_{01}T_{10}$, times the square power of R_1. As the prediction for first-order multiples of the leading-order attenuator, eq. (6), is already of third order in the data, the square power of R_1 means that to predict $T_{01}T_{10} * R_1^2 * (IM)_{j=1}$, the second term of the LOIMES should come from a term that is of fifth order in the data. That is, it must be somewhere within the fifth term in the ISS:

$$V_5 = -(V_1G_0V_1G_0V_1G_0V_1G_0V_1 + V_2G_0V_1G_0V_1G_0V_1 + V_1G_0V_2G_0V_1G_0V_1$$

$$+V_1G_0V_1G_0V_2G_0V_1 + V_1G_0V_1G_0V_1G_0V_2 + V_3G_0V_1G_0V_1 + V_1G_0V_3G_0V_1$$

$$+V_1G_0V_1G_0V_3 + V_4G_0V_1 + V_1G_0V_4). \tag{11}$$

The second step towards isolation of the portion of V_5 that contains $T_{01}T_{10} * R_1^2 * (IM)_{j=1}$, is to notice that the selected part should match the exact travel time of the true internal multiple. Using this argument, and upon some inspection of the terms in V_5 provided in Ramírez (2007), it is recognized that the correct term within V_5 should reside in the lower-higher-lower contribution of $V_1G_0V_3G_0V_1$, and in particular the contribution to V_3 coming from $V_1G_0V_1G_0V_1$ needs to be further selected. In other words we are looking for an expression like

$$b_5^{IM}(k) \equiv \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} F[b_1(z')] \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{12}$$

where, as is common for subseries of the ISS, the integrals have been expressed in terms of water-speed migrated data; i.e., in terms of $b_1(z)$. $F[b_1(z')]$ is the portion of $V_1G_0V_1G_0V_1$, expressed in terms of $b_1(z)$, that provides the two extra contributions R_1 we are looking for. As R_1 arises in the data as a result of interactions of the wave with the shallowest interface, to obtain $F[b_1(z')]$ we must split $V_1G_0V_1G_0V_1$ in a way that these interactions become explicit.

On the other hand, after isolating the model-type independent contribution of the term $V_1G_0V_1G_0V_1$, and expressing the result in terms of the water-speed migrated data, we arrive at the following expression:

$$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{13}$$

which is the same term from which the leading-order attenuator is extracted, when we are working with V_3. The next step is to introduce, in order to extract the desired interactions from eq. (13), the same parameter ϵ included in the leading-order attenuator, eq. (3), and then to break the two right integrals in eq. (13) as

$$\int_{-\infty}^{\infty} dz' = \int_{-\infty}^{z-\epsilon} dz' + \int_{z-\epsilon}^{z+\epsilon} dz' + \int_{z+\epsilon}^{\infty} dz'$$

$$\int_{-\infty}^{\infty} dz'' = \int_{-\infty}^{z'-\epsilon} dz'' + \int_{z'-\epsilon}^{z'+\epsilon} dz'' + \int_{z'+\epsilon}^{\infty} dz''. \tag{14}$$

By using eq. (14), we arrive at the following expansion of eq. (13):

$$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{\infty} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{\infty} dz'' e^{ikz''} b_1(z'') =$$

$$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z+\epsilon}^{\infty} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{z'-\epsilon} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z+\epsilon}^{\infty} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{z'-\epsilon} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{-\infty}^{z'-\epsilon} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z+\epsilon}^{\infty} dz' e^{-ikz'} b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} b_1(z'')$$

$$+ \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} b_1(z'') =$$

$$B_{31}(k) + B_{32}(k) + B_{33}(k) + B_{34}(k)$$

$$+B_{35}(k) + B_{36}(k) + B_{37}(k) + B_{38}(k) + B_{39}(k). \tag{15}$$

From (15), we further select the fifth term $B_{35}(k)$, as this is the term containing the interactions with the first reflector: $z'' = z'$ and $z' = z$. In this way we have isolated the interactions and their neighborhood. As this neighborhood is small, we expect we have done enough to reach our goal of elimination of internal multiples of first order with their downward reflection at the shallowest interface. It is interesting that the parameter ϵ is applied in this context to include the self-interactions, rather than to avoid them, as is the case for the leading-order attenuator.

The last step is to define $F[b_1(z)]$ as the inverse Fourier transform of $B_{35}(k)$:

$$F[b_1(z)] = \mathcal{F}^{-1}\left[\int_{-\infty}^{\infty} dze^{ikz}b_1(z)\int_{z-\epsilon}^{z+\epsilon} dz'e^{-ikz'}b_1(z')\int_{z'-\epsilon}^{z'+\epsilon} dz''e^{ikz''}b_1(z'')\right]. \tag{16}$$

In this way, we arrive at the second contribution towards elimination of internal multiples of first order with their downward reflection at the shallowest interface:

$$b_5^{IM}(k) = \int_{-\infty}^{\infty} dze^{ikz}b_1(z)\int_{-\infty}^{z-\epsilon} dz'e^{-ikz'}F[b_1(z')]\int_{z'+\epsilon}^{\infty} dz''e^{ikz''}b_1(z''). \tag{17}$$

In Appendix $A.2$ we show in detail how to perform the integrals in eq. (17), for the same model as the one in Figure 2 in Section 2.2. In the time domain the result is

$$b_5^{IM}(t) = R_1^3(R_2')^2\delta(t-(2t_2-t_1)) + 2R_2'R_1^3R_3'\delta(t-(t_2+t_3-t_1))+$$

$$R_3'R_1^3R_3'\delta(t-(2t_3-t_1)) + (R_2')^3(R_3')^2\delta(t-(2t_3-t_2)), \tag{18}$$

which can be expressed in terms of eqs. (7) and (8) as

$$b_5^{IM}(t) = -T_{01}T_{10} * R_1^2 * (IM)_{j=1} - (T_{01}T_{10})^2 * T_{12}T_{21} * (R_2')^2(IM)_{j=2}. \tag{19}$$

If we now add eq. (19) to the effect of the leading order attenuator; i.e., to eq. (9), we get

$$b_1(t) + b_3(t) + b_5^{IM}(t) = primaries + [1 - T_{01}T_{10}(1+R_1^2)](IM)_{j=1}+$$

$$[1-(T_{01}T_{10})^2 * T_{12}T_{21} * (1+(R_2')^2)](IM)_{j=2} + ... \tag{20}$$

Let's restrict our attention to the amplitude of the internal multiples generated at the shallowest reflector, i.e., to the coefficient of $(IM)_{j=1}$ in eq. (20). In this case the attenuation factor $T_{01}T_{10}$

is changed to $T_{01}T_{10}(1+R_1^2)$. This new contribution contains the first and second terms of the geometric series on the right-hand side of eq. (10). Hence, the integral proposed for b_5^{IM}, eq. (17), correctly reproduces the expected amplitude contribution to take the attenuation of first-order internal multiples with their downward reflection at the shallowest reflector closer to elimination.

To isolate higher-order contributions of the LOIMES, a process analogous to the isolation of $b_5^{IM}(k)$, eq. (12), is necessary. For example, the term following $b_5^{IM}(k)$, denoted as $b_7^{IM}(k)$, will be contained in the seventh term of the ISS. Specifically it will be in $V_1G_0V_5G_0V_1$, from which the part of V_5 corresponding to $V_1G_0V_1G_0V_1G_0V_1G_0V_1$ is further selected, followed by an expansion analogous to eq. (15). The difference is that in this case, there will be four integrals whose intervals of integration need to split. After computing a few higher-order terms, we can write, upon some formal definitions, a compact form for b_{LO}^{IM}:

$$b_{LO}^{IM}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} \times$$

$$\mathcal{F}^{-1}\left(\int_{-\infty}^{\infty} dz' e^{ikz'} b_1(z') \frac{1}{1-\int\int b_1(z')}\right) \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''), \tag{21}$$

where $\mathcal{F}^{-1}$ means the inverse Fourier transform and

$$\frac{1}{1-\int\int b_1(z')} \equiv 1 + \int\int b_1(z') + \left(\int\int b_1(z')\right)^2 + \left(\int\int b_1(z')\right)^3 + \ldots, \tag{22}$$

with

$$\left(\int\int b_1(z')\right)^n \equiv \int_{z'-\epsilon}^{z'+\epsilon} dz_1 e^{-ikz_1} b_1(z_1) \int_{z_1-\epsilon}^{z_1+\epsilon} dz_2 e^{ikz_2} b_1(z_2) \times$$

$$\int_{z_2-\epsilon}^{z_2+\epsilon} dz_3 e^{-ikz_3} b_1(z_3) \int_{z_3-\epsilon}^{z_3+\epsilon} dz_4 e^{ikz_4} b_1(z_4) \cdots \times$$

$$\int_{z_{(2n-2)}-\epsilon}^{z_{(2n-2)}+\epsilon} dz_{(2n-1)} e^{-ikz_{(2n-1)}} b_1(z_{(2n-1)}) \int_{z_{(2n-1)}-\epsilon}^{z_{(2n-1)}+\epsilon} dz_{2n} e^{ikz_{2n}} b_1(z_{2n}), \qquad n>0. \tag{23}$$

$$\left(\int\int b_1(z')\right)^n \equiv 1, \qquad n=0. \tag{24}$$

Finally, it can also be seen from eq. (20) that $b_5^{IM}(t)$ further attenuates the internal multiple of first order generated at the second reflector. However, the LOIMES by itself will not match the amplitude of this event. For that to occur, another subseries needs to be isolated such that, in cooperation with the LOIMES, the elimination takes place. Earlier work on this direction was also reported in Ramírez (2007).

4 Application of the LOIMES to an analytic model

As was mentioned in the introduction, one motivation for the surgical elimination of internal multiples is that in some situations current techniques such as the energy-minimization adaptive subtraction are no longer suitable and attenuation of internal multiples is not enough. An example of such a situation is present when an internal multiple is interfering destructively with a primary. On the other hand, as the LOIMES exactly predicts both the travel time and amplitude of the original internal multiple, it can be considered to be an example of a method for surgical removal of internal multiples, because it does not modify any other event. In this section we will use an analytic model in which an internal multiple of first order is interfering destructively with a primary, and the attenuation provided by the leading-order attenuator is not enough for correct interpretation of the primary. We will use this example to show the usefulness of the LOIMES by surgically removing the internal multiple.

The analytic model is the three-interface model of Figure 2, with the specific values for the parameters shown in Figure 3. We will use the notation P_i for the primary generated at the reflector Z_i. First-order internal multiples are denoted as IM_{ijk}, for $i, j, k = 1, 2, 3$, with j indicating the reflector in which the downward reflection is generated; i and k indicate the reflectors in which the upward reflections are generated.

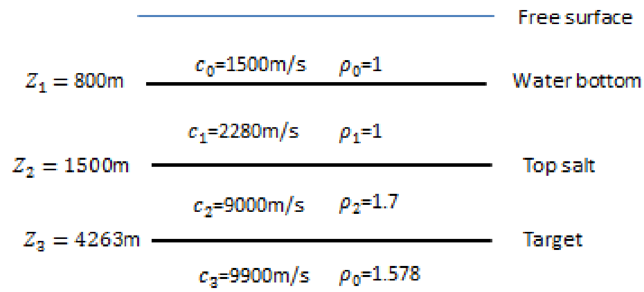

Figure 3: Specific analytic model. This model has the same configuration as that presented in Section 2, with the specified values for the depths, velocities, and densities.

The interfering events are the primary P_3 and the internal multiple IM_{212}, whose common travel time is 2.2947s. The amplitudes for P_3 and IM_{212} are 0.0045 and -0.1084, respectively. A trace is shown in Figure 4, from which the amplitude of the combined event $P_3 + IM_{212}$ can be read as -0.1039: the polarity is opposite that of the primary.

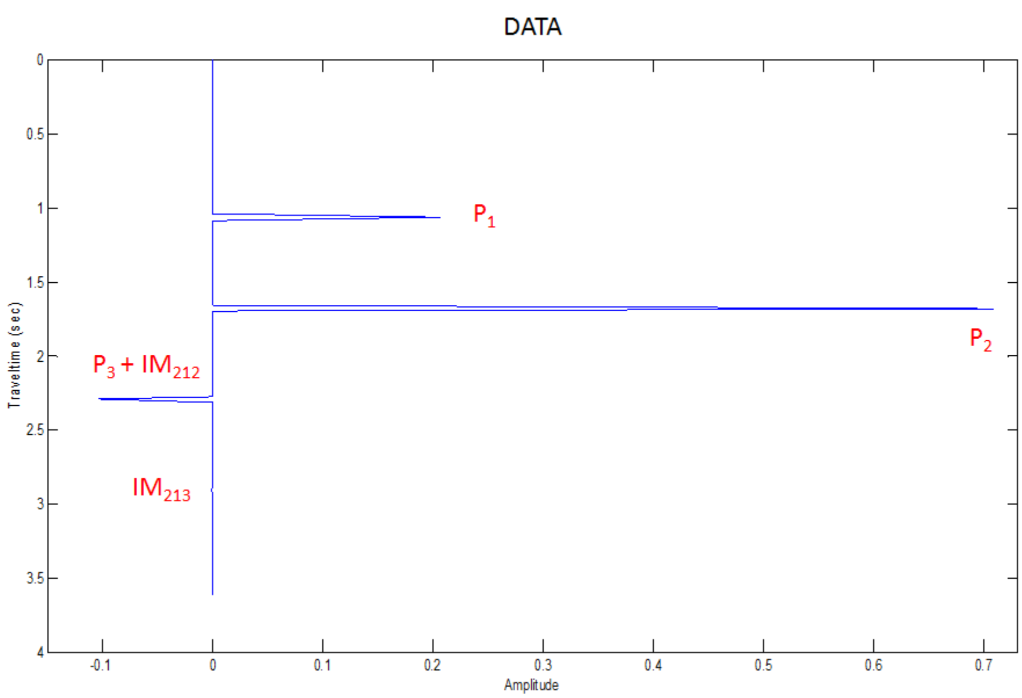

Figure 4: Data of the model. These data include primaries and the relevant internal multiples of first order.

The next step would be the application of $b_3(t)$ to attenuate internal multiples of first order. The result is shown in Figure 5, in which a small time window containing the travel time of the interfering event is shown with an increased scale, in order to make visible the attenuated amplitude. It can also be seen from the right side of Figure 5 that the primaries P_1 and P_2 are not affected at all. From the left side, we can see that the amplitude attenuation is not enough to change the polarity of the interfering event. This might lead to assignment of the incorrect polarity to the primary.

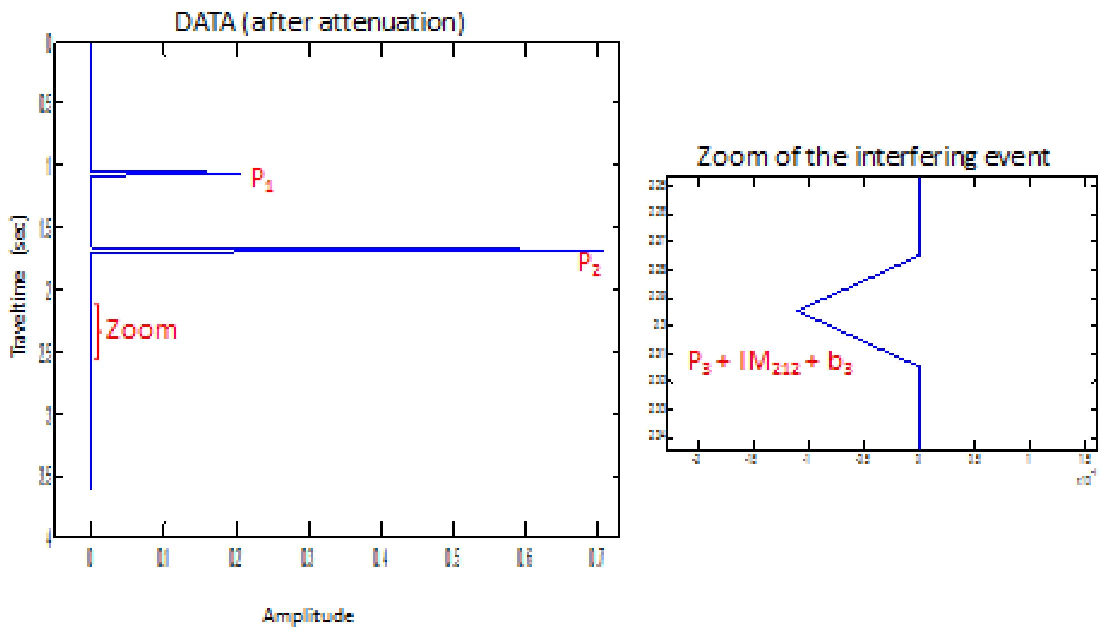

Figure 5: Data after the action of the leading-order attenuator, $b_3(t)$

From the above paragraph it is evident that an improvement in the predicted amplitude for IM_{212} is needed. As was explained in Section 3, this can be done if we include further terms from the LOIMES. This is shown in Figure 6, in which the effect of the second term, $b_5^{IM}(t)$, has been added to that of $b_3(t)$.

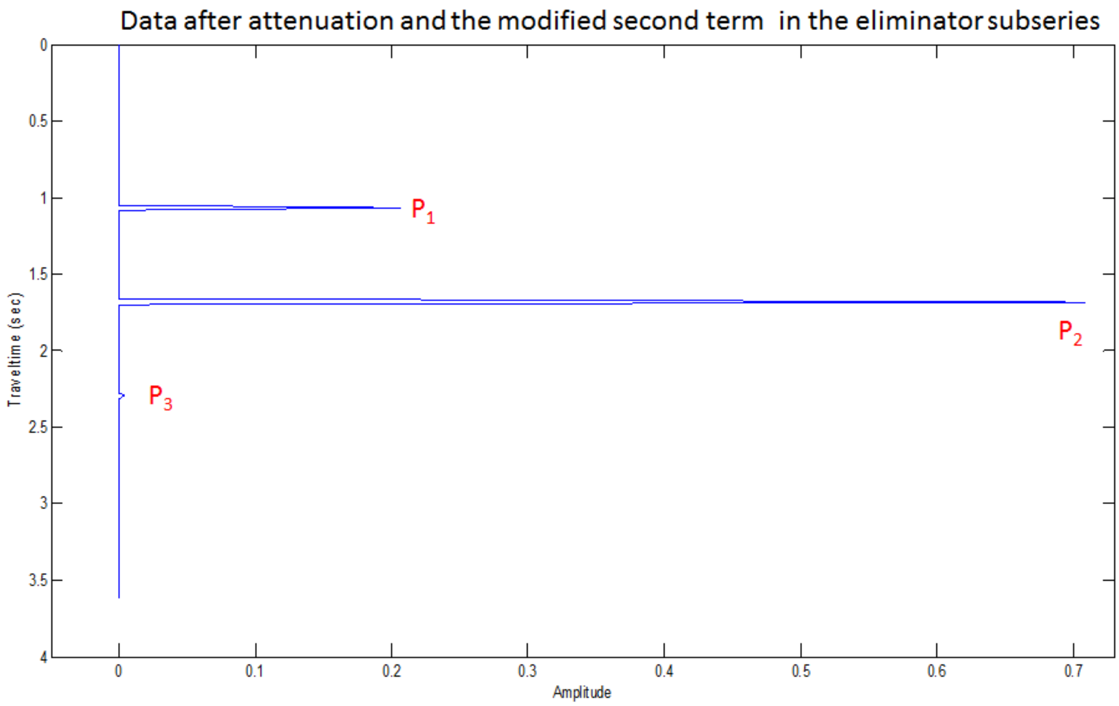

Figure 6: Data after the action of both the leading-order attenuator and $b_5^{IM}(t)$.

In this case, it can be seen that the primary P_3 appears with its original amplitude and polarity, 0.0045, which means that the interfering internal multiple has been removed. This illustrates, at least for the present model, the high rate of convergence of the LOIMES. However, for more complex models the convergence can be slower, and more terms might be needed. Also, from Figure 6, it can be noticed that neither the travel times nor the amplitudes of the primaries P_1 and P_2 are influenced or changed, as expected from a method for surgical removal of internal multiples.

5 Discussion and conclusions

In this work we have isolated a subseries whose task is to eliminate first-order internal multiples with their downward reflections at the shallowest interface. A generic term of this subseries is given by eqs. (21)-(24). This subseries is called the leading-order internal multiple elimination subseries (LOIMES). This elimination subseries predicts the phase and the exact amplitude of the internal multiples and does not modify any primary. Therefore, the surgical removal of such internal multiples is achieved.

We have also applied the LOIMES to an analytic example with three interfaces. The configuration is set up to produce an internal multiple (with downward reflection at the shallowest reflector) interfering destructively with the primary generated at the third reflector, in a way that the leading-order attenuator is not enough to let the primary show up in the data with its correct polarity. We show how the action of the third-order and fifth-order contributions of the algorithm remove the interfering internal multiple, making the primary to appear in the trace with its original amplitude and polarity. In practice however, it is not possible to know a priori the number of terms that are necessary to eliminate the interfering internal multiple. The recipe is to apply to the data one term at a time until no change is noticed in the primary. Although higher-order terms will imply an increased computational cost (more integrals need to be calculated), if the interfering primary is suspected to be the target, then the investment might be worthwhile, as a situation involving a drilling or no drilling decision might be involved and processing costs pale compared to drilling dry holes.

Interfering events are common in onshore exploration, but they may also occur offshore. Therefore, the algorithm in this work may provide added value in those challenging geologic configurations in which techniques such as the energy-minimization adaptive subtraction fails.

So far, we have assumed that the earth is acoustic. It would be interesting to study the properties of the LOIMES, with the assumption of the more realistic situation of an elastic earth, in which the internal multiple can include S-waves.

Further research in this topic includes extending the method beyond the normal incidence assumption of the present work, and to derive the corresponding multidimensional version of the subseries presented here. Additionally, current challenges in exploration seismology might also require the removal of other internal multiples of first-order, generated beneath the shallowest reflector. Hence, a more general research goal is to isolate a subseries, with the specific task of the elimination of first-order internal multiples generated at all reflectors.

So far, we have assumed that the earth is acoustic. It would be interesting to study the properties of the LOIMES, with the assumption of the more realistic situation of an elastic earth, in which the internal multiple can include S-waves.

A Calculation of the leading-order attenuator, $b_3(t)$

Now we will show the key steps involved in calculation of eq. (3). We will use the 1D model with three interfaces shown in Figure 2, with data generated by a spike wave with normal incidence, i.e., when the input is given by eq. (5). We will follow the procedure described in Weglein et al. (2003), in which a 1D model with two interfaces is presented.

A.1 Preparing the input for the Leading-order attenuator

The first task is to obtain $b_1(z)$ from the data of the model, eq. (4), which for convenience is repeated here:

$$D(t) = R_1\delta(t - t_1) + R_2'\delta(t - t_2) + R_3'\delta(t - t_3) + IM,$$

where R_2', R_3' and t_i are as in Section 3.

As it was mentioned in the main body of this work, formally $b_1(z)$ is obtained by Stolt's migration of eq. (4) using the water speed. However the procedure is captured, in this case, by a simple set of rules:

1. Perform a temporal Fourier transform

$$D(\omega) = R_1e^{i\omega t_1} + R_2'e^{i\omega t_2} + R_3'e^{i\omega t_3} + ...$$

2. Define the vertical wavenumber and pseudodepths

$$k = 2\frac{\omega}{c_0} \qquad\qquad z_i = \frac{c_0 t_i}{2}.$$

   Now D can be written as

$$D(k) = R_1e^{ikz_1} + R_2'e^{ikz_2} + R_3'e^{ikz_3} + ...$$

3. Perform a Fourier transform on k and denote the result as $b_1(z)$:

$$b_1(z) \equiv D(z) = R_1\delta(z - z_1) + R_2'\delta(z - z_2) + R_3'\delta(z - z_3) + ...$$

   In the general case, $b_1(z)$ is $D(z)$ times an obliquity factor. In our case, this factor is not needed as we are considering normal incidence of a plane wave; i.e., $b_1(z) \equiv D(z)$. The role of the obliquity factor in more general situations is to produce a plane wave in the Fourier domain (see Weglein et al. 2003).

The equation in item 3 is the input for the leading-order attenuator, eq. (3), and it matches exactly eq. (5) in Section 2.2.

A.2 Explicit calculation of the analytic expression for $b_3(t)$

We will now insert eq. (5) into eq. (3), which for convenience is repeated here:

$$\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z'').$$

We start the evaluation of the above expression with the integral on the right (we only take into account the primaries):

$$\int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z) = \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} [R_1\delta(z''-z_1) + R_2'\delta(z''-z_2) + R_3'\delta(z''-z_2)] =$$

$$\int_{-\infty}^{\infty} dz'' e^{ikz''} [R_1\delta(z''-z_1)H(z''-(z'+\epsilon)) + R_2'\delta(z''-z_2)H(z''-(z'+\epsilon)) +$$

$$R_3'\delta(z''-z_2)H(z''-(z'+\epsilon))] =$$

$$R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) + R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) + R_3' e^{ikz_3} H(z_3-(z'+\epsilon)). \tag{25}$$

As it will be used repeatedly throughout the present and the next appendices, it is worthwhile to say some words about the procedure to go from the second term to the third one in eq. (25). The interval of integration is extended from $z'-\epsilon$ to ∞, but Heaviside functions are introduced at each term of the integrand, with each Heaviside function having the appropriate argument to avoid the modification the original integral.

Substituting eq. (25) into the second integral of eq. (3) , we get

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} [R_1\delta(z'-z_1) + R_2'\delta(z'-z_2) + R_3'\delta(z'-z_3)] \times$$

$$[R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) + R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) + R_3' e^{ikz_3} H(z_3-(z'+\epsilon))]$$

$$= \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_1 e^{ikz_1} H(z_1-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_2' e^{ikz_2} H(z_2-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_3' e^{ikz_3} H(z_3-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_2' \delta(z'-z_2) R_1 e^{ikz_1} H(z_1-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_2' \delta(z'-z_2) R_2' e^{ikz_2} H(z_2-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_2' \delta(z'-z_2) R_3' e^{ikz_3} H(z_3-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_3' \delta(z'-z_3) R_1 e^{ikz_1} H(z_1-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_3' \delta(z'-z_3) R_2' e^{ikz_2} H(z_2-(z'+\epsilon))+$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_3' \delta(z'-z_3) R_3' e^{ikz_3} H(z_3-(z'+\epsilon)) =$$

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9. \tag{26}$$

Evaluating each of the integrals in eq. (26) we get

$$I_1 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) = R_1^4 \underbrace{H(z_1-(z_1+\epsilon))}_{=0} H((z-\epsilon)-z_1) = 0,$$

$$I_2 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) =$$

$$R_1 R_2' e^{ik(z_2 - z_1)} H(z_2 - (z_1 + \epsilon)) H((z - \epsilon) - z_1),$$

$$I_3 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1 \delta(z' - z_1) R_3' e^{ikz_3} H(z_3 - (z' + \epsilon)) =$$

$$R_1 R_3' e^{ik(z_3 - z_1)} H(z_3 - (z_1 + \epsilon)) H((z - \epsilon) - z_1),$$

$$I_4 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_2' \delta(z' - z_2) R_1 e^{ikz_1} H(z_1 - (z' + \epsilon)) =$$

$$R_1 R_2' e^{ik(z_1 - z_2)} \underbrace{H(z_1 - (z_2 + \epsilon))}_{=0} H((z - \epsilon) - z_2) = 0,$$

$$I_5 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_2' \delta(z' - z_2) R_2' e^{ikz_2} H(z_2 - (z' + \epsilon)) =$$

$$R_2' \underbrace{H(z_2 - (z_2 + \epsilon))}_{=0} H((z - \epsilon) - z_2) = 0,$$

$$I_6 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_2' \delta(z' - z_2) R_3' e^{ikz_3} H(z_3 - (z' + \epsilon)) =$$

$$R_2' R_3' e^{ik(z_3 - z_2)} H((z - \epsilon) - z_2),$$

$$I_7 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_3' \delta(z' - z_3) R_1 e^{ikz_1} H(z_1 - (z' + \epsilon)) =$$

$$R_1 R_3' e^{ik(z_1 - z_3)} \underbrace{H(z_1 - (z_3 + \epsilon))}_{=0},$$

$$I_8 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_3' \delta(z' - z_3) R_2' e^{ikz_2} H(z_2 - (z' + \epsilon)) =$$

$$R_2' R_3' e^{ik(z_2 - z_3)} \underbrace{H(z_2 - (z_3 + \epsilon))}_{=0},$$

$$I_9 = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_3' \delta(z' - z_3) R_3' e^{ikz_3} H(z_3 - (z' + \epsilon)) =$$

$$R_3' \underbrace{H(z_3 - (z_3 + \epsilon))}_{=0}.$$

Hence, the result of the second integral in eq. (3) is

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9$$

$$= R_1 R_2' e^{ik(z_2 - z_1)} H((z-\epsilon)-z_1) + R_1 R_3' e^{ik(z_3 - z_1)} H((z-\epsilon)-z_1) + R_2' R_3' e^{ik(z_3 - z_2)} H((z-\epsilon)-z_2). \quad (27)$$

Substituting eq. (27), into the last integral of eq. (3), we finally have

$$b_3(k) = \int_{-\infty}^{\infty} dz e^{ikz} [R_1 \delta(z - z_1) + R_2' \delta(z - z_2) + R_3' \delta(z - z_3)] \times$$

$$[R_1 R_2' e^{ik(z_2 - z_1)} H((z - \epsilon) - z_1) + R_1 R_3' e^{ik(z_3 - z_1)} H((z - \epsilon) - z_1) + (R_2') R_3' e^{ik(z_3 - z_2)} H((z - \epsilon) - z_2)]$$

$$= \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z - z_1) R_1 R_2' e^{ik(z_2 - z_1)} H((z - \epsilon) - z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z - z_1) R_1 R_3' e^{ik(z_3 - z_1)} H((z - \epsilon) - z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z - z_1) R_2' R_3' e^{ik(z_3 - z_2)} H((z - \epsilon) - z_2) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z - z_2) R_1 R_2' e^{ik(z_2 - z_1)} H((z - \epsilon) - z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z - z_2) R_1 R_3' e^{ik(z_3 - z_1)} H((z - \epsilon) - z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_2' R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_2' R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) =$$

$$I_1' + I_2' + I_3' + I_4' + I_5' + I_6' + I_7' + I_8' + I_9'. \tag{28}$$

Evaluating now the integrals in (28), we get

$$I_1' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) =$$

$$R_1^2 R_2' e^{ik(z_2-2z_1)} \underbrace{H((z_1-\epsilon)-z_1)}_{=0},$$

$$I_2' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) =$$

$$R_1^2 R_3' e^{ik(z_3-2z_1)} \underbrace{H((z_1-\epsilon)-z_1)}_{=0},$$

$$I_3' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_2' R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) =$$

$$R_1 R_2' R_3' e^{ik(z_1+z_3-z_2)} \underbrace{H((z_1-\epsilon)-z_2)}_{=0},$$

$$I_4' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1)$$

$$R_1(R_2')^2 e^{ik(2z_2-z_1)},$$

$$I_5' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) =$$

$$R_2' R_1 R_3' e^{ik(z_2+z_3-z_1)},$$

$$I_6' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_2' R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) =$$

$$R_2' R_3' e^{ik(z_3-2z_2)} \underbrace{H((z_2-\epsilon)-z_2)}_{=0},$$

$$I_7' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) =$$

$$R_3' R_1 R_2' e^{ik(z_3+z_2-z_1)},$$

$$I_8' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) =$$

$$R_3' R_1^3 R_3' e^{ik(2z_3-z_1)},$$

$$I_9' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_2' R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) =$$

$$R_2'(R_3')^2 e^{ik(2z_3-z_2)}.$$

With the results above, the sum of the integrals in eq. (28) gives

$$b_3(k) = R_1(R_2')^2 e^{ik(2z_2-z_1)} + 2R_2' R_1 R_3' e^{ik(z_2+z_3-z_1)} +$$

$$= R_1(R_2')^2e^{ik(2z_2-z_1)} + 2R_2'R_1R_3'e^{ik(z_2+z_3-z_1)} + R_3'R_1R_3'e^{ik(2z_3-z_1)} + R_2'(R_3')^2e^{ik(2z_3-z_2)},$$

which in the time domain is expressed as

$$b_3(t) = R_1(R_2')^2\delta(t-(2t_2-t_1)) + 2R_2'R_1R_3'\delta(t-(t_2+t_3-t_1)) + R_3'R_1R_3'\delta(t-(2t_3-t_1)) + R_2'(R_3')^2\delta(t-(2t_3-t_2))$$

$$b_3(t) = -T_{01}T_{10} * (IM)_{j=1} - (T_{01}T_{10})^2 * T_{12}T_{21} * (IM)_{j=2}.$$

The above expression is exactly eq. (6). $(IM)_{j=1}$ and $(IM)_{j=2}$ represent the contributions of the internal multiples (of first order) with their downward reflection originating at the first (shallowest) and second reflectors, respectively. Their analytic expressions are given by eqs. (7) and (8):

$$(IM)_{j=1} = -T_{01}R_2R_1R_2T_{10}\delta(t-(2t_2-t_1))$$

$$-2T_{01}R_2R_1T_{21}R_3T_{12}T_{10}\delta(t-(t_2+t_3-t_1)) - T_{01}T_{12}^2R_3R_1R_3T_{21}^2\delta(t-(2t_3-t_1)).$$

$$(IM)_{j=2} = -T_{01}T_{12}R_3R_2R_3T_{10}T_{21}\delta(t-(2t_3-t_1)).$$

B Explicit calculation of the expression for $b_5^{IM}(t)$

In this appendix we will provide the details of the calculation of $b_5^{IM}(k)$, using the second term of the LOIMES, which is presented here for convenience:

$$b_5^{IM}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} F[b_1(z')] \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z''),$$

where

$$F[b_1(z)] = \mathcal{F}^{-1}\left[\int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z') \int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} b_1(z'')\right].$$

We start with the evaluation of $F[b_1(z)]$. First we insert eq. (5) into the right integral of $F[b_1(z)]$. Then, by extension of the interval of integration and insertion of the convenient Heaviside functions, as in Appendix A, we get

$$\int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} b_1(z'') = \int_{z'-\epsilon}^{z'+\epsilon} dz'' e^{ikz''} [R_1\delta(z''-z_1) + R'_2\delta(z''-z_2) + R'_3\delta(z''-z_3)] =$$

$$R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1) + R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)+$$

$$R'_3 e^{ikz_3} H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3). \tag{29}$$

Substituting eq. (29) into the second integral in $F[b_1(z')]$, we have

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} b_1(z')[R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$$

$$R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2) + R'_3 e^{ikz_3} H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3)]$$

$$= \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} [R_1\delta(z'-z_1) + R'_2\delta(z'-z_2) + R'_3\delta(z'-z_3)]\times$$

$$[R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1) + R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)$$

$$+R'_3 e^{ikz_3} H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3)] =$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1) R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1) R'_2 e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2))+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1\delta(z'-z_1) R'_3 e^{ikz_3} H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3))+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_2' \delta(z'-z_2) R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_2' \delta(z'-z_2) R_2' e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_2' \delta(z'-z_2) R_3' e^{ikz_3} H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3)+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_3' \delta(z'-z_3) R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1)+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_3' \delta(z'-z_3) R_2' e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2)+$$

$$\int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_3' \delta(z'-z_3) R_3' e^{ikz_3} H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3)$$

$$I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9. \tag{30}$$

The integrals in (30) are evaluated as follows:

$$I_1 = \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_1 e^{ikz_1} H(z_1-(z'-\epsilon))H((z'+\epsilon)-z_1) =$$

$$R_1^2 e^{-ikz_1} e^{ikz_1} H(z_1-(z-\epsilon))H((z+\epsilon)-z_1) \underbrace{H(z_1-(z_1-\epsilon))}_{=1} \underbrace{H((z_1+\epsilon)-z_1)}_{=1} =$$

$$R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1),$$

$$I_2 = \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_2' e^{ikz_2} H(z_2-(z'-\epsilon))H((z'+\epsilon)-z_2) =$$

$$R_1 R_2' e^{-ikz_1} e^{ikz_2} H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)H(z_2-(z_1-\epsilon)) \underbrace{H((z_1+\epsilon)-z_2)}_{=0} = 0,$$

$$I_3 = \int_{z-\epsilon}^{z+\epsilon} dz' e^{-ikz'} R_1 \delta(z'-z_1) R_3' e^{ikz_3} H(z_3 - (z'-\epsilon)) H((z'+\epsilon) - z_3) =$$

$$R_1 R_3' e^{-ikz_1} e^{ikz_3} H(z_1 - (z-\epsilon)) H((z+\epsilon) - z_1) H(z_3 - (z_1 - \epsilon)) \underbrace{H((z_1+\epsilon) - z_3)}_{=0} = 0,$$

$$I_4 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R_2' \delta(z'-z_2) R_1 e^{ikz_1} H(z_1 - (z'-\epsilon)) H((z'+\epsilon) - z_1) =$$

$$R_1 R_2' e^{-ikz_2} e^{ikz_1} H(z_2 - (z-\epsilon)) H((z+\epsilon) - z_2) \underbrace{H(z_1 - (z_2 - \epsilon))}_{=0} H((z_2+\epsilon) - z_1) = 0,$$

$$I_5 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R_2' \delta(z'-z_2) R_2' e^{ikz_2} H(z_2 - (z'-\epsilon)) H((z'+\epsilon) - z_2) =$$

$$(R_2')^2 e^{-ikz_2} e^{ikz_2} H(z_2 - (z-\epsilon)) H((z+\epsilon) - z_2) H(z_2 - (z_2 - \epsilon)) H((z_2+\epsilon) - z_2) =$$

$$(R_2')^2 H(z_2 - (z-\epsilon)) H((z+\epsilon) - z_2),$$

$$I_6 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R_2' \delta(z'-z_2) R_3' e^{ikz_3} H(z_3 - (z'-\epsilon)) H((z'+\epsilon) - z_3) =$$

$$R_3' R_2' e^{-ikz_2} e^{ikz_3} H(z_2 - (z-\epsilon)) H((z+\epsilon) - z_2) \underbrace{H(z_3 - (z_2 - \epsilon))}_{=0} H((z_2+\epsilon) - z_3) = 0,$$

$$I_7 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R_3' \delta(z'-z_3) R_1 e^{ikz_1} H(z_1 - (z'-\epsilon)) H((z'+\epsilon) - z_1) =$$

$$R_1 R_3' e^{-ikz_3} e^{ikz_1} H(z_3 - (z-\epsilon)) H((z+\epsilon) - z_3) \underbrace{H(z_1 - (z_3 - \epsilon))}_{=0} H((z_3+\epsilon) - z_1) = 0,$$

$$I_8 = \int_{z-\epsilon}^{z+\epsilon} dz e^{-ikz'} R_3' \delta(z'-z_3) R_2' e^{ikz_2} H(z_2 - (z'-\epsilon)) H((z'+\epsilon) - z_2) =$$

$$R_3'R_2'e^{-ikz_3}e^{ikz_2}H(z_3-(z-\epsilon))H((z+\epsilon)-z_3)\underbrace{H(z_2-(z_3-\epsilon))}_{=0}H((z_3+\epsilon)-z_2)=0,$$

$$I_9=\int_{z-\epsilon}^{z+\epsilon}dze^{-ikz'}R_3'\delta(z'-z_3)R_3'e^{ikz_3}H(z_3-(z'-\epsilon))H((z'+\epsilon)-z_3)=$$

$$(R_3')^2e^{-ikz_3}e^{ikz_3}H(z_3-(z-\epsilon))H((z+\epsilon)-z_3)H(z_3-(z_3-\epsilon))H((z_3+\epsilon)-z_3)=$$

$$(R_3')^2H(z_3-(z-\epsilon))H((z+\epsilon)-z_3). \tag{31}$$

Upon substitution of the integrals just calculated, we get

$$I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8+I_9=$$

$$R_1^2H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)+(R_2')^2H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)+$$

$$(R_3')^2H(z_3-(z-\epsilon))H((z+\epsilon)-z_3). \tag{32}$$

Finally, substituting eq. (32) into the third integral in $F[b_1(z')]$, and using the notation of eq. (15), we end up with

$$\int_{-\infty}^{\infty}dze^{ikz}[R_1\delta(z-z_1)+R_2'\delta(z-z_2)+R_3'\delta(z-z_3)]\times$$

$$[R_1^2H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)+(R_2')^2H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)+$$

$$(R_3')^2H(z_3-(z-\epsilon))H((z+\epsilon)-z_3)]=$$

$$\int_{-\infty}^{\infty}dze^{ikz}R_1\delta(z-z_1)R_1^2H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1)(R_2')^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1)(R_3')^2 H(z_3-(z-\epsilon))H((z+\epsilon)-z_3)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2)(R_2')^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2)(R_3')^2 H(z_3-(z-\epsilon))H((z+\epsilon)-z_3)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3)(R_2')^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3)(R_3')^2 H(z_3-(z-\epsilon))H((z+\epsilon)-z_3)=$$

$$I_1' + I_2' + I_3' + I_4' + I_5' + I_6' + I_7' + I_8' + I_9'. \tag{33}$$

Evaluating the integrals above, we have

$$I_1' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1)=$$

$$R_1^3 e^{ikz_1} H(z_1-(z_1-\epsilon))H((z_1+\epsilon)-z_1) = R_1^3 e^{ikz_1},$$

$$I_2' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1)(R_2')^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2)=$$

$$R_1(R_2')^2 e^{ikz_1} H(z_2-(z_1-\epsilon)) \underbrace{H((z_1+\epsilon)-z_2)}_{=0} = 0,$$

$$I_3' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1)(R_3')^2 H(z_3-(z-\epsilon))H((z+\epsilon)-z_3) =$$

$$R_1(R_3')^2 e^{ikz_1} H(z_3-(z_1-\epsilon)) \underbrace{H((z_1+\epsilon)-z_3)}_{=0} = 0,$$

$$I_4' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1) =$$

$$R_1^2 R_2' e^{ikz_2} \underbrace{H(z_1-(z_2-\epsilon))}_{=0} H((z_2+\epsilon)-z_1) = 0,$$

$$I_5' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2)(R_2')^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2) =$$

$$(R_2')^3 e^{ikz_2} H(z_2-(z_2-\epsilon))H((z_2+\epsilon)-z_2) = (R_2')^3 e^{ikz_2}, \tag{34}$$

$$I_6' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2)(R_3')^2 H(z_3-(z-\epsilon))H((z+\epsilon)-z_3) =$$

$$R_2'(R_3')^2 e^{ikz_2} H(z_3-(z_2-\epsilon)) \underbrace{H((z_2+\epsilon)-z_3)}_{=0} = 0,$$

$$I_7' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1^2 H(z_1-(z-\epsilon))H((z+\epsilon)-z_1) =$$

$$R_1^2 R_3' e^{ikz_3} \underbrace{H(z_1-(z_3-\epsilon))}_{=0} H((z_3+\epsilon)-z_1) = 0,$$

$$I_8' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3)(R_2')^2 H(z_2-(z-\epsilon))H((z+\epsilon)-z_2) =$$

$$(R_2')^2 R_3' e^{ikz_3} \underbrace{H(z_2-(z_3-\epsilon))}_{=0} H((z_2+\epsilon)-z_1)=0,$$

$$I_9' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_2)(R_3')^2 H(z_3-(z-\epsilon))H((z+\epsilon)-z_3) =$$

$$(R_3')^3 e^{ikz_3} H(z_3-(z_3-\epsilon))H((z_3+\epsilon)-z_3) = (R_3')^3 e^{ikz_3}. \tag{35}$$

Adding the integrals above, we finally have

$$B_{35}(k) = R_1^3 e^{ikz_1} + (R_2')^3 e^{ikz_2} + (R_3')^3 e^{ikz_2}, \tag{36}$$

where notation from eq. (15) has been used. When transformed to the pseudodepth domain, eq. (36) becomes

$$F[b_1(z)] = R_1^3 \delta(z-z_1) + (R_2')^3 \delta(z-z_2) + (R_3')^3 \delta(z-z_3). \tag{37}$$

Now we will evaluate $b_5^{IM}(k)$,the second term in b_{LO}^{IM}, using eq. (37):

$$b_5^{IM}(k) = \int_{-\infty}^{\infty} dz e^{ikz} b_1(z) \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} F[b_1(z')] \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z). \tag{38}$$

The 1st integral in the above expression is

$$\int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} b_1(z) = \int_{z'+\epsilon}^{\infty} dz'' e^{ikz''} [R_1 \delta(z''-z_1) + (R_2')\delta(z''-z_2) +$$

$$(R_3')\delta(z''-z_3)] =$$

$$R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) + R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) + R_3' e^{ikz_3} H(z_3-(z'+\epsilon)). \tag{39}$$

Substituting eq. (39) in the second integral of eq. (38), we get

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} [R_1^3 \delta(z'-z_1) + (R_2')^3 \delta(z'-z_2) + (R_3')^3 \delta(z'-z_3)] \times$$

$$[R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) + R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) + R_3' e^{ikz_3} H(z_3-(z'+\epsilon))]$$

$$= \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3 \delta(z'-z_1) R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3 \delta(z'-z_1) R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3 \delta(z'-z_1) R_3' e^{ikz_3} H(z_3-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3 \delta(z'-z_2) R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3 \delta(z'-z_2) R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3 \delta(z'-z_2) R_3' e^{ikz_3} H(z_3-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_3')^3 \delta(z'-z_3) R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_3')^3 \delta(z'-z_3) R_2' e^{ikz_2} H(z_2-(z'+\epsilon)) +$$

$$\int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_3')^3 \delta(z'-z_3) R_3' e^{ikz_3} H(z_3-(z'+\epsilon)) =$$

$$I_1'' + I_2'' + I_3'' + I_4'' + I_5'' + I_6'' + I_7'' + I_8'' + I_9''. \tag{40}$$

Evaluating each of the integrals in eq. (40), we have

$$I_1'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3 \delta(z'-z_1) R_1 e^{ikz_1} H(z_1-(z'+\epsilon)) = R_1^4 \underbrace{H(z_1-(z_1+\epsilon))}_{=0} H((z-\epsilon)-z_1) = 0,$$

$$I_2'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3 \delta(z' - z_1) R_2' e^{ikz_2} H(z_2 - (z' + \epsilon)) =$$

$$R_1^3 R_2' e^{ik(z_2 - z_1)} H(z_2 - (z_1 + \epsilon)) H((z - \epsilon) - z_1),$$

$$I_3'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} R_1^3 \delta(z' - z_1) R_3' e^{ikz_3} H(z_3 - (z' + \epsilon)) =$$

$$R_1^3 R_3' e^{ik(z_3 - z_1)} H(z_3 - (z_1 + \epsilon)) H((z - \epsilon) - z_1),$$

$$I_4'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3 \delta(z' - z_2) R_1 e^{ikz_1} H(z_1 - (z' + \epsilon)) =$$

$$R_1 (R_2')^3 e^{ik(z_1 - z_2)} \underbrace{H(z_1 - (z_2 + \epsilon))}_{=0} H((z - \epsilon) - z_2) = 0,$$

$$I_5'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3 \delta(z' - z_2) R_2' e^{ikz_2} H(z_2 - (z' + \epsilon)) =$$

$$(R_2')^4 \underbrace{H(z_2 - (z_2 + \epsilon))}_{=0} H((z - \epsilon) - z_2) = 0,$$

$$I_6'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_2')^3 \delta(z' - z_2) R_3' e^{ikz_3} H(z_3 - (z' + \epsilon)) =$$

$$(R_2')^3 R_3' e^{ik(z_3 - z_2)} H((z - \epsilon) - z_2),$$

$$I_7'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_3')^3 \delta(z' - z_3) R_1 e^{ikz_1} H(z_1 - (z' + \epsilon)) =$$

$$R_1 (R_3')^3 e^{ik(z_1 - z_3)} \underbrace{H(z_1 - (z_3 + \epsilon))}_{=0},$$

$$I_8'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_3')^3 \delta(z' - z_3) R_2' e^{ikz_2} H(z_2 - (z' + \epsilon)) =$$

$$R_2'(R_3')^3 e^{ik(z_2-z_3)} \underbrace{H(z_2-(z_3+\epsilon))}_{=0},$$

$$I_9'' = \int_{-\infty}^{z-\epsilon} dz' e^{-ikz'} (R_3')^3 \delta(z'-z_3) R_3' e^{ikz_3} H(z_3-(z'+\epsilon)) =$$

$$(R_3')^4 \underbrace{H(z_3-(z_3+\epsilon))}_{=0}.$$

Hence, the value of eq. (40) is

$$I_1'' + I_2'' + I_3'' + I_4'' + I_5'' + I_6'' + I_7'' + I_8'' + I_9''$$

$$= R_1^3 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) + R_1^3 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) + (R_2')^3 R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2). \tag{41}$$

Substituting eq. (41) in the last integral of eq. (37), we finally have

$$b_5^{IM}(k) = \int_{-\infty}^{\infty} dz e^{ikz} [R_1 \delta(z-z_1) + R_2' \delta(z-z_2) + R_3' \delta(z-z_3)] \times$$

$$[R_1^3 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) + R_1^3 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) + (R_2')^3 R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2)] =$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^3 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^3 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) (R_2')^3 R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1^3 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) +$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) R_1^3 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z-z_2) (R_2')^3 R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1^3 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) R_1^3 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1)+$$

$$\int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z-z_3) (R_2')^3 R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) =$$

$$I_1''' + I_2''' + I_3''' + I_4''' + I_5''' + I_6''' + I_7''' + I_8''' + I_9''' \tag{42}$$

The integrals in eq. (42) are calculated as usual:

$$I_1''' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^3 R_2' e^{ik(z_2-z_1)} H((z-\epsilon)-z_1) =$$

$$R_1^4 R_2' e^{ik(z_2-2z_1)} \underbrace{H((z_1-\epsilon)-z_1)}_{=0}.$$

$$I_2''' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) R_1^3 R_3' e^{ik(z_3-z_1)} H((z-\epsilon)-z_1) =$$

$$R_1^4 R_3' e^{ik(z_3-2z_1)} \underbrace{H((z_1-\epsilon)-z_1)}_{=0}.$$

$$I_3''' = \int_{-\infty}^{\infty} dz e^{ikz} R_1 \delta(z-z_1) (R_2')^3 R_3' e^{ik(z_3-z_2)} H((z-\epsilon)-z_2) =$$

$$R_1 (R_2')^3 R_3' e^{ik(z_1+z_3-z_2)} \underbrace{H((z_1-\epsilon)-z_2)}_{=0}.$$

$$I_4''' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z - z_2) R_1^3 R_2' e^{ik(z_2 - z_1)} H((z - \epsilon) - z_1)$$

$$R_1^3 (R_2')^2 e^{ik(2z_2 - z_1)}.$$

$$I_5''' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z - z_2) R_1^3 R_3' e^{ik(z_3 - z_1)} H((z - \epsilon) - z_1) =$$

$$R_2' R_1^3 R_3' e^{ik(z_2 + z_3 - z_1)}.$$

$$I_6''' = \int_{-\infty}^{\infty} dz e^{ikz} R_2' \delta(z - z_2) (R_2')^3 R_3' e^{ik(z_3 - z_2)} H((z - \epsilon) - z_2) =$$

$$(R_2')^3 R_3' e^{ik(z_3 - 2z_2)} \underbrace{H((z_2 - \epsilon) - z_2)} = 0,$$

$$I_7''' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z - z_3) R_1^3 R_2' e^{ik(z_2 - z_1)} H((z - \epsilon) - z_1) =$$

$$R_3' R_1^3 R_2' e^{ik(z_3 + z_2 - z_1)},$$

$$I_8''' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z - z_3) R_1^3 R_3' e^{ik(z_3 - z_1)} H((z - \epsilon) - z_1) =$$

$$R_3' R_1^3 R_3' e^{ik(2z_3 - z_1)},$$

$$I_9''' = \int_{-\infty}^{\infty} dz e^{ikz} R_3' \delta(z - z_3) (R_2')^3 R_3' e^{ik(z_3 - z_2)} H((z - \epsilon) - z_2) =$$

$$(R_2')^3 (R_3')^2 e^{ik(2z_3 - z_2)}.$$

The sum of the integrals above gives

$$b_5^{IM} = R_1^3(R_2')^2 e^{ik(2z_2-z_1)} + 2R_2'R_1^3R_3' e^{ik(z_2+z_3-z_1)} +$$

$$R_3'R_1^3R_3' e^{ik(2z_3-z_1)} + (R_2')^3(R_3')^2 e^{ik(2z_3-z_2)}. \tag{43}$$

Upon Fourier transformation, eq. (43) becomes:

$$b_5^{IM}(t) = R_1^3(R_2')^2\delta(t-(2t_2-t_1)) + 2R_2'R_1^3R_3'\delta(t-(t_2+t_3-t_1)) +$$

$$R_3'R_1^3R_3'\delta(t-(2t_3-t_1)) + (R_2')^3(R_3')^2\delta(t-(2t_3-t_2)),$$

which is exactly eq. (18) in Section 3.

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