Calculation and imaging of the non-linear 2D wavefield at depth in terms of the data and without any assumptions about the medium

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Abstract

We show the steps involved in the calculation and imaging of the second order wavefield at depth using the inverse scattering series. In the calculations we employ only the part of the perturbation operator which does not depend on a medium's model type. The calculations and result only require the recorded data and the Green's function of the homogeneous background without any a priori assumptions on the medium that's being investigated.

1 Introduction

Inverse scattering series provides the opportunity to determine a multidimensional unknown medium directly from the measured data without making any intermediate determinations of, or assumptions on, the medium under investigation. The inversion process can be thought of as a sequence of independent tasks (1) free surface multiple removal (2) internal multiple removal (3) imaging the reflectors at depth and (4) identifying the medium properties changing across the reflectors. Each task can be associated with a subseries of the full inverse scattering series which only provides the respective capability without affecting the other tasks. For a description of the logic and the history of the subseries method see Weglein et al. (2003).

For the first two tasks, free surface and internal multiple elimination, model type independent subseries and algorithms have been found and applied extensively in the oil industry (see e.g. Weglein et al. (1997) and Weglein et al. (2003) and references therein). For the third task of imaging the reflectors at depth, algorithms have been found and tested for 1D and multi-D acoustic media (Shaw, 2001; Weglein et al., 2001; Shaw, 2002; Shaw and Weglein, 2003; Shaw et al., 2003; Shaw, 2003; Weglein et al., 2003; Shaw et al., 2004; Shaw, 2005; Liu et al., 2005, 2006, 2007).

This research investigates a possible model type independent methodology to calculate and image the wavefield at depth from the inverse scattering series using only recorded data. Roughly speaking, the method uses the calculated and task specific separated perturbation operator V, in the forward scattering series, to calculate different orders of the scattered field at any depth. The subsequent imaging of this wavefield at depth is performed similarly to the imaging step in $f-k$ depth migration algorithms. For an acoustic medium, the method was first described in Weglein et al. (2000) where the first order wavefield at depth was calculated. In this paper we investigate the possibility of performing these calculations in a model type independent environment for the first and the second orders wavefield at depth.

One important conclusions that comes out of this calculations is that, when the actual medium is unknown, calculating the wavefield at depth for one frequency requires all frequencies in the wavefield on the measurement surface (data). This is fundamentally different from migration algorithms, which assume the medium is known, and which can extrapolate, at depth, each frequency individually (by performing a phase-shift for example). This was originally noted in Weglein et al. (2000) for the first order of the acoustic wavefield at depth calculated using the inverse scattering series. Here we discover the same characteristic for the wavefield at depth without a specified model type for both first and second order.

The paper starts with the necessary background, definitions and description of the method, in Section 2, and then proceeds with the calculation of the first and second orders of the wavefield at depth in Sections 3 and 4 respectively. Section 5 shows how this part of the field can be imaged. We end the paper with conclusions and discussion of future research directions. Throughout the paper we use the following conventions for Fourier transforming over the space and time coordinates. For the Fourier transform over the horizontal variable x, we are going to use the different sign convention for the transformation over the source and receiver coordinates. Accordingly, the forward Fourier transform of a real function f over the horizontal source coordinate x_s is going to be

$$f(k_{xs}) = \int_{-\infty}^{\infty} f(x_s) e^{ik_{xs}x_s} dx_s, \tag{1}$$

where k_{xs} is the associated horizontal wavenumber. The forward Fourier transform of f over the horizontal receiver coordinate x_g is going to be

$$f(k_{xg}) = \int_{-\infty}^{\infty} f(x_g) e^{-ik_{xg}x_g} dx_g, \tag{2}$$

where k_{xg} is, same as before, the associated horizontal wavenumber. The corresponding inverse Fourier transforms are

$$f(x_s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k_{xs}) e^{-ik_{xs}x_s} dk_{xs} \tag{3}$$

and

$$f(x_g) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(k_{xg}) e^{ik_{xg}x_g} dk_{xg} \tag{4}$$

respectively. There will be no such distinction for the vertical coordinates/wavenumbers. The Fourier transform of, say, $f(z)$ will be

$$f(q) = \int_{-\infty}^{\infty} f(z) e^{iqz} dz \tag{5}$$

and the inverse Fourier transform of $f(q)$ will be

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(q) e^{-iqz} dq \tag{6}$$

The forward Fourier transform over the time coordinate t is

$$f(\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt, \tag{7}$$

where ω is the temporal frequency. Its corresponding inverse Fourier transform will be given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega. \tag{8}$$

2 Background

In operator form, the differential equations describing wave propagation in an actual and a reference medium can be written as

$$\mathbf{LG} = -\mathbf{I} \tag{9}$$

and

$$\mathbf{L}_0\mathbf{G}_0 = -\mathbf{I}, \tag{10}$$

where $\mathbf{L}$, $\mathbf{L}_0$ and $\mathbf{G}$, $\mathbf{G}_0$ are the actual and reference differential and Green's operators, respectively, for a single temporal frequency and $\mathbf{I}$ is the identity operator. The above equations (9) and (10) assume that the source and receiver signatures have been deconvolved. The perturbation, $\mathbf{V}$, and the scattered field operator, ψ_s, are defined as

$$\mathbf{V} = \mathbf{L} - \mathbf{L}_0, \tag{11}$$

$$\psi_{\mathbf{s}} = \mathbf{G} - \mathbf{G}_0. \tag{12}$$

The fundamental equation of scattering theory, the Lippmann–Schwinger equation, relates ψ_s, $\mathbf{G}_0$, $\mathbf{V}$, and $\mathbf{G}$ (see, e.g., Taylor (1972)):

$$\psi_s = \mathbf{G} - \mathbf{G}_0 = \mathbf{G}_0\mathbf{VG}. \tag{13}$$

The Lippmann-Schwinger equation (13) is valid everywhere, inside or outside the support of $\mathbf{V}$. Expressions for $\mathbf{L}$, $\mathbf{L}_0$ and $\mathbf{V}$, in the case of a pressure wavefield propagating in inhomogeneous acoustic and elastic media, have been given in Clayton and Stolt (1981) and Stolt and Weglein (1985). Equation (13) can be expanded in an infinite series by substituting $\mathbf{G} = \mathbf{G}_0 - \mathbf{G}_0\mathbf{VG}$ into the right-hand side repeatedly to obtain

$$\begin{aligned} \psi_s &= \mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G} \qquad (14)\\ \psi_s &= \mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}\\ &\vdots \end{aligned}$$

and so on. By repeating this process an infinite number of times we imagine that we can drop the last term containing the Green's function of the actual medium, $\mathbf{G}$, in favor of an infinite series, and write the scattered field as

$$\psi_s \equiv \mathbf{G} - \mathbf{G}_0 = \mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \cdots \quad . \tag{15}$$

When convergent (see e.g. Matson (1996) and Nita et al. (2004)), this series, the forward scattering series, constructs the scattered field operator ψ_s, everywhere inside or outside the medium, as a sum of terms representing propagations in the reference medium ($\mathbf{G}_0$) and interactions with the inhomogeneity represented by the perturbation operator $\mathbf{V}$. For example, one could use this expression to calculate the reflected or transmitted response of the medium everywhere. The data recorded in a seismic experiment is usually considered to be the scattered field on the measurement surface

$$(\psi_s)_{MS} = \mathbf{D} \tag{16}$$

Next we consider the expansion of the perturbation V and the scattered field ψ_s as a series in orders of the data $\mathbf{D}$ and write

$$\mathbf{V} = \mathbf{V_1} + \mathbf{V_2} + \mathbf{V_3} + \ldots \tag{17}$$

and

$$\psi_s = \psi_s^1 + \psi_s^2 + \psi_s^3 + \ldots \tag{18}$$

respectively, where V_i and ψ_s^i are terms of order i in the data $\mathbf{D}$. Notice, for example, that, on the measurement surface, we have

$$\begin{aligned}(\psi_s^1)_{MS} &= \mathbf{D} \\ (\psi_s^i)_{MS} &= 0, \;\; i \geq 2.\end{aligned} \tag{19}$$

Plugging the series in (17) and (18) into the forward scattering series (15) we find

$$\begin{aligned}\psi_s^1 + \psi_s^2 + \psi_s^3 + \ldots = \; & \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_3}\mathbf{G}_0 \ldots \\ & + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 + \ldots \\ & + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 + \ldots \\ & + \ldots \; .\end{aligned} \tag{20}$$

Equating like orders in the data in the equation above we find

$$\psi_s^1 = \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 \tag{21}$$
$$\psi_s^2 = \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 \tag{22}$$
$$\psi_s^3 = \mathbf{G}_0\mathbf{V_3}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 \tag{23}$$
$$\vdots \, .$$

On the measurements surface, and because of (19), equations (21)-(23) provide an algorithm for computing V_i, $i \geq 1$

$$D = (\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0)_{ms} \tag{24}$$
$$0 = (\mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0)_{ms} \tag{25}$$
$$0 = (\mathbf{G}_0\mathbf{V_3}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0)_{ms} \tag{26}$$
$$\vdots \, .$$

One can then calculate $\mathbf{V}$ as a series $\mathbf{V} = \mathbf{V_1} + \mathbf{V_2} + \mathbf{V_3} + ...$ This approach and task specific subseries associated with the series for $\mathbf{V}$ were studied (see e.g. Weglein et al. (2003) and references therein) and continue to be studied. Most importantly, the subseries method provided model type independent algorithms for free surface and internal multiple elimination. Subseries for moving reflectors at the correct depth have been found for the acoustic case (Shaw (2005), Liu et al. (2006)) and research efforts are under way to generalize the subseries to a model type independent algorithm (Ramírez et al. (2007)).

However equations (21)-(23) also provide a different method for computing the wavefield at depth in two steps (see also Weglein et al. (2000) and Weglein et al. (2006)). In the first step we restrict these equations to the measurement surface to obtain (24)-(26). From these equations we can calculate $\mathbf{V_i}$, $i \geq 1$. In a second step we use the fully unrestricted equations (21)-(23), now with known $\mathbf{V_i}$'s, to calculate ψ_s^i, $i \geq 1$, at any depth. The connection between the two steps is realized through the perturbation operator $\mathbf{V}$, a quantity which only depends on the actual medium (for a fixed reference medium) and does not change from one equation to another. We emphasize that this calculation of the wavefield at depth is performed starting with the data and a reference medium and does not require any features of the actual medium. There are several important features of this calculation that will be evident in the following section. First, in the calculation of $\mathbf{V_i}$, only the model type independent part (as described in Weglein et al. (2003), Ramírez et al. (2007)) is retained and the part that depends on the medium properties is ignored. Second, since what we are trying to achieve is the extrapolation of the wavefield at depth without any change in amplitude, we only use the terms in the series that correct for the reflectors mislocation, as found in Shaw (2005).

In the following sections we calculate the first and second orders of the wavefield at depth in terms of the data and the reference medium only.

3 The calculation of the first order wavefield at depth ψ_s^1

We start with equation (21)

$$\psi_s^1 = \mathbf{G}_0 \mathbf{V_1} \mathbf{G}_0 \tag{27}$$

or, in a coordinate system,

$$\psi_s^1(x_1, z_1, x_2, z_2; \omega_1) = \int dx' dx'' dz' dz'' G_0(x_1, z_1, x', z'; \omega_1) V_1(x', z', x'', z'', \omega_1) G_0(x'', z'', x_2, z_2; \omega_1) \tag{28}$$

where (see Appendix 6)

$$G_0(x_1, z_1, x', z', \omega_1) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_g \int_{-\infty}^{\infty} dq_1 \frac{e^{ik_g(x_1 - x')} e^{iq_1(z' - z_1)}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \tag{29}$$

and

$$G_0(x'', z'', x_2, z_2, \omega_1) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_s \int_{-\infty}^{\infty} dq_2 \frac{e^{ik_s(x'' - x_2)} e^{iq_2(z'' - z_2)}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \tag{30}$$

and where x_1, z_1, x_2 and z_2 are arbitrary coordinates. Notice that in the equations for the Green's functions given above we have associated the space variables x_1, z_1, x_2 and z_2 with the wavenumbers k_g, q_1, k_s, q_2 satisfying the dispersion relations

$$k_g^2 + q_1^2 = \frac{\omega_1^2}{c_0^2}, \quad k_s^2 + q_2^2 = \frac{\omega_1^2}{c_0^2}. \tag{31}$$

Fourier transforming equation (28) over all space variables (i.e. applying on both sides the integral operators $\int_{-\infty}^{\infty} dx_1 e^{-ik'_g x_1}$, $\int_{-\infty}^{\infty} dx_2 e^{ik'_s x_2}$, $\int_{-\infty}^{\infty} dz_1 e^{iq'_1 z_1}$ and $\int_{-\infty}^{\infty} dz_2 e^{iq'_2 z_2}$) we find

$$\begin{aligned}\psi_s^1(k'_g, q'_1, k'_s, q'_2; \omega_1) = &\left(\frac{1}{2\pi}\right)^4 \int dx' dx'' dz' dz'' \int_{-\infty}^{\infty} dk_g \int_{-\infty}^{\infty} dx_1 e^{ix_1(k_g - k'_g)} \int_{-\infty}^{\infty} dk_s \int_{-\infty}^{\infty} dx_1 e^{-ix_2(k_s - k'_s)} \\ &\times \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dz_1 e^{iz_1(q'_1 - q_1)} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dz_2 e^{iz_2(q'_2 - q_2)} \\ &\times \frac{e^{-ix' k_g} e^{ix'' k_s}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x', z', x'', z'', \omega_1) \frac{e^{iz' q_1} e^{iz'' q_2}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}\end{aligned} \tag{32}$$

which becomes

$$\begin{aligned}\psi_s^1(k'_g, q'_1, k'_s, q'_2; \omega_1) = &\left(\frac{1}{2\pi}\right)^4 \int dx' dx'' dz' dz'' \int_{-\infty}^{\infty} dk_g \delta(k_g - k'_g) \int_{-\infty}^{\infty} dk_s \delta(k_s - k'_s) \\ &\times \int_{-\infty}^{\infty} dq_1 \delta(q'_1 - q_1) \int_{-\infty}^{\infty} dq_2 \delta(q'_2 - q_2) \\ &\times \frac{e^{-ix' k_g} e^{ix'' k_s}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x', z', x'', z'', \omega_1) \frac{e^{iz' q_1} e^{iz'' q_2}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}\end{aligned} \tag{33}$$

or, after simplifying the delta functions and eliminating the primed notation from the wavenumbers,

$$\psi_s^1(k_g, q_1, k_s, q_2; \omega_1) = \int dx' dx'' dz' dz'' \frac{e^{-ix' k_g} e^{ix'' k_s}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x', z', x'', z'', \omega_1) \frac{e^{iz' q_1} e^{iz'' q_2}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}. \tag{34}$$

The last four integrals can also be regarded as Fourier transforms over x', x'', z' and z'' so that the last expression can be written as

$$\psi_s^1(k_g, q_1, k_s, q_2; \omega_1) = \frac{1}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(k_g, -q_1, k_s, -q_2, \omega_1) \frac{1}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}. \tag{35}$$

From Appendix 6 equation (146) we have that

$$V_1(k_g, -q_g, -k_s, -q_s, \omega) = -4 q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, \omega) \tag{36}$$

where we emphasize that while the horizontal wavenumbers are the same in the two equations, the vertical wavenumbers are different and are related to the horizontal ones through different frequencies ω and ω_1. To avoid confusions (see also Appendix 6) we rewrite the last equation as

$$V_1(k_g, -q_g, -k_s, -q_s, \omega) = -4 q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, q_g + q_s). \tag{37}$$

It is important to notice that the V_1 on the left is the 3-dimensional projection of the fully 5-dimensional V_1 operator (so chosen by ignoring the P.V. part of the Green's function, see Appendix 6). The independent variables on the left are k_g, k_s and $q_g + q_s$. With this in mind we write

$$V_1(k_g, -q_1, -k_s, -q_2, \omega_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(-q_g - q_s)\delta(q_g + q_s - q_1 - q_2)V_1(k_g, -q_g, -k_s, -q_s, \omega) \quad (38)$$

or

$$V_1(k_g, -q_1, -k_s, -q_2, \omega_1) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} d(-q_g - q_s)\delta(q_g + q_s - q_1 - q_2)q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, \omega). \quad (39)$$

Equation (39) leads to a relationship between the sums of the vertical wavenumbers,

$$q_g + q_s = q_1 + q_2, \quad (40)$$

which, in turn, allows us to calculate ω_1 in terms of ω as (see Appendix 6 equation (152)) as

$$\frac{\omega_1^2}{c_0^2} = \frac{\left[(q_g + q_s)^2 + k_g^2 + k_s^2\right]^2 - 4k_g^2 k_s^2}{4(q_g + q_s)^2}. \quad (41)$$

With this particular ω_1, equation (39) for V_1 becomes

$$V_1(k_g, -q_1, k_s, -q_2, \omega_1) = -4q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, \omega) \quad (42)$$

so the final expression for ψ_s^1

$$\psi_s^1(k_g, q_1, k_s, q_2; \omega_1) = \frac{-4q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, \omega)}{\left(k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon\right)\left(k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon\right)} \quad (43)$$

in which the variables are related by the dispersion relationships

$$k_g^2 + q_1^2 = \frac{\omega_1^2}{c_0^2}, \quad k_s^2 + q_2^2 = \frac{\omega_1^2}{c_0^2} \quad (44)$$

$$k_g^2 + q_g^2 = \frac{\omega^2}{c_0^2}, \quad k_s^2 + q_s^2 = \frac{\omega^2}{c_0^2} \quad (45)$$

and

$$\frac{\omega_1^2}{c_0^2} = \frac{\left[(q_g + q_s)^2 + k_g^2 + k_s^2\right]^2 - 4k_g^2 k_s^2}{4(q_g + q_s)^2}. \quad (46)$$

For the acoustic case, equation (43) was obtained in Weglein et al. (2000) and it was discussed in Weglein et al. (2006). In the next section we calculate the second order wavefield at depth, ψ_s^2.

4 The calculation of the second order wavefield at depth ψ_s^2

We start with equation (22)

$$\psi_s^2 = \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0. \tag{47}$$

For convenience, we will denote

$$\psi_s^{21} = \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 \tag{48}$$

$$\psi_s^{22} = \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0. \tag{49}$$

4.1 The calculation of ψ_s^{21}

We start with

$$\psi_s^{21} = \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 \tag{50}$$

or, with coordinates,

$$\psi_s^{21}(x_1, z_1, x_2, z_2; \omega_1) = \int dx' dx'' dz' dz'' G_0(x_1, z_1, x', z'; \omega_1) V_1(x', z', x'', z'', \omega_1)$$
$$\times \int dx''' dx^{iv} dz''' dz^{iv} G_0(x'', z'', x''', z'''; \omega_1) V_1(x''', z''', x^{iv}, z^{iv}, \omega_1) G_0(x^{iv}, z^{iv}, x_s, z_s; \omega_1) \tag{51}$$

where x_1, z_1, x_2 and z_2 are arbitrary coordinates and, as before, the Green's functions have the expressions (see Appendix 6)

$$G_0(x_1, z_1, x', z', \omega_1) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_g \int_{-\infty}^{\infty} dq_1 \frac{e^{ik_g(x_1-x')} e^{iq_1(z'-z_1)}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}, \tag{52}$$

$$G_0(x^{iv}, z^{iv}, x_2, z_2, \omega_1) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk_s \int_{-\infty}^{\infty} dq_2 \frac{e^{ik_s(x^{iv}-x_2)} e^{iq_2(z^{iv}-z_2)}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}. \tag{53}$$

and

$$G_0(x'', z'', x''', z''', \omega_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_{\lambda_1} \frac{e^{ik_{\lambda_1}(x''-x''')} e^{iq_{\lambda_1}|z''-z'''|}}{2iq_{\lambda_1}}. \tag{54}$$

where $q_{\lambda_1} = \sqrt{\frac{\omega_1^2}{c_0^2} - k_{\lambda_1}^2}$. In the expressions of the Green's functions above we have associated the spatial variables x_1, z_1, x_2 and z_2 with the wavenumbers k_g, q_1, k_s and q_2 respectively. The wavenumbers satisfy the dispersion relations

$$k_g^2 + q_1^2 = \frac{\omega_1^2}{c_0^2}, \quad k_s^2 + q_2^2 = \frac{\omega_1^2}{c_0^2}, \quad k_{\lambda_1}^2 + q_{\lambda_1}^2 = \frac{\omega_1^2}{c_0^2}. \tag{55}$$

Fourier transforming equation (51) over all spatial arguments of ψ_s^{21} (i.e. applying on both sides the integral operators $\int_{-\infty}^{\infty} dx_1 e^{-ik'_g x_1}$, $\int_{-\infty}^{\infty} dx_2 e^{ik'_s x_2}$, $\int_{-\infty}^{\infty} dz_1 e^{iq'_1 z_1}$ and $\int_{-\infty}^{\infty} dz_2 e^{iq'_2 z_2}$) we find

$$\psi_s^{21}(k'_g, q'_1, k'_s, q'_2; \omega_1) = \left(\frac{1}{2\pi}\right)^5 \int dk_{\lambda_1} \frac{1}{2iq_{\lambda_1}} \int dx' dx'' dz' dz'' dx''' dx^{iv} dz''' dz^{iv}$$
$$\int dk_g \int dx_1 e^{ix_1(k_g - k'_g)} \int dk_s \int dx_1 e^{-ix_2(k_s - k'_s)} \int dq_1 \int dz_1 e^{iz_1(q'_1 - q_1)} \int dq_2 \int dz_2 e^{iz_2(q'_2 - q_2)}$$
$$\times \frac{e^{-ix' k_g} e^{ix'' k_{\lambda_1}}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x', z', x'', z'', \omega_1) \frac{e^{-ix''' k_{\lambda_1}} e^{ix^{iv} k_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x''', z''', x^{iv}, z^{iv}, \omega_1) e^{iz' q_1} e^{iq_{\lambda_1}|z'' - z'''|} e^{iz^{iv} q_2} \tag{56}$$

which becomes

$$\psi_s^{21}(k'_g, q'_1, k'_s, q'_2; \omega_1) = \left(\frac{1}{2\pi}\right)^5 \int dk_{\lambda_1} \frac{1}{2iq_{\lambda_1}} \int dx' dx'' dz' dz'' dx''' dx^{iv} dz''' dz^{iv} \int dk_g \delta(k_g - k'_g)$$
$$\times \int dk_s \delta(k_s - k'_s) \int dq_1 \delta(q'_1 - q_1) \int dq_2 \delta(q'_2 - q_2) V_1(x', z', x'', z'', \omega_1)$$
$$\times \frac{e^{-ix' k_g} e^{ix'' k_{\lambda_1}}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{-ix''' k_{\lambda_1}} e^{ix^{iv} k_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x''', z''', x^{iv}, z^{iv}, \omega_1) e^{iz' q_1} e^{iq_{\lambda_1}|z'' - z'''|} e^{iz^{iv} q_2} \tag{57}$$

or, after simplifying the delta functions and eliminating the primed notation on the wavenumbers,

$$\psi_s^{21}(k_g, q_1, k_s, q_2; \omega_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_{\lambda_1} \frac{1}{2iq_{\lambda_1}} \int dx' dx'' dz' dz'' dx''' dx^{iv} dz''' dz^{iv} e^{iz' q_1} e^{iq_{\lambda_1}|z'' - z'''|} e^{iz^{iv} q_2}$$
$$\times \frac{e^{-ix' k_g} e^{ix'' k_{\lambda_1}}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{-ix''' k_{\lambda_1}} e^{ix^{iv} k_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_1(x', z', x'', z'', \omega_1) V_1(x''', z''', x^{iv}, z^{iv}, \omega_1). \tag{58}$$

The integrals over x', x'', x''', x^{iv}, z' and z^{iv} are Fourier transform so we can further simplify into

$$\psi_s^{21}(k_g, q_1, k_s, q_2; \omega_1) = \frac{1}{4\pi i} \frac{1}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{1}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \int dk_{\lambda_1} \frac{1}{q_{\lambda_1}} \int dz''$$
$$\times V_1(k_g, -q_1, -k_{\lambda_1}, z'', \omega_1) \int dz''' e^{iq_{\lambda_1}|z'' - z'''|} V_1(k_{\lambda_1}, z''', -k_s, -q_2, \omega_1). \tag{59}$$

Next we use the Heaviside step function H to express the absolute values and write

$$e^{iq_{\lambda_1}|z'' - z'|} = e^{iq_{\lambda_1}(z' - z'')} H(z' - z'') + e^{iq_{\lambda_1}(z'' - z')} H(z'' - z'). \tag{60}$$

Moreover we use the integral representation of H (reference)

$$H(z) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \frac{1}{i(p - i\epsilon)} e^{-ipz}. \tag{61}$$

With this, the expression of ψ_s^{21} becomes

$$\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1)=\frac{1}{4\pi i}\frac{1}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{1}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int dk_{\lambda_1}\frac{1}{q_{\lambda_1}}\int dz''V_1(k_g,-q_1,-k_{\lambda_1},z'',\omega_1)$$
$$\times\left[\int dz'''e^{iq_{\lambda_1}(z''-z''')}\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\frac{1}{i(p-i\epsilon)}e^{-ip(z''-z''')}V_1(k_{\lambda_1},z''',-k_s,-q_2,\omega_1)\right.$$
$$\left.+\int dz'''e^{iq_{\lambda_1}(z'''-z'')}\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty}dp\frac{1}{i(p-i\epsilon)}e^{-ip(z'''-z'')}V_1(k_{\lambda_1},z''',-k_s,-q_2,\omega_1)\right] \tag{62}$$

Rearranging the order of integration and solving the Fourier transforms over dz'' and dz''' we find

$$\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1)=\frac{1}{8\pi^2 i}\frac{1}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{1}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty}dk_{\lambda_1}\frac{1}{q_{\lambda_1}}$$
$$\times\left[\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}dp\frac{V_1(k_g,-q_1,-k_{\lambda_1},-q_{\lambda_1}+p,\omega_1)V_1(k_{\lambda_1},q_{\lambda_1}-p,-k_s,-q_2,\omega_1)}{i(p-i\epsilon)}\right.$$
$$\left.+\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}dp\frac{V_1(k_g,-q_1,-k_{\lambda_1},q_{\lambda_1}-p,\omega_1)V_1(k_{\lambda_1},-q_{\lambda_1}+p,-k_s,-q_2,\omega_1)}{i(p-i\epsilon)}\right] \tag{63}$$

The two dp integrals can be separated into a principal value and a contribution from contour integrals around the pole $p=i\epsilon$. The portion of V_2 which depends on the principal value part of that integral, is not computable in terms of the data without specifying a model type. In conclusion we will exclude that part from the computation. The contribution from integrating around the contour integrals around the pole leads to

$$\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1)=\frac{1}{8\pi^2 i}\frac{1}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{1}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty}dk_{\lambda_1}\frac{1}{q_{\lambda_1}}$$
$$\times\left[\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}dp\ i\pi\delta(p-i\epsilon)V_1(k_g,-q_1,-k_{\lambda_1},-q_{\lambda_1}+p,\omega_1)V_1(k_{\lambda_1},q_{\lambda_1}-p,-k_s,-q_2,\omega_1)\right.$$
$$\left.+\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}dp\ i\pi\delta(p-i\epsilon)V_1(k_g,-q_1,-k_{\lambda_1},q_{\lambda_1}-p,\omega_1)V_1(k_{\lambda_1},-q_{\lambda_1}+p,-k_s,-q_2,\omega_1)\right] \tag{64}$$

or

$$\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1)=\frac{1}{8\pi}\frac{1}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{1}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty}dk_{\lambda_1}\frac{1}{q_{\lambda_1}}$$

$$\times\left[V_1(k_g,-q_1,-k_{\lambda_1},-q_{\lambda_1},\omega_1)V_1(k_{\lambda_1},q_{\lambda_1},-k_s,-q_2,\omega_1)\right.$$
$$\left.+V_1(k_g,-q_1,-k_{\lambda_1},q_{\lambda_1},\omega_1)V_1(k_{\lambda_1},-q_{\lambda_1},-k_s,-q_2,\omega_1)\right]. \tag{65}$$

Next we relate V_1 in vertical numbers q_1, q_2 and V_1 in vertical numbers q_g, q_s. As noted before, this leads to two sets of relationships between the sums of the vertical wavenumbers,

$$q_g + q_\lambda = q_1 + q_{\lambda_1} \tag{66}$$
$$q_\lambda - q_s = q_{\lambda_1} - q_2 \tag{67}$$

and

$$q_g - q_\lambda = q_1 - q_{\lambda_1} \tag{68}$$
$$q_\lambda + q_s = q_{\lambda_1} + q_2. \tag{69}$$

where

$$q_{\lambda_1} = \sqrt{\frac{\omega_1^2}{c_0^2} - k_\lambda^2}, \quad q_\lambda = \sqrt{\frac{\omega^2}{c_0^2} - k_\lambda^2}. \tag{70}$$

Each of the two sets of equations for the vertical wavenumbers has to be satisfied simultaneously. Notice that these equations provide a unique and consistent formula for ω_1, in terms of ω, which can be discovered by, for example, subtracting equations (66) and (67) and adding equations (68) and (69). The relationship is (see also equation (40))

$$q_g + q_s = q_1 + q_2, \tag{71}$$

which leads to (see Appendix 6 equation (152) and also Section 3)

$$\frac{\omega_1^2}{c_0^2} = \frac{\left[(q_g+q_s)^2 + k_g^2 + k_s^2\right]^2 - 4k_g^2k_s^2}{4(q_g+q_s)^2}. \tag{72}$$

With this particular ω_1, equation (65) for ψ_s^{21} becomes

$$\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1) = \frac{1}{8\pi}\frac{1}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{1}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int dk_\lambda \frac{1}{q_{\lambda_1}}$$
$$\times\left[V_1(k_g,-q_g,-k_\lambda,-q_\lambda,\omega)V_1(k_\lambda,q_\lambda,-k_s,-q_s,\omega) + V_1(k_g,-q_g,-k_\lambda,q_\lambda,\omega)V_1(k_\lambda,-q_\lambda,-k_s,-q_2,\omega)\right]. \tag{73}$$

Next we plug in the expressions for V_1's in terms of the measured data. From equations (172), (173), (174) and (175) in Appendix 6 we have

$$V_1(k_g,-q_g,-k_\lambda,-q_\lambda,\omega) = -4q_gq_\lambda e^{iq_gz_g}e^{iq_\lambda z_s}D(k_g,k_\lambda,q_g+q_\lambda), \tag{74}$$

$$V_1(k_\lambda,q_\lambda,-k_s,-q_s,\omega) = 4q_\lambda q_s e^{-iq_\lambda z_g}e^{iq_sz_s}D(k_\lambda,k_s,-q_\lambda+q_s) \tag{75}$$

$$V_1(k_g,-q_g,-k_\lambda,q_\lambda,\omega) = 4q_gq_\lambda e^{iq_gz_g}e^{-iq_\lambda z_s}D(k_g,k_\lambda,q_g-q_\lambda) \tag{76}$$

and

$$V_1(k_\lambda, -q_\lambda, -k_s, -q_2, \omega) = -4q_\lambda q_s e^{iq_\lambda z_g} e^{iq_s z_s} D(k_\lambda, k_s, q_\lambda + q_s) \tag{77}$$

and so equation (73) becomes

$$\psi_s^{21}(k_g, q_1, k_s, q_2; \omega_1) = -\frac{2q_g q_s}{\pi} \frac{e^{iq_g z_g}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{iq_s z_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}$$
$$\times \int_{-\infty}^{\infty} dk_\lambda \frac{q_\lambda^2}{q_{\lambda_1}} \Big[e^{iq_\lambda(z_s - z_g)} D(k_g, k_\lambda, q_g + q_\lambda) D(k_\lambda, k_s, -q_\lambda + q_s)$$
$$+ e^{iq_\lambda(z_g - z_s)} D(k_g, k_\lambda, q_g - q_\lambda) D(k_\lambda, k_s, q_\lambda + q_s) \Big]. \tag{78}$$

Similar to what is described in Appendix 6 we are going to separate the expression of ψ_s^{21} into an imaging part and an inversion part and use the former and discard the latter for our calculation of the second order wavefield at depth. To separate, we write the data terms as Fourier integrals over vertical wavenumbers as

$$D(k_g, k_\lambda, q_g + q_\lambda) = \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g + q_\lambda)} D(k_g, k_\lambda, z_1), \tag{79}$$

$$D(k_\lambda, k_s, -q_\lambda + q_s) = \int_{-\infty}^{\infty} dz_2 e^{iz_2(-q_\lambda + q_s)} D(k_\lambda, k_s, z_2) \tag{80}$$

$$D(k_g, k_\lambda, q_g - q_\lambda) = \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \tag{81}$$

$$D(k_\lambda, k_s, q_\lambda + q_s) = \int_{-\infty}^{\infty} dz_4 e^{iz_4(q_\lambda + q_s)} D(k_\lambda, k_s, z_4), \tag{82}$$

and rewrite equation (78) as

$$\psi_s^{21}(k_g, q_1, k_s, q_2; \omega_1) = -\frac{2q_g q_s}{\pi} \frac{e^{iq_g z_g}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{iq_s z_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \int_{-\infty}^{\infty} dk_\lambda \frac{q_\lambda^2}{q_{\lambda_1}}$$
$$\Bigg[e^{iq_\lambda(z_s - z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g + q_\lambda)} D(k_g, k_\lambda, z_1) \int_{-\infty}^{\infty} dz_2 e^{iz_2(-q_\lambda + q_s)} D(k_\lambda, k_s, z_2)$$
$$+ e^{iq_\lambda(z_g - z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \int_{-\infty}^{\infty} dz_4 e^{iz_4(q_\lambda + q_s)} D(k_\lambda, k_s, z_4) \Bigg]. \tag{83}$$

Depending on the relative position of the two pseudo-depths z_1, z_2, z_3 and z_4 we can further separate the last expression into

$$
\begin{aligned}
\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1) = &-\frac{2q_gq_s}{\pi}\frac{e^{iq_gz_g}}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{e^{iq_sz_s}}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty}dk_\lambda\frac{q_\lambda^2}{q_{\lambda_1}}e^{iq_\lambda(z_s-z_g)} \\
&\left[\int_{-\infty}^{\infty}dz_1e^{iz_1(q_g+q_\lambda)}D(k_g,k_\lambda,z_1)\int_{-\infty}^{\infty}dz_2e^{iz_2(-q_\lambda+q_s)}D(k_\lambda,k_s,z_2)\delta(z_2-z_1)\right. \\
&+\int_{-\infty}^{\infty}dz_1e^{iz_1(q_g+q_\lambda)}D(k_g,k_\lambda,z_1)\int_{z_1+\epsilon}^{\infty}dz_2e^{iz_2(-q_\lambda+q_s)}D(k_\lambda,k_s,z_2) \\
&\left.+\int_{-\infty}^{\infty}dz_1e^{iz_1(q_g+q_\lambda)}D(k_g,k_\lambda,z_1)\int_{-\infty}^{z_1-\epsilon}dz_2e^{iz_2(-q_\lambda+q_s)}D(k_\lambda,k_s,z_2)\right] \\
&-\frac{2q_gq_s}{\pi}\frac{e^{iq_gz_g}}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{e^{iq_sz_s}}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty}dk_\lambda\frac{q_\lambda^2}{q_{\lambda_1}}e^{iq_\lambda(z_g-z_s)} \\
&\left[\int_{-\infty}^{\infty}dz_3e^{iz_3(q_g-q_\lambda)}D(k_g,k_\lambda,z_3)\int_{-\infty}^{\infty}dz_4e^{iz_4(q_\lambda+q_s)}D(k_\lambda,k_s,z_4)\delta(z_3-z_4)\right. \\
&+\int_{-\infty}^{\infty}dz_3e^{iz_3(q_g-q_\lambda)}D(k_g,k_\lambda,z_3)\int_{z_3+\epsilon}^{\infty}dz_4e^{iz_4(q_\lambda+q_s)}D(k_\lambda,k_s,z_4) \\
&\left.+\int_{-\infty}^{\infty}dz_3e^{iz_3(q_g-q_\lambda)}D(k_g,k_\lambda,z_3)\int_{-\infty}^{z_3-\epsilon}dz_4e^{iz_4(q_\lambda+q_s)}D(k_\lambda,k_s,z_4)\right]
\end{aligned}
\tag{84}
$$

or, after solving the integral containing the delta function,

$$
\begin{aligned}
\psi_s^{21}(k_g,q_1,k_s,q_2;\omega_1) = &-\frac{2q_gq_s}{\pi}\frac{e^{iq_gz_g}}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{e^{iq_sz_s}}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty}dk_\lambda\frac{q_\lambda^2}{q_{\lambda_1}}e^{iq_\lambda(z_s-z_g)} \\
&\left[2\pi\int_{-\infty}^{\infty}dz_1e^{iz_1(q_g+q_s)}D(k_g,k_\lambda,z_1)D(k_\lambda,k_s,z_1)\right. \\
&+\int_{-\infty}^{\infty}dz_1e^{iz_1(q_g+q_\lambda)}D(k_g,k_\lambda,z_1)\int_{z_1+\epsilon}^{\infty}dz_2e^{iz_2(-q_\lambda+q_s)}D(k_\lambda,k_s,z_2) \\
&\left.+\int_{-\infty}^{\infty}dz_1e^{iz_1(q_g+q_\lambda)}D(k_g,k_\lambda,z_1)\int_{-\infty}^{z_1-\epsilon}dz_2e^{iz_2(-q_\lambda+q_s)}D(k_\lambda,k_s,z_2)\right]
\end{aligned}
$$

$$
- \frac{2q_g q_s}{\pi} \frac{e^{iq_g z_g}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{iq_s z_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \int_{-\infty}^{\infty} dk_\lambda \frac{q_\lambda^2}{q_{\lambda_1}} e^{iq_\lambda (z_g - z_s)}
$$

$$
\Bigg[2\pi \int_{-\infty}^{\infty} dz_3 e^{iz_3 (q_g + q_s)} D(k_g, k_\lambda, z_3) D(k_\lambda, k_s, z_3)
$$

$$
+ \int_{-\infty}^{\infty} dz_3 e^{iz_3 (q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \int_{z_3+\epsilon}^{\infty} dz_4 e^{iz_4 (q_\lambda + q_s)} D(k_\lambda, k_s, z_4)
$$

$$
+ \int_{-\infty}^{\infty} dz_3 e^{iz_3 (q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4 (q_\lambda + q_s)} D(k_\lambda, k_s, z_4) \Bigg]. \tag{85}
$$

As also noted in Appendix 6, the first term in each square bracket in equation (85) is similar to an amplitude corrector and it will be ignored in the following calculations. The second term in the first square bracket and the third in the second square bracket are similar to depth correctors (see e.g. Shaw (2005), Liu et al. (2006) Ramirez and Otnes (2007)). For the purpose of this paper we will only keep these (imaging) terms and arrive to our final expression

$$
\psi_s^{21_{IM}}(k_g, q_1, k_s, q_2; \omega_1) = - \frac{2q_g q_s}{\pi} \frac{e^{iq_g z_g}}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{iq_s z_s}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \int_{-\infty}^{\infty} dk_\lambda \frac{q_\lambda^2}{q_{\lambda_1}}
$$

$$
\Bigg[e^{iq_\lambda (z_s - z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1 (q_g + q_\lambda)} D(k_g, k_\lambda, z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2 (-q_\lambda + q_s)} D(k_\lambda, k_s, z_2)
$$

$$
+ e^{iq_\lambda (z_g - z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3 (q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4 (q_\lambda + q_s)} D(k_\lambda, k_s, z_4) \Bigg], \tag{86}
$$

in which the variables are related by the dispersion relationships

$$
k_g^2 + q_1^2 = \frac{\omega_1^2}{c_0^2} \quad k_s^2 + q_2^2 = \frac{\omega_1^2}{c_0^2} \tag{87}
$$

$$
k_g^2 + q_g^2 = \frac{\omega^2}{c_0^2} \quad k_s^2 + q_s^2 = \frac{\omega^2}{c_0^2} \tag{88}
$$

and

$$
\frac{\omega_1^2}{c_0^2} = \frac{\left[(q_g + q_s)^2 + k_g^2 + k_s^2 \right]^2 - 4k_g^2 k_s^2}{4(q_g + q_s)^2}. \tag{89}
$$

4.2 The calculation of ψ_s^{22}

For ψ_s^{22} the calculations are similar to the ones in Section 3 and we can find

$$\psi_s^{22}(k_g, q_1, k_s, q_2; \omega_1) = \frac{1}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} V_2(k_g, -q_1, -k_s, -q_2, \omega_1) \frac{1}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon}. \tag{90}$$

It is again important to notice that the V_2 on the right is the 3-dimensional projection of the fully 5-dimensional V_2 operator. The independent variables on the right are k_g, k_s and $q_1 + q_2$. With this in mind we write (see also Appendix 6)

$$V_2(k_g, -q_1, -k_s, -q_2, \omega_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(-q_g - q_s) \delta(q_g + q_s - q_1 - q_2) V_2(k_g, -q_g, -k_s, -q_s, \omega). \tag{91}$$

This last equation leads to the same relationship between the sums of the vertical wavenumbers,

$$q_g + q_s = q_1 + q_2, \tag{92}$$

which, in turn, allows us to calculate ω_1 in terms of ω as (see Appendix 6 equation (152))

$$\frac{\omega_1^2}{c_0^2} = \frac{\left[(q_g + q_s)^2 + k_g^2 + k_s^2\right]^2 - 4k_g^2 k_s^2}{4(q_g + q_s)^2}. \tag{93}$$

Note that this value of ω_1 is consistent with the previous values obtained in the calculation of V_1 and ψ_s^1. With this particular ω_1, the equation for V_2 becomes

$$V_2(k_g, -q_1, -k_s, -q_2, \omega_1) = V_2(k_g, -q_g, -k_s, -q_s, \omega). \tag{94}$$

In this expression, consistent with our previous remarks, we will only use the imaging part of V_2 as calculated in equation (185) in Appendix 6

$$V_2^{IM}(k_g, -q_1, -k_s, -q_2, \omega_1) = V_2^{IM}(k_g, -q_g, -k_s, -q_s, \omega) = \frac{2q_g q_s e^{i(q_g z_g + q_s z_s)}}{\pi} \int_{-\infty}^{\infty} dk_\lambda q_\lambda$$
$$\times \left[e^{iq_\lambda(z_s - z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g + q_\lambda)} D(k_g, k_\lambda, z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda + q_s)} D(k_\lambda, k_s, z_2) \right.$$
$$\left. + e^{iq_\lambda(z_g - z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda + q_s)} D(k_\lambda, k_s, z_4) \right]. \tag{95}$$

The final expression for $\psi_s^{22_{IM}}$ is then

$$\psi_s^{22_{IM}}(k_g, q_1, k_s, q_2; \omega_1) = \frac{2}{\pi} \frac{q_g q_s}{k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \frac{e^{i(q_g z_g + q_s z_s)}}{k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon} \int_{-\infty}^{\infty} dk_\lambda q_\lambda$$

$$\times \left[e^{iq_\lambda(z_s-z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g,k_\lambda,z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda,k_s,z_2) \right.$$
$$\left. + e^{iq_\lambda(z_g-z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4) \right]. \tag{96}$$

in which the variables are related by the dispersion relationships

$$k_g^2+q_1^2=\frac{\omega_1^2}{c_0^2}, \quad k_s^2+q_2^2=\frac{\omega_1^2}{c_0^2}, \tag{97}$$

$$k_g^2+q_g^2=\frac{\omega^2}{c_0^2}, \quad k_s^2+q_s^2=\frac{\omega^2}{c_0^2}, \quad k_\lambda^2+q_\lambda^2=\frac{\omega^2}{c_0^2} \tag{98}$$

and

$$\frac{\omega_1^2}{c_0^2}=\frac{\left[(q_g+q_s)^2+k_g^2+k_s^2\right]^2-4k_g^2k_s^2}{4(q_g+q_s)^2}. \tag{99}$$

4.3 Solution for $\psi_s^{2_{IM}}$

Combining equations (86) and (96) we find

$$\psi_s^{2_{IM}}(k_g,q_1,k_s,q_2;\omega_1)=\frac{2q_gq_s}{\pi}\frac{e^{iq_gz_g}}{k_g^2+q_1^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\frac{e^{iq_sz_s}}{k_s^2+q_2^2-\frac{\omega_1^2}{c_0^2}-i\epsilon}\int_{-\infty}^{\infty} dk_\lambda \frac{q_\lambda}{q_{\lambda_1}}(q_{\lambda_1}-q_\lambda)$$
$$\times \left[e^{iq_\lambda(z_s-z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g,k_\lambda,z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda,k_s,z_2) \right.$$
$$\left. + e^{iq_\lambda(z_g-z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4) \right] \tag{100}$$

in which, again, the variables are related by the dispersion relationships

$$k_g^2+q_1^2=\frac{\omega_1^2}{c_0^2}, \quad k_s^2+q_2^2=\frac{\omega_1^2}{c_0^2}, \quad k_\lambda^2+q_{\lambda_1}^2=\frac{\omega_1^2}{c_0^2} \tag{101}$$

$$k_g^2+q_g^2=\frac{\omega^2}{c_0^2}, \quad k_s^2+q_s^2=\frac{\omega^2}{c_0^2}, \quad k_\lambda^2+q_\lambda^2=\frac{\omega^2}{c_0^2} \tag{102}$$

and

$$\frac{\omega_1^2}{c_0^2}=\frac{\left[(q_g+q_s)^2+k_g^2+k_s^2\right]^2-4k_g^2k_s^2}{4(q_g+q_s)^2}. \tag{103}$$

5 Imaging the wavefield at depth

Equation (18) provides a formula for the scattered wavefield everywhere inside or outside the actual medium

$$\psi_s = \psi_s^1 + \psi_s^2 + \psi_s^3 + \ldots . \tag{104}$$

Plugging in the expressions we found for the first and second orders, ψ_s^1 and ψ_s^2, in equations (43) and (100) respectively, we find

$$\psi_s^{2nd}(k_g, q_1, k_s, q_2; \omega_1) = \frac{-4q_g q_s e^{iq_g z_g} e^{iq_s z_s}}{\left(k_g^2 + q_1^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon\right)\left(k_s^2 + q_2^2 - \frac{\omega_1^2}{c_0^2} - i\epsilon\right)}$$
$$\Bigg\{ D(k_g, z_g, k_s, z_s, \omega) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_\lambda \frac{q_\lambda}{q_{\lambda_1}}(q_\lambda - q_{\lambda_1})$$
$$\times \Bigg[e^{iq_\lambda(z_s - z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g + q_\lambda)} D(k_g, k_\lambda, z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda + q_s)} D(k_\lambda, k_s, z_2)$$
$$+ e^{iq_\lambda(z_g - z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g - q_\lambda)} D(k_g, k_\lambda, z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda + q_s)} D(k_\lambda, k_s, z_4) \Bigg] \Bigg\}. \tag{105}$$

To image this wavefield we first transform it into the depth domain by inverse Fourier transforming over the vertical wavenumbers to obtain

$$\psi_s^{2nd}(k_g, k_s, z; \omega_1) = \int_{-\infty}^{\infty} dk_z \; e^{ik_z z} \psi_s^{2nd}(k_g, q_1, k_s, q_2; \omega_1) \tag{106}$$

where $k_z = q_1 + q_2$. Then we integrate over all temporal frequencies, which amounts in applying the imaging condition, to obtain

$$I(k_g, k_s, z) = \int_{-\infty}^{\infty} d\omega_1 \psi_s^{2nd}(k_g, k_s, z; \omega_1), \tag{107}$$

and finally we transform over the horizontal wavenumbers to obtain the image in the space domain

$$I(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(k_g - k_s) e^{-i(k_g - k_s)x} I(k_g, k_s, z). \tag{108}$$

6 Conclusions

In this report we describe an approach to calculating and imaging the wavefield at depth using the inverse scattering series. The method does not make any assumptions on the medium under investigations and only inputs the recorded data on the measurement surface and a background acoustic Green's function. Roughly speaking, the method uses the calculated and task specific separated perturbation operator **V**, in the forward scattering series, to calculate different orders of the scattered field at any depth. For an acoustic medium, the method was presented in Weglein et al. (2000) where the first order wavefield was calculated. Here we proceed without specifying an actual model type. The main results of this paper are the calculated first and second orders wavefield at depth, equations (43) and (100) respectively.

It is important to notice that the calculation of the second order wavefield at depth uses the formula (see equation (22))

$$\psi_s^2 = \mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0 \tag{109}$$

and hence $\mathbf{V_1}$ and $\mathbf{V_2}$ are required for the calculation (see theri calculated expressions in the Appendices). There are two important related choices that we made in this calculation and that are worth mentioning. First, instead of putting through the equation the full expression of $\mathbf{V_2}$, we separated it and determined just the piece which corrects for the wrong depth and used that part only. Second, consistently with the first choice, instead of using the full second term on the right side of the above equation for ψ_s^2 we, again, separated the term, determined the part which corrects for the wrong depth and used that part only. The motivation behind these choices is simple: the full expression of **V** will construct the full wavefield at depth, including primaries and multiples (as shown for example in Matson (1996)). Since what we are trying to construct is the image at depth of data containing primaries only, it was reasonable to assume that this will be achieved by the part in V which only corrects for depth. Further analytical and numerical examples will verify this hypothesis.

We emphasize one important conclusion that comes out of the two expressions of the first and second orders wavefield at depth. When using inverse scattering methods, the actual medium is assumed to be unknown and no a priori assumptions are made about its properties. As a consequence, calculating the wavefield at depth for one frequency requires all frequencies in the data. This is fundamentally different from well known migration algorithms, which assume the velocity profile of the medium can be a priori found, and which, hence, can extrapolate, at depth, each frequency individually (by performing a phase-shift in the wavenumber-frequency domain for example). This was originally noted in Weglein et al. (2000) for the first order of the acoustic wavefield at depth calculated using the inverse scattering series. Here we discovered the same characteristic for the wavefield at depth without a specified model type for both first and second order.

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Appendices

A. Green's function in an infinite homogeneous space

Consider the homogeneous acoustic wave equation (see also equation (10))

$$\nabla^2\phi(\mathbf{x},t) - \frac{1}{c_0^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} = -\delta(\mathbf{x})\delta(t) \tag{110}$$

where $\mathbf{x} = (x_g - x_s, z_g - z_s)$ is the vector connecting the source of the wave to the point where the wave is measured (the receiver) and where we assume that the source goes off at time $t = 0$. In the frequency domain, the solution to equation (110) in an infinite homogeneous space is (see e.g. Aki and Richards (2002))

$$\phi(\mathbf{x},\omega) = \frac{1}{R}e^{i\omega\left(\frac{R}{c_0}\right)} \tag{111}$$

where $R = |\mathbf{x}| = \sqrt{(x_g - x_s)^2 + (z_g - z_s)^2}$. This is the Green's function of the acoustic wave equation in an infinite homogeneous space and it is usually denoted by

$$G_0(x_g, z_g, x_s, z_s, \omega) = \frac{1}{R}e^{i\omega\left(\frac{R}{c_0}\right)} \tag{112}$$

In the following we will use ϕ and G_0 interchangeably.

In terms of its Fourier transform over all space coordinates, we can also write ϕ as

$$\phi(\mathbf{x},t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq\phi(\mathbf{k},t)e^{i\mathbf{k}\cdot\mathbf{x}} \tag{113}$$

where $\mathbf{k} = (k, q)$ is the wavenumber vector, with horizontal and vertical components, associated with $\mathbf{x}$. Notice that we also have

$$\nabla^2\phi(\mathbf{x},t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq\phi(\mathbf{k},t)(-k_x^2 - q^2)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{114}$$

$$-\frac{1}{c_0^2}\frac{\partial^2\phi(\mathbf{x},t)}{\partial t^2} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq\phi(\mathbf{k},t)\frac{\omega^2}{c_0^2}e^{i\mathbf{k}\cdot\mathbf{x}} \tag{115}$$

and

$$-\delta(\mathbf{x}) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{116}$$

Putting these last three expressions back into equation (110) and transforming to frequency domain we find

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq\phi(\mathbf{k},\omega)\left(\frac{\omega^2}{c^2} - k^2 - q^2\right)e^{i\mathbf{k}\cdot\mathbf{x}} = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{117}$$

By equating the integrands we find

$$G_0(k,q,\omega) = \phi(k,q,\omega) = \frac{1}{k^2 + q^2 - \frac{\omega^2}{c_0^2}}. \tag{118}$$

Then from (112) and the double inverse Fourier transform of (118)

$$G_0(x_g,z_g,x_s,z_s,\omega) = \frac{1}{R} e^{i\omega\left(\frac{R}{c_0}\right)} = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dq \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k^2 + q^2 - \frac{\omega^2}{c_0^2}}. \tag{119}$$

The first expression is a cylindrical wave propagating from the source to the receiver with speed c_0. The right side represents a superposition of planewaves over the entire range of wavenumbers k and q. These planewaves have the arbitrary velocity $\frac{\omega}{|\mathbf{k}|}$ which varies from 0 to ∞. In order to write the expression on the right as a superposition of planewaves traveling at the same speed c_0, we have to perform one of the integrations with respect to one of the two wavenumbers. We will do this over the vertical wavenumber q, then comment on this calculation to obtain equivalent forms for the Green's function.

The Weyl Integral form of the Green's function

To integrate the right side of equation (119) with respect to q we apply residue theorem. We complexify q and notice that the poles of the integrand are at

$$q = \pm\sqrt{\frac{\omega^2}{c_0^2} - k^2} \tag{120}$$

with some of them lying on the real q axis, i.e. along the integration path. To make the integrand analytic along the real q axis, a small attenuation is introduced through an imaginary part in the velocity c_0 (see Aki and Richards (2002)) so that the new velocity c_0^{new} is

$$\frac{1}{c_0^{new}} = \frac{1}{c_0} + i\varepsilon \tag{121}$$

with ε being a small parameter such that $\varepsilon > 0$ for $\omega > 0$. This attenuation effects in a shift of the poles away from the real q axis and into the first and the third quadrant in the complex q plane. We define the poles in the first quadrant as

$$q = +\sqrt{\frac{\omega^2}{c_0^2} - k^2} \tag{122}$$

and the poles in the third quadrant as

$$q = -\sqrt{\frac{\omega^2}{c_0^2} - k^2}. \tag{123}$$

Notice that in the first quadrant both the imaginary and the real part of q are positive, while in the third quadrant both the imaginary and the real part of q are negative. We now apply Cauchy's theorem to calculate the integral.

For positive $z_g - z_s$ a factor of $e^{iq(z_g - z_s)}$ will cancel the integrand if taken around a sufficiently large semicircle in the upper half complex q-plane. This implies that adding this semicircle to the integration path will not change the value of the integral and hence it can be used to close the integration path. Cauchy's theorem implies

$$\phi = P.V. + i\pi\delta\left(q - \sqrt{\frac{\omega^2}{c_0^2} - k^2}\right) = P.V. + \frac{e^{-i\omega t}}{2\pi}\int_{-\infty}^{\infty} dk_x \frac{e^{i[k(x_g - x_s) + iq(z_g - z_s)]}}{2iq}, \tag{124}$$

where k and q now satisfy the dispersion relation

$$k^2 + q^2 = \frac{\omega^2}{c_0^2}. \tag{125}$$

For negative $z_g - z_s$ the same factor $e^{iq(z_g - z_s)}$ will cancel the integrand if taken around a sufficiently large semicircle in the lower half complex q-plane. We add the semicircle to close the integration path and obtain, from Cauchy's theorem,

$$\phi = P.V + i\pi\delta\left(q + \sqrt{\frac{\omega^2}{c_0^2} - k^2}\right) = P.V. + \frac{e^{-i\omega t}}{2\pi}\int_{-\infty}^{\infty} dk_x \frac{e^{i[k(x_g - x_s) + iq(z_s - z_g)]}}{2iq}. \tag{126}$$

where, again, k and q satisfy the dispersion relation

$$k^2 + q^2 = \frac{\omega^2}{c_0^2}. \tag{127}$$

The results in equations (124) and (126) can be summarized in the Weyl integral

$$G(x_g, z_g, x_s, z_s; \omega) = P.V. + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_x \frac{e^{i[k(x_g - x_s) + iq|z_g - z_s|]}}{2iq} \tag{128}$$

where

$$q = \sqrt{\frac{\omega^2}{c_0^2} - k^2} \tag{129}$$

and the sign of q is chosen such that the $Im\ q > 0$.

In the history of the development of a model type independent internal multiple algorithm it was determined that the portion of V_2 which depends on the principal value part of the contribution from G_0 is not computable from surface data without assuming a model type. For this reason we will also ignore the principal value part of the Green's function and investigate the usefulness of a wavefield at depth formula derived by considering the model type independent part of V_2 only. The Green's function that we are going to use hence is

$$G(x_g, z_g, x_s, z_s; \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_x \frac{e^{i[k(x_g - x_s) + iq|z_g - z_s|]}}{2iq}. \tag{130}$$

An alternative formula for the Green's function

It is also useful to have an alternative expression for the Green's function, e.g. in the wavenumber / frequency domain. Recall from equation (118) that such a form is close to

$$G_0(k, q, \omega) = \frac{1}{k^2 + q^2 - \frac{\omega^2}{c_0^2}}. \tag{131}$$

However, because of the dispersion relations, which we now have to impose, and the poles located on the real q axis we have to rewrite it as

$$G_0(k, q, \omega) = \frac{1}{k^2 + q^2 - \frac{\omega^2}{c_0^2} - i\epsilon} \tag{132}$$

where the selection of $\pm\epsilon$ leads to a causal/anticausal Green's function (here chosen as causal) and where, as before,

$$q = \sqrt{\frac{\omega^2}{c_0^2} - k^2} \tag{133}$$

and the sign of q is chosen such that the $Im\ q > 0$. If we wanted to work with this form in the space domain we would have to double inverse Fourier transform over the horizontal and vertical wavenumbers and obtain

$$G(x_g, z_g, x_s, z_s; \omega) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dq \frac{e^{-ik(x_g - x_s)} e^{-iq(z_g - z_s)}}{k^2 + q^2 - \frac{\omega^2}{c_0^2} - i\epsilon}. \tag{134}$$

It is worth mentioning, even though this does not appear explicitly, that this Green's function represents only the part equivalent to the $i\pi\delta$ contribution described by formula (130) and with the principal value discarded.

B. The calculation of V_1

Start with equation (24)

$$D = (\mathbf{G}_0 \mathbf{V_1} \mathbf{G}_0)_{ms} \tag{135}$$

where D is the data, G_0 is the Greens function of the reference medium and $V_1(x', z', x'', z'', \omega)$ is the first order component of the perturbation V. In coordinates, this equation can be written as

$$D(x_g, x_s, \omega) = \int dx' dx'' dz' dz'' G_0(x_g, z_g, x', z', \omega) V_1(x', z', x'', z'', \omega) G_0(x'', z'', x_s, z_s, \omega) \tag{136}$$

where x_g, z_g, x_s and z_s are the spatial coordinates of the source of the wave and the receiver used to record the data and where we have omitted the vertical coordinates arguments in the data since they are fixed given numbers (not actual variables). In this equation we will use the following expressions for the Green's functions

$$G_0(x_g, z_g, x', z', \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_g \frac{e^{ik_g(x_g - x')} e^{iq_g|z' - z_g|}}{2iq_g} \tag{137}$$

and

$$G_0(x'', z'', x_s, z_s, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_s \frac{e^{ik_s(x'' - x_s)} e^{iq_s|z'' - z_s|}}{2iq_s} \tag{138}$$

where k_g, q_g, k_s and q_s are the wavenumbers associated with x_g, z_g, x_s and z_s respectively and which satisfy the dispersion relations

$$k_g^2 + q_g^2 = \frac{\omega^2}{c_0^2}, \quad k_s^2 + q_s^2 = \frac{\omega^2}{c_0^2}. \tag{139}$$

Plugging these expressions of the Green's functions into equation (136) we find

$$D(x_g, x_s, \omega) = \int dx' dx'' dz' dz'' \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_g \frac{e^{ik_g(x_g - x')} e^{iq_g|z' - z_g|}}{2iq_g} V_1(x', z', x'', z'', \omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_s \frac{e^{ik_s(x'' - x_s)} e^{iq_s|z'' - z_s|}}{2iq_s}. \tag{140}$$

Next we apply Fourier transforms on x_g and x_s, i.e. we apply, on both sides, the integral operators $\int_{-\infty}^{\infty} dx_g e^{-ik_g' x_g}$ and $\int_{-\infty}^{\infty} dx_s e^{ik_s' x_s}$ and obtain

$$D(k_g', k_s', \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_g \int_{-\infty}^{\infty} dx_g e^{-ix_g(k_g' - k_g)} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_s \int_{-\infty}^{\infty} dx_s e^{ix_s(k_s' - k_s)} \tag{141}$$
$$\times \int dx' dx'' dz' dz'' \frac{e^{-ik_g x'} e^{iq_g|z' - z_g|}}{2iq_g} V_1(x', z', x'', z'', \omega) \frac{e^{ik_s x''} e^{iq_s|z'' - z_s|}}{2iq_s}$$

or

$$D(k_g', k_s', \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_g \delta(k_g' - k_g) \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_s \delta(k_s' - k_s) \tag{142}$$
$$\times \int dx' dx'' dz' dz'' \frac{e^{-ik_g x'} e^{iq_g|z' - z_g|}}{2iq_g} V_1(x', z', x'', z'', \omega) \frac{e^{ik_s x''} e^{iq_s|z'' - z_s|}}{2iq_s}$$

After solving the first two integrals and changing the notation for the wavenumbers from prime to non primed quantities (for simplicity) we obtain

$$D(k_g, k_s, \omega) = \int dx' dx'' dz' dz'' \frac{e^{-ik_g x'} e^{iq_g|z' - z_g|}}{2iq_g} V_1(x', z', x'', z'', \omega) \frac{e^{ik_s x''} e^{iq_s|z'' - z_s|}}{2iq_s} \tag{143}$$

and, after factoring and additional assumption that $z' > z_g$ and $z' > z_s$ (which is reasonable since the depth of the scatterer is always larger than the depth of the sources and receivers in a surface seismic experiment and when the positive z-axis points downward), we find

$$D(k_g, k_s, \omega) = \frac{e^{-iq_g z_g} e^{-iq_s z_s}}{-4q_g q_s} \int dx' dx'' dz' dz'' e^{iz' q_g} e^{iq_s z''} e^{-ix' k_g} e^{ik_s x''} V_1(x', z', x'', z'', \omega), \qquad (144)$$

and finally

$$D(k_g, k_s, \omega) = \frac{e^{-iq_g z_g} e^{-iq_s z_s}}{-4q_g q_s} V_1(k_g, -q_g, -k_s, -q_s, \omega). \qquad (145)$$

From here we can calculate V_1 in the wavenumbers domain to be

$$V_1(k_g, -q_g, -k_s, -q_s, \omega) = -4q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, \omega). \qquad (146)$$

C. Relation between ω_1 and ω

Here we show how we can calculate ω_1 in terms of ω such that equation (40)

$$q_g + q_s = q_1 + q_2, \qquad (147)$$

is satisfied. Squaring both sides of

$$q_g + q_s = \sqrt{\frac{\omega_1^2}{c_0^2} - k_g^2} + \sqrt{\frac{\omega_1^2}{c_0^2} - k_s^2} \qquad (148)$$

we find, after rearranging terms,

$$(q_g + q_s)^2 + k_g^2 + k_s^2 - 2\frac{\omega_1^2}{c_0^2} = 2\sqrt{\left(\frac{\omega_1^2}{c_0^2} - k_g^2\right)\left(\frac{\omega_1^2}{c_0^2} - k_s^2\right)}. \qquad (149)$$

After squaring one more time we find

$$\left[(q_g + q_s)^2 + k_g^2 + k_s^2 - 2\frac{\omega_1^2}{c_0^2}\right]^2 = 4\left(\frac{\omega_1^2}{c_0^2} - k_g^2\right)\left(\frac{\omega_1^2}{c_0^2} - k_s^2\right) \qquad (150)$$

or, after some cancellations,

$$\left[(q_g + q_s)^2 + k_g^2 + k_s^2\right]^2 - 4\frac{\omega_1^2}{c_0^2}(q_g + q_s)^2 = 4k_g^2 k_s^2. \qquad (151)$$

From here we obtain

$$\frac{\omega_1^2}{c_0^2} = \frac{\left[(q_g + q_s)^2 + k_g^2 + k_s^2\right]^2 - 4k_g^2 k_s^2}{4(q_g + q_s)^2} \qquad (152)$$

which is the desired formula.

D. The calculation and separation of V_2

Start with equation (25)

$$0 = (\mathbf{G}_0\mathbf{V_2}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0)_{ms} \tag{153}$$

or

$$(-\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0)_{ms} = (\mathbf{G}_0\mathbf{V_2}\mathbf{G}_0)_{ms}. \tag{154}$$

For the right hand-side of equation (154) we find similarly to the calculation of V_1 (see equation (145))

$$RHS = G_0V_2G_0 = \frac{e^{-i(q_gz_g+q_sz_s)}}{-4q_gq_s}V_2(k_g, -q_g, -k_s, -q_s, \omega), \tag{155}$$

where, as before, k_g, q_g, k_s and q_s are the wavenumbers associated with x_g, z_g, x_s and z_s (source and receiver coordinates) respectively and which satisfy the dispersion relations

$$k_g^2 + q_g^2 = \frac{\omega^2}{c_0^2}, \quad k_s^2 + q_s^2 = \frac{\omega^2}{c_0^2}. \tag{156}$$

The left hand-side of equation (154) is

$$LHS = -(\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0\mathbf{V_1}\mathbf{G}_0) = -\int dx'dx''dz'dz''G_0(x_g, z_g, x', z', \omega)V_1(x', x'', z', dz'', \omega)$$
$$\times \int dx'''dx^{iv}dz'''dz^{iv}G_0(x'', z'', x''', z''', \omega)V_1(x''', x^{iv}, z''', dz^{iv}, \omega)G_0(x^{iv}, z^{iv}, x_s, z_s, \omega) \tag{157}$$

where the Green's functions are (see Appendix 6)

$$G_0(x_g, z_g, x', z', \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_g\frac{e^{ik_g(x_g-x')}e^{iq_g|z'-z_g|}}{2iq_g}, \tag{158}$$

$$G_0(x'', z'', x''', z''', \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_\lambda\frac{e^{ik_\lambda(x''-x''')}e^{iq_\lambda|z'''-z''|}}{2iq_\lambda} \tag{159}$$

and

$$G_0(x^{iv}, z^{iv}, x_s, z_s, \omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk_s\frac{e^{ik_s(x^{iv}-x_s)}e^{iq_s|z^{iv}-z_s|}}{2iq_s}. \tag{160}$$

Plugging these expressions into equation (157) and then Fourier transforming it over x_g and x_s (i.e. applying on both sides the integral operators $\int_{-\infty}^{\infty} dx_g e^{-ik_gx_g}$ and $\int_{-\infty}^{\infty} dx_s e^{ik_sx_s}$) we find

$$LHS = \frac{e^{-i(q_gz_g+q_sz_s)}}{16\pi iq_gq_s}\int dk_\lambda\frac{1}{q_\lambda}\int dz''dz'''e^{iq_\lambda|z''-z'''|}V_1(k_g, -q_g, -k_\lambda, z'', \omega)V_1(k_\lambda, z''', -k_s, -q_s, \omega). \tag{161}$$

Next we use the Heaviside step function H to express the absolute values and write

$$e^{iq_\lambda|z''-z'''|} = e^{iq_\lambda(z''-z''')}H(z''-z''') + e^{iq_\lambda(z'''-z'')}H(z'''-z''). \tag{162}$$

Moreover we use the integral representation of H (reference)

$$H(z) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} dp \frac{1}{i(p-i\epsilon)}e^{-ipz}. \tag{163}$$

With these considerations, the LHS term becomes

$$\begin{aligned} LHS = &\frac{1}{2\pi}\frac{e^{-iq_g z_g}e^{-iq_s z_s}}{8iq_g q_s}\int dk_\lambda \frac{1}{q_\lambda}\int dz'' dz''' V_1(k_g,-q_g,-k_\lambda,z'',\omega)V_1(k_\lambda,z''',-k_s,-q_s,\omega) \\ &\times\left(e^{iq_\lambda(z''-z''')}\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\frac{1}{i(p-i\epsilon)}e^{-ip(z''-z''')} + e^{iq_\lambda(z'''-z'')}\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^{\infty} dp\frac{1}{i(p-i\epsilon)}e^{-ip(z'''-z'')}\right) \end{aligned} \tag{164}$$

or

$$\begin{aligned} LHS = &\left(\frac{1}{2\pi}\right)^2 \frac{e^{-iq_g z_g}e^{-iq_s z_s}}{8iq_g q_s}\left[\int dk_\lambda \frac{1}{q_\lambda}\lim_{\epsilon\to 0}\int dp\frac{1}{i(p-i\epsilon)}\int dz'' dz''' \right. \\ &\times e^{iq_\lambda(z''-z''')}e^{-ip(z''-z''')}V_1(k_g,-q_g,-k_\lambda,z'',\omega)V_1(k_\lambda,z''',-k_s,-q_s,\omega) \\ &+\int dk_\lambda \frac{1}{q_\lambda}\lim_{\epsilon\to 0}\int dp\frac{1}{i(p-i\epsilon)}\int dz'' dz''' \\ &\left.\times e^{iq_\lambda(z'''-z'')}e^{-ip(z'''-z'')}V_1(k_g,-q_g,-k_\lambda,z'',\omega)V_1(k_\lambda,z''',-k_s,-q_s,\omega)\right]. \end{aligned} \tag{165}$$

Next we treat the dz' and dz'' integrals as Fourier transforms and obtain

$$\begin{aligned} LHS = &\left(\frac{1}{2\pi}\right)^2 \frac{e^{-iq_g z_g}e^{-iq_s z_s}}{8iq_g q_s} \\ &\times\left[\int_{-\infty}^{\infty} dk_\lambda \frac{1}{q_\lambda}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dp\frac{V_1(k_g,-q_g,-k_\lambda,-q_\lambda+p,\omega)V_1(k_\lambda,q_\lambda-p,-k_s,-q_s,\omega)}{i(p-i\epsilon)}\right. \\ &\left.+\int_{-\infty}^{\infty} dk_\lambda \frac{1}{q_\lambda}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dp\frac{V_1(k_g,-q_g,-k_\lambda,q_\lambda-p,\omega)V_1(k_\lambda,-q_\lambda+p,-k_s,-q_s,\omega)}{i(p-i\epsilon)}\right] \end{aligned} \tag{166}$$

The two dp integrals can be separated into a principal value and a contribution from contour integrals around the pole $p = i\epsilon$. The portion of V_2 which depends on the principal value part of that integral, is not computable in terms of the data without specifying a model type. In conclusion we will exclude that part from the computation. The contribution from integrating around the contour integrals around the pole leads to

$$LHS = \left(\frac{1}{2\pi}\right)^2 \frac{e^{-iq_g z_g}e^{-iq_s z_s}}{8iq_g q_s}$$

$$\times \left[\int dk_\lambda \frac{1}{q_\lambda} \lim_{\epsilon \to 0} \int dp \; i\pi\delta(p - i\epsilon) V_1(k_g, -q_g, -k_\lambda, -q_\lambda + p, \omega) V_1(k_\lambda, q_\lambda - p, -k_s, -q_s, \omega) \right.$$
$$\left. + \int dk_\lambda \frac{1}{q_\lambda} \lim_{\epsilon \to 0} \int dp \; i\pi\delta(p - i\epsilon) V_1(k_g, -q_g, -k_\lambda, q_\lambda - p, \omega) V_1(k_\lambda, -q_\lambda + p, -k_s, -q_s, \omega) \right] \quad (167)$$

or

$$LHS = \frac{1}{4\pi} \frac{e^{-iq_g z_g} e^{-iq_s z_s}}{8 q_g q_s} \int dk_\lambda \frac{1}{q_\lambda} \left[V_1(k_g, -q_g, -k_\lambda, -q_\lambda, \omega) V_1(k_\lambda, q_\lambda, -k_s, -q_s, \omega) \right.$$
$$\left. + V_1(k_g, -q_g, -k_\lambda, q_\lambda, \omega) V_1(k_\lambda, -q_\lambda, -k_s, -q_s, \omega) \right]. \quad (168)$$

Next, we relate V_1 to the data. From equation (146) obtained in Appendix 6 we have

$$V_1(k_g, -q_g, -k_s, -q_s, \omega) = -4 q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, \omega). \quad (169)$$

To avoid confusion we will relate the temporal frequency ω with the sum of the vertical wavenumbers and write

$$V_1(k_g, -q_g, -k_s, -q_s, \omega) = -4 q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, q_g + q_s). \quad (170)$$

To calculate the first V_1 in the equation (168) we write

$$V_1(k_g, -q_g, x, z, \omega) = \frac{1}{2\pi} \int d(-k_s) e^{-i(-k_s)x} \frac{1}{2\pi} \int d(-q_s) e^{i(-q_s)z} V_1(k_g, -q_g, -k_s, -q_s, \omega), \quad (171)$$

and then

$$V_1(k_g, -q_g, -k_\lambda, -q_\lambda, \omega) = \int dx e^{i(-k_\lambda)x} \int dz e^{-i(-q_\lambda)z} V_1(k_g, -q_g, x, z, \omega)$$
$$= \int dx e^{i(-k_\lambda)x} \int dz e^{-i(-q_\lambda)z} \frac{1}{2\pi} \int d(-k_s) e^{-i(-k_s)x} \frac{1}{2\pi} \int d(-q_s) e^{i(-q_s)z} V_1(k_g, -q_g, -k_s, -q_s, \omega)$$
$$= \frac{1}{2\pi} \int d(-k_s) \delta(k_s - k_\lambda) \frac{1}{2\pi} \int d(-q_s) \delta(q_s - q_\lambda) \left[-4 q_g q_s e^{iq_g z_g} e^{iq_s z_s} D(k_g, k_s, q_g + q_s) \right]$$
$$= -4 q_g q_\lambda e^{iq_g z_g} e^{iq_\lambda z_s} D(k_g, k_\lambda, q_g + q_\lambda). \quad (172)$$

Similarly we find

$$V_1(k_\lambda, q_\lambda, -k_s, -q_s, \omega) = 4 q_\lambda q_s e^{-iq_\lambda z_g} e^{iq_s z_s} D(k_\lambda, k_s, -q_\lambda + q_s), \quad (173)$$

$$V_1(k_g, -q_g, -k_\lambda, q_\lambda, \omega) = 4 q_g q_\lambda e^{iq_g z_g} e^{-iq_\lambda z_s} D(k_g, k_\lambda, q_g - q_\lambda) \quad (174)$$

and

$$V_1(k_\lambda, -q_\lambda, -k_s, -q_s, \omega) = -4 q_\lambda q_s e^{iq_\lambda z_g} e^{iq_s z_s} D(k_\lambda, k_s, q_\lambda + q_s). \quad (175)$$

With these expressions, equation (168) becomes

$$LHS = -\frac{1}{2\pi} \int dk_\lambda q_\lambda \left[e^{iq_\lambda(z_s - z_g)} D(k_g, k_\lambda, q_g + q_\lambda) D(k_\lambda, k_s, -q_\lambda + q_s) \right.$$
$$\left. + e^{iq_\lambda(z_g - z_s)} D(k_g, k_\lambda, q_g - q_\lambda) D(k_\lambda, k_s, q_\lambda + q_s) \right]. \quad (176)$$

To separate this expression into imaging-only and inversion-only parts we write all data terms as Fourier integrals over vertical wavenumbers as

$$D(k_g, k_\lambda, q_g + q_\lambda) = \int dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g, k_\lambda, z_1), \tag{177}$$

$$D(k_\lambda, k_s, -q_\lambda + q_s) = \int dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda, k_s, z_2) \tag{178}$$

$$D(k_g, k_\lambda, q_g - q_\lambda) = \int dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g, k_\lambda, z_3) \tag{179}$$

$$D(k_\lambda, k_s, q_\lambda + q_s) = \int dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda, k_s, z_4), \tag{180}$$

then we rewrite equation (176) as

$$\begin{aligned} LHS = -\frac{1}{2\pi} \int dk_\lambda q_\lambda \Big[& e^{iq_\lambda(z_s-z_g)} \int dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g, k_\lambda, z_1) \int dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda, k_s, z_2) \\ & + e^{iq_\lambda(z_g-z_s)} \int dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g, k_\lambda, z_3) \int dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda, k_s, z_4) \Big]. \end{aligned} \tag{181}$$

Depending on the relative position of the two pseudo-depths z_1, z_2, z_3 and z_4 we can further separate the last expression into

$$\begin{aligned} LHS = & -\frac{1}{2\pi} \int dk_\lambda q_\lambda e^{iq_\lambda(z_s-z_g)} \\ & \Big[\int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g, k_\lambda, z_1) \int_{-\infty}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda, k_s, z_2) \delta(z_2 - z_1) \\ & + \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g, k_\lambda, z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda, k_s, z_2) \\ & + \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g, k_\lambda, z_1) \int_{-\infty}^{z_1-\epsilon} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda, k_s, z_2) \Big] \\ & - \frac{1}{4\pi} \int_{-\infty}^{\infty} dk_\lambda q_\lambda e^{iq_\lambda(z_g-z_s)} \\ & \Big[\int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g, k_\lambda, z_3) \int_{-\infty}^{\infty} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda, k_s, z_4) \delta(z_3 - z_4) \\ & + \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g, k_\lambda, z_3) \int_{z_3+\epsilon}^{\infty} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda, k_s, z_4) \end{aligned}$$

$$+\int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4)\Bigg] \tag{182}$$

or, after solving the integrals containing the delta function,

$$\begin{aligned}LHS = &-\frac{1}{2\pi}\int_{-\infty}^{\infty} dk_\lambda q_\lambda e^{iq_\lambda(z_s-z_g)} \Bigg[2\pi \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_s)} D(k_g,k_\lambda,z_1) D(k_\lambda,k_s,z_1)\\ &+\int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g,k_\lambda,z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda,k_s,z_2)\\ &+\int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g,k_\lambda,z_1) \int_{-\infty}^{z_1-\epsilon} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda,k_s,z_2)\Bigg]\\ &-\frac{1}{4\pi}\int_{-\infty}^{\infty} dk_\lambda q_\lambda e^{iq_\lambda(z_g-z_s)} \Bigg[2\pi \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g+q_s)} D(k_g,k_\lambda,z_3) D(k_\lambda,k_s,z_3)\\ &+\int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{z_3+\epsilon}^{\infty} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4)\\ &+\int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4)\Bigg].\end{aligned} \tag{183}$$

The first term in each square bracket in equation (183) is similar to an amplitude corrector (see e.g. Shaw (2005)) and it will be ignored for the purpose of this paper. The second term in the first square bracket and the third in the second square bracket are similar to depth correctors (see e.g. Shaw (2005), Liu et al. (2006) Ramirez and Otnes (2007)). For the purpose of this paper we will only keep these (imaging) terms and arrive to our final expression

$$\begin{aligned}LHS = -\frac{1}{2\pi}\int dk_\lambda q_\lambda &\Bigg[e^{iq_\lambda(z_s-z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g,k_\lambda,z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda,k_s,z_2)\\ &+e^{iq_\lambda(z_g-z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4)\Bigg].\end{aligned} \tag{184}$$

Combining equations (155) and (184) we find the imaging part of V_2 to be

$$V_2^{IM}(k_g,-q_g,-k_s,-q_s,\omega) = \frac{2q_g q_s e^{i(q_g z_g + q_s z_s)}}{\pi} \int_{-\infty}^{\infty} dk_\lambda q_\lambda$$

$$\times \left[e^{iq_\lambda(z_s-z_g)} \int_{-\infty}^{\infty} dz_1 e^{iz_1(q_g+q_\lambda)} D(k_g,k_\lambda,z_1) \int_{z_1+\epsilon}^{\infty} dz_2 e^{iz_2(-q_\lambda+q_s)} D(k_\lambda,k_s,z_2) \right.$$

$$\left. + e^{iq_\lambda(z_g-z_s)} \int_{-\infty}^{\infty} dz_3 e^{iz_3(q_g-q_\lambda)} D(k_g,k_\lambda,z_3) \int_{-\infty}^{z_3-\epsilon} dz_4 e^{iz_4(q_\lambda+q_s)} D(k_\lambda,k_s,z_4) \right] . \qquad (185)$$