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FORWARD SCATTERING SERIES AND SEISMIC EVENTS: FAR FIELD APPROXIMATIONS, CRITICAL AND POSTCRITICAL EVENTS*

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Abstract. Inverse scattering series is the only nonlinear, direct inversion method for the multidimensional, acoustic or elastic equation. Recently developed techniques for inverse problems based on the inverse scattering series [Weglein et al., *Geophys.*, 62 (1997), pp. 1975–1989; *Top. Rev. Inverse Problems*, 19 (2003), pp. R27–R83] were shown to require two mappings, one associating nonperturbative description of seismic events with their forward scattering series description and a second relating the construction of events in the forward to their treatment in the inverse scattering series. This paper extends and further analyzes the first of these two mappings, introduced, for 1D normal incidence, in Matson [*J. Seismic Exploration*, 5 (1996), pp. 63–78] and later extended to two dimensions in Matson [*An Inverse Scattering Series for Attenuating Elastic Multiples from Multicomponent Land and Ocean Bottom Seismic Data*, Ph.D. thesis, Department of Earth and Ocean Sciences, University of British Columbia, Vancouver, BC, Canada, 1997]. It brings a new and more rigorous understanding of the mathematics and physics underlying the calculation of terms in the forward scattering series and the events in the seismic model. The convergence of the series for 1D acoustic models is examined, and the earlier precritical analysis is extended to critical and postcritical reflections. An explanation is proposed for the divergence of the series for postcritical incident planewaves.

Key words. scattering theory, forward problem, critical reflections, postcritical reflections

AMS subject classifications. 34L25, 47A40

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1. Introduction.

Scattering theory is a form of perturbation theory. In seismic exploration, it relates the propagation of a wave in an actual medium with the propagation of the wave in a reference medium and a perturbation operator which describes the difference between the two media. The forward problem (or forward modeling) is to construct the actual wave-field, given the reference wave-field and the perturbation operator; the inverse problem is to construct the perturbation operator, given the reference wave-field everywhere and the actual wave-field on a measurement surface. The relation between these three quantities is nonlinear and cannot be given, at least so far, in a closed form in either the forward or the inverse problem. This relationship takes the form of a series which, when convergent, constructs the actual wave-field and the perturbation operator.

Inverse scattering series is the only nonlinear, direct inversion method for the multidimensional, acoustic or elastic equation. Early tests on the convergence of the entire series for an acoustic medium by Carvalho [4] were not favorable for real world application. Weglein and collaborators then developed the "subseries method" for the inverse problem (for a description and a complete history, see Weglein et al. [13] and references therein). The overall undertaking of the inverse scattering series was broken up into four tasks, which otherwise would be performed simultaneously by the series acting upon the input data. The four tasks are 1. elimination of the free surface

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multiples, 2. elimination of the internal multiples, 3. locating where rapid changes in the medium properties occur (imaging), and 4. determining the changes at those locations (inversion). These tasks were associated with subseries of the full series, subseries which, if identified, would perform their job as if no other task existed in the series. Two immediate advantages of this separation of tasks are the favorable convergence properties of the subseries and the ability to judge the effectiveness of each step before proceeding on to the next. To facilitate the identification of the task-specific subseries in the inverse series, two maps have to be constructed (see [12]): one map associates seismic events with their forward scattering series description, while the second relates the construction of events in the forward to their usage in the inverse scattering series. In this paper we advance the analysis of the first of these mappings, introduced, for 1D (one-dimensional) normal incidence, by Matson [6] and later extended by Matson to two dimensions [7].

The forward series takes as input the information about the wave-field propagating through the reference medium and about the perturbation operator and outputs the wave-field everywhere in the actual medium. This process can be regarded as creating data (primaries, free surface multiples, internal multiples) for a given model; in practice, the forward series is never used for this purpose due to its inefficiency: it takes an infinite number of terms to create any single event. The events recorded in a seismic experiment are used by the inverse series to find the perturbation and, although the relation between their creation in the forward and their exploitation in the inverse series is not one-to-one, certain analogies could provide useful hints or at least point to where various activities reside in the inverse series. The forward series does not hint at whether events will be signal or noise in the full inverse series; it only suggests where one might look for that answer in the subseries. Take multiples, for example: it turns out that the inverse scattering subseries made of terms that mimic the diagrams for multiples in the forward series is responsible for attenuating/removing such multiples from the data [1].

The forward scattering series models seismic events in a fundamentally different way from conventional nonperturbative theory, where seismic waves propagate through the medium with different velocities and are reflected and transmitted at media boundaries. To construct one event alone, the forward series needs a sequence of terms which can be viewed as a succession of propagations in the reference medium separated by different orders of scattering interactions with a point scatterer; the different terms in the perturbation series correspond to the number of scattering interactions a wave experiences. Even with these differences taken into account, the wave-field output by the forward scattering series has to agree, when the series converges, with the well-known nonperturbative results for any given seismic experiment. Precritical data has been studied by Matson [7], who showed that the expected (from wave-theory) reflected wave-field is constructed by the convergent forward scattering series in a 2D (two-dimensional) experiment. This study brings new understanding about the physical interpretation of these previous results; it also shows that the same forward series converges for critical angles and diverges for postcritical, and an explanation of this divergence is proposed.

The plan for this paper is as follows. In section 2 we present the mathematical description for the forward scattering series for a 3D (three-dimensional) earth, both in operator and nonoperator form; in section 3, following Matson [7], we apply this description to a specific, 2D seismic model, and discuss the convergence of the forward scattering series for that model. Section 4 presents an alternative method for solving for the terms in the series using saddle point analysis, which, in this setting, is equivalent to far field approximation. Section 5 presents the physical interpretation of the approximations performed in section 4. Section 6 shows the convergence of the forward scattering series for this model at the critical angle, and section 7 proposes an explanation for the divergence of the series for postcritical events. Some conclusions are given in section 8. Although in this paper we mainly focus on application of the scattering theory to seismic exploration, we mention that the same methods and discussions apply to other areas of explorative sciences like medical imaging, whole earth exploration, etc.

2. Forward scattering series. In operator form, the differential equations describing wave propagation in an actual and a reference medium can be written as

$$\mathbf{L}\mathbf{G} = -\mathbf{I} \tag{2.1}$$

and

$$\mathbf{L}_0\mathbf{G}_0 = -\mathbf{I}, \tag{2.2}$$

where $\mathbf{L}$, $\mathbf{L}_0$ and $\mathbf{G}$, $\mathbf{G}_0$ are the actual and reference differential and Green's operators, respectively, for a single temporal frequency and $\mathbf{I}$ is the identity operator. The above equations (2.1) and (2.2) assume that the source and receiver signatures have been deconvolved. The perturbation, $\mathbf{V}$, and the scattered field operator, ψ_s, are defined as

$$\mathbf{V} = \mathbf{L} - \mathbf{L}_0, \tag{2.3}$$

$$\psi_{\mathbf{s}} = \mathbf{G}{-}\mathbf{G}_0. \tag{2.4}$$

The fundamental equation of scattering theory, the Lippmann–Schwinger equation, relates ψ_s, $\mathbf{G}_0$, $\mathbf{V}$, and $\mathbf{G}$ (see, e.g., [10]):

$$\psi_s = \mathbf{G}{-}\mathbf{G}_0 = \mathbf{G}_0\mathbf{V}\mathbf{G}. \tag{2.5}$$

When $\mathbf{G}$ corresponds to the pressure field in an inhomogeneous acoustic medium, an example of $\mathbf{L}$, $\mathbf{L}_0$, and $\mathbf{V}$ is (see, e.g., [5])

$$\mathbf{L} = \frac{\omega^2}{\kappa} + \nabla \cdot \left(\frac{1}{\rho}\nabla\right), \tag{2.6}$$

$$\mathbf{L}_0 = \frac{\omega^2}{\kappa_0} + \nabla \cdot \left(\frac{1}{\rho_0}\nabla\right), \tag{2.7}$$

and

$$\mathbf{V} = \omega^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa_0}\right) + \nabla \cdot \left[\left(\frac{1}{\rho} - \frac{1}{\rho_0}\right)\nabla\right], \tag{2.8}$$

where κ, κ_0, ρ, and ρ_0 are the actual and reference bulk moduli and densities, respectively. If the density is constant $(\rho = \rho_0 = const.)$, the above expressions become

$$\mathbf{L} = \frac{\omega^2}{\kappa}, \tag{2.9}$$

$$\mathbf{L}_0 = \frac{\omega^2}{\kappa_0}, \tag{2.10}$$

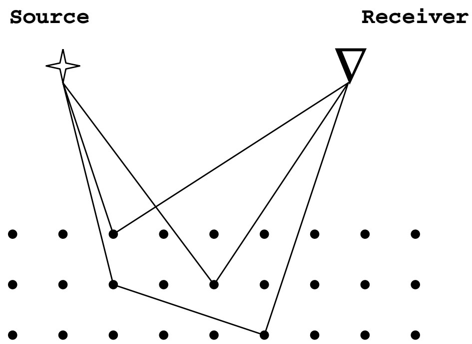

FIG. 2.1. *Graphical representation of the terms in the forward scattering series: the first term is an integral over all* 1*-interaction events, the second term is an integral over all* 2*-interactions events, etc.*

and

$$\mathbf{V} = \omega^2 \left(\frac{1}{\kappa} - \frac{1}{\kappa_0} \right). \tag{2.11}$$

For an elastic isotropic actual and a homogeneous reference medium, the expressions for $\mathbf{L}$, $\mathbf{L}_0$, and $\mathbf{V}$ are different and given, e.g., in [9].

Equation (2.5) can be expanded in an infinite series by repeatedly substituting $\mathbf{G} = \mathbf{G}_0 - \mathbf{G}_0\mathbf{V}\mathbf{G}$ into the right-hand side to obtain

$$\psi_s \equiv \mathbf{G} - \mathbf{G}_0 = \mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \mathbf{G}_0\mathbf{V}\mathbf{G}_0\mathbf{V}\mathbf{G}_0 + \cdots \quad . \tag{2.12}$$

This series constructs the scattered field operator ψ_s as a series of terms formed as propagations in the reference medium ($\mathbf{G}_0$) and interactions with the inhomogeneity ($\mathbf{V}$). Note that the nth term in this series is of order n in the perturbation operator $\mathbf{V}$ and, in fact, can be written as $(\psi_s)_n \equiv \mathbf{G}_0 (\mathbf{V}\mathbf{G}_0)^n$.

For the previous example (constant density case), define $k_0 = \frac{\omega}{c_0}$ and $\alpha = \left(1 - \frac{c_1^2}{c_0^2}\right)$, where c_1 and c_0 are the actual and the reference medium velocities, respectively; the series becomes

$$\begin{aligned} \psi_s(\mathbf{r}_g|\mathbf{r}_s;\omega) = & \int_{\mathbf{V}} G_0(\mathbf{r}_g|\mathbf{r}';\omega)\, k_0^2 \alpha(\mathbf{r}')\, G_0(\mathbf{r}'|\mathbf{r}_s;\omega)\, d\mathbf{r}' \\ & + \int_{\mathbf{V}} G_0(\mathbf{r}_g|\mathbf{r}';\omega)\, k_0^2 \alpha(\mathbf{r}') \int_{\mathbf{V}} G_0(\mathbf{r}'|\mathbf{r}'';\omega)\, k_0^2 \alpha(\mathbf{r}'')\, G_0(\mathbf{r}''|\mathbf{r}_s;\omega)\, d\mathbf{r}'' d\mathbf{r}' \\ & + \cdots, \end{aligned} \tag{2.13}$$

where the integrals are 3D volume integrals taken over the inhomogeneity $\mathbf{V}$. For an easy physical interpretation of this series, consider the perturbation $\mathbf{V}$ to be composed of point scatterers separated by the reference medium. The first term in the series for the scattered field (2.13) represents a summation over all 1-interaction events, i.e., events formed from a wave propagating from the source location $\mathbf{r}_s$ to the scatterer location at $\mathbf{r}'$, $G_0(\mathbf{r}'|\mathbf{r}_s;\omega)$, interacting with the scatterer at $\mathbf{r}'$, $k_0^2\alpha(\mathbf{r}')$, and propagating to the receiver location at $\mathbf{r}_g$, $G_0(\mathbf{r}_g|\mathbf{r}';\omega)$. The second term represents a summation over all 2-interaction events and so on. Note that, as stated before, the propagations between source, receiver, and scatterers occur only in the reference medium, i.e., with the Green's function G_0, even though the speed of the wave in the actual medium is different from the speed of the wave in the reference medium. A picture of the physical interpretation of these terms is shown in Figure 2.1.

3. A 2D seismic profile. Matson [6, 7] describes the propagation of a wave-field in a given 1D or 2D medium, using the forward scattering series. We use the same 2D model in this paper to give an alternate derivation, and a physical interpretation for that derivation, for the Matson [7] result. The model is a half-space earth with no lateral variance and an interface at z_1; the scattering perturbation for this model is, therefore,

$$V(z') = k_0^2 \alpha H(z' - z_1), \tag{3.1}$$

where, as before, $\alpha = 1 - c_0^2/c_1^2$, c_1 is the velocity in the second medium, c_0 is the velocity in the reference medium, and H is the Heaviside step function.

The propagations in the reference medium are described by the 2D Green's function (see, e.g., [2])

$$G_0(x_g, z_g|x_s, z_s; \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik_s(x_g - x_s)} e^{i\nu_{0s}|z_g - z_s|}}{2i\nu_{0s}} dk_s, \tag{3.2}$$

where k_s and ν_{0s} are the horizontal and the vertical wavenumber, respectively, of the reference medium ($\nu_{0s}^2 + k_s^2 = \omega^2/c_0^2$). Rewriting G_0 as

$$G_0(x_g, z_g|x_s, z_s; \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-ik_s x_s}}{2i\nu_{0s}} \phi_0(x_g, z_g|k_s, z_s; \omega) dk_s \tag{3.3}$$

with $\phi_0(x_g, z_g|k_s, z_s; \omega) = e^{i(k_s x_g + \nu_{0s}|z_g - z_s|)}$, it is apparent that G_0 represents a superposition of weighted planewaves. This motivates the use of a planewave component as the incident wave with the remark that one can construct solutions for point sources from planewave solutions by performing the above-mentioned weighted integration. Denote by P the actual wave-field and by P_0, P_1, etc., the corresponding term in the forward scattering series. For simplicity consider the source location to be (0, 0); the Born series takes the form

$$\begin{aligned} P(x_g, z_g|k; \omega) &= e^{i(kx_g + \nu_0 z_g)} \\ &\quad + \int_{z_1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}|z_g - z'|}}{2i\nu_0} dk_g k_0^2 \alpha P_0(x', z'|k; \omega) dx' dz' \\ &\quad + \int_{z_1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}|z_g - z'|}}{2i\nu_0} dk_g k_0^2 \alpha P_1(x', z' > z_1|k; \omega) dx' dz' \\ &\quad + \cdots . \end{aligned} \tag{3.4}$$

Note that the incoming wave hits all the scatterers at once; each scatterer then emits a cylindrical wave which propagates to the receiver or to another scatterer. Each term in the forward series represents the response, at the receiver, after a certain number of interactions: the zeroth term represents the direct arrival, the first term represents the wave-field after one interaction with a point scatterer, and so on. To construct even the simplest event, one needs an infinite number of terms in the forward series. To obtain the total wave-field at the receiver we have to solve the integrals in the previous expression. Following Matson [7], we solve for the first term in the series

$$P_1(x_g, z_g|k; \omega) = \int_{z_1}^{\infty} \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}|z_g - z'|}}{2i\nu_0} dk_g k_0^2 \alpha e^{i(kx' + \nu_0 z')} dx' dz'. \tag{3.5}$$

Begin by switching the order of integration so that the integration with respect to dx' is performed first. Hence

(3.6)
$$P_1(x_g, z_g|k;\omega) = \frac{1}{2\pi}\int_{z_1}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i(k-k_g)x'}dx'\right)e^{ikgx_g}e^{i\nu_{0g}|z_g-z'|}e^{i\nu_0 z'}\frac{k_0^2\alpha}{2i\nu_{0g}}dk_g dz'.$$

Using

$$\int_{-\infty}^{\infty} e^{i(k-k_g)x'}dx' = 2\pi\delta(k_g - k), \tag{3.7}$$

P_1 becomes

$$P_1(x_g, z_g|k;\omega) = \int_{z_1}^{\infty}\int_{-\infty}^{\infty}\delta(k_g - k)e^{ikgx_g}e^{i\nu_{0g}|z_g-z'|}e^{i\nu_0 z'}\frac{k_0^2\alpha}{2i\nu_{0g}}dk_g dz'. \tag{3.8}$$

Using the properties of the delta function, we see that the inside integral switches $k_g \to k$ and hence $\nu_{0g} \to \nu_0$, and so the expression becomes

$$P_1(x_g, z_g|k;\omega) = \frac{k_0^2\alpha}{2i\nu_0}e^{ikx_g}\int_{z_1}^{\infty} e^{i\nu_0|z_g-z'|}e^{i\nu_0 z'}dz'. \tag{3.9}$$

There are two cases to be considered at this point: $z_g < z_1$ for the reflected P_1 and $z_g > z_1$ for the transmitted part. The first enters into the series for the total reflected field, while the second is used either in the series for transmitted wave-field or for the calculation of P_2 (reflected or transmitted). We have

$$P_1(x_g, z_g < z_1|k;\omega) = \frac{k_0^2\alpha}{2i\nu_0}e^{ikx_g}e^{-i\nu_0 z_g}\int_{z_1}^{\infty} e^{i\nu_0 2z'}dz'. \tag{3.10}$$

The last integral,

$$\int_{z_1}^{\infty} e^{i\nu_0 2z'}dz', \tag{3.11}$$

is not defined in the Riemannian sense because the integrand oscillates, preserving its amplitude towards infinity. We are going to define this integral to be the value of the antiderivative of the integrand calculated at its finite boundary z_1, i.e.,

$$\int_{z_1}^{\infty} e^{i\nu_0 2z'}dz' = -\frac{e^{i\nu_0 2z_1}}{2i\nu_0}. \tag{3.12}$$

This definition is consistent with considering that the reference medium is attenuating the wave-field which will vanish at infinity. The attenuation is introduced in the equations through an imaginary part in the velocity c_0 (see [2, Chapter 5, equations 5.87 and 5.88]) so that the new velocity c_0^{new} is

$$\frac{1}{c_0^{new}} = \frac{1}{c_0} + i\varepsilon,$$

with ε being a small parameter such that $\varepsilon > 0$ for $\omega > 0$. It is easy to see that, with this new effective velocity, the value of the integral is indeed the one defined in (3.12).

The final expression for P_1 is hence

$$P_1(x_g, z_g < z_1|k;\omega) = \frac{k_0^2\alpha}{4\nu_0^2} e^{ikx_g} e^{i\nu_0(2z_1 - z_g)}. \tag{3.13}$$

The same integration procedure is used for the calculation of P_2, P_3, etc. The calculated series for the scattered field (also denoted by P) is

$$P(x_g, z_g < z_1|k;\omega) = e^{ikx_g} e^{i\nu_0(2z_1 - z_g)} \left[\frac{1}{4}\frac{k_0^2\alpha}{\nu_0^2} + \frac{1}{8}\left(\frac{k_0^2\alpha}{\nu_0^2}\right)^2 + \frac{5}{64}\left(\frac{k_0^2\alpha}{\nu_0^2}\right)^3 + \cdots \right] \tag{3.14}$$

and indicates a certain regularity after some algebraic operations: the series is recognized to be the Taylor series of $\sqrt{1 - \frac{k_0^2\alpha}{\nu_0^2}}$ about $\frac{k_0^2\alpha}{\nu_0^2} = 0$ (a rigorous proof is given in the appendix). The ratio test indicates that the series converges for $\left|\frac{k_0^2\alpha}{\nu_0^2}\right| < 1$. By writing $\nu_0 = k_0 \cos\theta$, with θ being the incidence angle of the incoming planewave, this condition becomes

$$\sin\theta < \frac{c_0}{c_1} < \left(1 + \cos^2\theta\right)^{1/2}. \tag{3.15}$$

This last relation can be viewed in the following two ways:

1. First, for a fixed incidence angle θ, this is a restriction on the velocity contrast between the reference and the actual medium. In particular, for $\theta = 0$ (normal incidence) the left inequality is satisfied for any two velocities; the right inequality becomes $c_0 < \sqrt{2}c_1$, a result obtained in Matson [6].
2. Second, for a fixed velocity model, the restriction is on the incident angle. Note that, given any two velocities c_0 and c_1, one of the two inequalities is automatically satisfied. For $c_0 > c_1$, the condition reads $\frac{c_0}{c_1} < (1 + \cos^2\theta)^{1/2}$ or $\sin^2\theta < 1 + \alpha$ with $\alpha < 0$.

For $c_0 < c_1$, the condition becomes $\sin\theta < \frac{c_0}{c_1}$ or $\theta < \theta_c$, where θ_c is the critical angle $\theta_c = \sin^{-1}(c_0/c_1)$. When the series converges, the limit is

$$2\frac{\nu_0^2}{k_0^2\alpha}\left[1 - \sqrt{1 - \frac{k_0^2\alpha}{\nu_0^2}}\right] - 1 = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1}, \tag{3.16}$$

and so the final expression for the reflected field is

$$P(x_g, z_g < z_1|k;\omega) = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1} e^{ikx_g} e^{i\nu_0(2z_1 - z_g)}, \tag{3.17}$$

which is the expected result from nonperturbative theory (see, e.g., [2]).

4. An alternative derivation using saddle point approximations. The calculation of

$$P_1(x_g, z_g|k;\omega) = \int_{z_1}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}|z_g - z'|}}{2i\nu_0} dk_g\right) k_0^2\alpha e^{i(kx' + \nu_0 z')} dx'dz' \tag{4.1}$$

contains a reordering of integrals: in the original expression the dk_g integral should be solved first, then the dx', and finally the dz' integral. As we saw in the previous section, the calculations are greatly simplified if the dx' integration is performed first, then the dk_g, and finally the dz' integration. However, this kind of operation has to be performed with great care since it might impose some restrictions, which might change the result obtained from solving the integrals in the original order.

The theorem which deals with interchanging integrals is Fubini's theorem. It states that when a function f is integrable on $R^n = R^k \times R^m$, the iterated integrals of f over R^k and R^m exist and

$$\int_{R^n} f = \int_{R^k}\int_{R^m} f(x,y)dydx = \int_{R^m}\int_{R^k} f(x,y)dxdy. \tag{4.2}$$

The theorem gives sufficient conditions for interchanging the order of integrals, but those conditions are not necessary. For example, you can have a function non-integrable over R^n for which the integration in both directions would yield the same result. The only way to show that the interchange of integrals does not hold is to calculate the integrals in both direction and obtain different results. However, to calculate the dk_g integral first in the expression (4.1) means to find a closed form for the Green's function (3.2), which is not possible. For an in-depth analysis of the cylindrical functions, see [11].

In this section we show that the interchange of integrals yields the same result as the far field approximation of the integrals in question. The Fubini theorem does not apply here because the function to be doubly integrated is not integrable. To be more specific, the integral representation of the Dirac delta function,

$$\int_{-\infty}^{\infty} e^{i(k-k_g)x'}dx', \tag{4.3}$$

is meaningless in the strict Riemannian sense.

Recalculate P_1 using saddle point approximations for the two integrals involved without switching the order of integration, and show that the result is the one obtained in Matson [7]. Saddle point or stationary phase approximation gives the leading asymptotic behavior of generalized Fourier integrals, i.e., of the form $\int_{-\infty}^{\infty} F(p)e^{\omega f(p)}dp$, having stationary points, i.e., points p_s such that $f'(p_s) = 0$. The idea of the method is to use the analyticity of the integrand to justify deforming the path of integration to a new path on which $f(p)$ has a constant imaginary path. How the contour is deformed depends on the singularities and branch cuts of the integrand. Once this has been done, the integral may be found asymptotically ($\omega \to \infty$) to be

$$\int_{-\infty}^{\infty} F(p)e^{\omega f(p)}dp \ \sim \left|\frac{2\pi}{\omega f''(p_s)}\right|^{1/2} F(p_s)e^{isign\left(f''(p_s)\right)\frac{\pi}{4}}\exp\left[\omega f(p_s)\right]. \tag{4.4}$$

To calculate P_1 in (4.1), start by rewriting

$$G_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik_g(x_g-x')}e^{i\nu_{0g}\left|z_g-z'\right|}}{2i\nu_0}dk_g \tag{4.5}$$

as

$$G_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)e^{\omega f(p)}dp, \tag{4.6}$$

where

$$F(p) = \frac{1}{2i\sqrt{1/c_0^2 - p^2}} \tag{4.7}$$

and

$$f(p) = i\left[p(x_g - x') + |z_g - z'|\sqrt{\frac{1}{c_0^2} - p^2}\right], \tag{4.8}$$

p being the horizontal slowness $p = \frac{k_g}{\omega}$. Note that, due to the square root, $F(p)$ defines two branch cuts in the complex p plane; the branch cuts are hyperbolas in the first and third quadrant and are running very close to the coordinate axis. (For a full discussion of the branch cuts of F, see [2, Box 6.2].) By definition, branch cuts are lines of discontinuities for $F(p)$ and here are given by $Im\sqrt{1/c_0^2 - p^2} = 0$. This means that when the new integration path (see Figure 6.6 in [2]) intersects these branch cuts, $F(p)$ is discontinuous and hence not analytic. This apparent problem can be avoided if we relax the condition $Im\sqrt{1/c_0^2 - p^2} \geq 0$ along the integration path. Instead we allow $Im\sqrt{1/c_0^2 - p^2}$ to change sign at each branch cut intersection which, for the integration path, is equivalent to a transition to a different Riemann sheet. The integrand looses physical interpretation while on another Riemann sheet but gains analyticity. However, the two intersections with the branch cut insure two sign changes and the emergence of the integrand with the correct sign at the saddle point. (Eventually the integrand is going to be expanded in a Taylor series at that point, and the rest of the path is going to be discarded.) To calculate the location of the saddle point, equate the derivative of f with zero; this gives

$$p_s = \frac{x_g - x'}{c_0 d'}, \tag{4.9}$$

with $d' = \sqrt{(z_g - z')^2 + (x_g - x')^2}$. Calculate

$$f(p_s) = i\frac{d'}{c_0}, \tag{4.10}$$

$$f''(p_s) = -\frac{ic_0 d'^3}{|z_g - z'|^2}, \tag{4.11}$$

$$F(p_s) = \frac{c_0 d'}{2i\,|z_g - z'|}, \tag{4.12}$$

and plug them into the above formula (4.4) to obtain

$$G_0 \sim \frac{1}{4\pi i}\left(\frac{2\pi c_0}{i\omega d'}\right)^{1/2} e^{ik_0 d'}. \tag{4.13}$$

(Compare with the approximation for $i\pi H_0^{(1)}(\omega/c_0 d')$, the Green's function for the 2D Helmholtz equation, where $H_0^{(1)}$ is the Hankel function of the first kind, given by formula (5.3.69) in [8].) With this approximation, expression (4.1) for P_1 becomes

$$P_1(x_g, z_g|k;\omega) = \frac{1}{4\pi i}\int_{z_1}^{\infty}\int_{-\infty}^{\infty} e^{ik_0 d'}\left(\frac{2\pi c_0}{i\omega d'}\right)^{1/2} k_0^2 \alpha e^{i(kx' + \nu_0 z')} dx' dz' \tag{4.14}$$

or

$$P_1(x_g, z_g|k;\omega) = \frac{k_0^{3/2}\alpha}{2\pi i}\sqrt{\frac{\pi}{2i}}\int_{z_1}^{\infty} e^{i\nu_0 z'}\int_{-\infty}^{\infty}\frac{e^{i\omega\left(\frac{d'}{c_0}+\frac{k}{\omega}x'\right)}}{\sqrt{d'}}dx'dz'. \tag{4.15}$$

Again, the innermost integral has the form

$$I = \int_{-\infty}^{\infty} F(x')e^{\omega f(x')}dx' \tag{4.16}$$

with $F(x') = \frac{1}{\sqrt{d'}}$ and $f(x') = i(\frac{d'}{c_0} + \frac{k}{\omega}x')$. Note that the integrand has no branch cuts this time since $d' = \sqrt{(z_g - z')^2 + (x_g - x')^2}$ is always positive; the saddle point is x'_s such that

$$x_g - x'_s = |z_g - z'|\,\frac{k}{\nu_0}, \tag{4.17}$$

and so we have

$$f(x'_s) = i\left(\frac{\nu_0}{\omega}\,|z_g - z'| + \frac{k}{\omega}x_g\right), \tag{4.18}$$

$$f''(x'_s) = \frac{ic_0^2\nu_0^3}{\omega^3\,|z_g - z'|}, \tag{4.19}$$

and

$$F(x'_s) = \frac{1}{\sqrt{|z_g - z'|}}\sqrt{\frac{c_0\nu_0}{\omega}}. \tag{4.20}$$

Using the same high frequency approximation (4.4), we find

$$\int_{-\infty}^{\infty}\frac{e^{i\omega\left(\frac{d'}{c_0}+\frac{k}{\omega}x'\right)}}{\sqrt{d'}}dx' \sim \frac{1}{\nu_0}\sqrt{\frac{2\pi i\omega}{c_0}}e^{i\left(\nu_0|z_g-z'|+kx_g\right)}. \tag{4.21}$$

Substituting this into the expression (4.15) for P_1, we obtain

$$P_1(x_g, z_g|k;\omega) = \frac{k_0^2\alpha}{2i\nu_0}e^{ikx_g}\int_{z_1}^{\infty} e^{i\nu_0|z_g-z'|}e^{i\nu_0 z'}dz', \tag{4.22}$$

which is the same result as that obtained before by switching the order of integration. The rest of the terms in the series for P can be similarly shown to resemble the expressions given by Matson [7].

5. Physical interpretation of the approximations. The two far field approximations performed in the previous derivation have an easily understandable physical interpretation. The approximation of the first integral in the expression of P_1 represents the most important contribution arriving at the receiver from each point scatterer (see Figure 5.1).

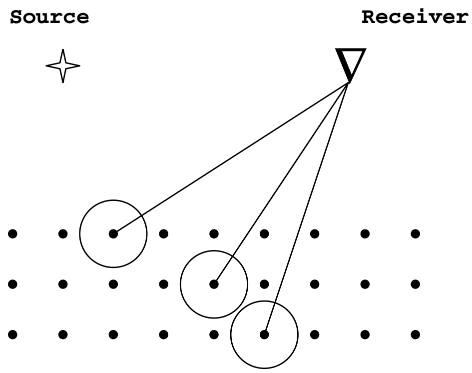

FIG. 5.1. *The physical interpretation of the approximation of the first integral in the calculation of P_1.*

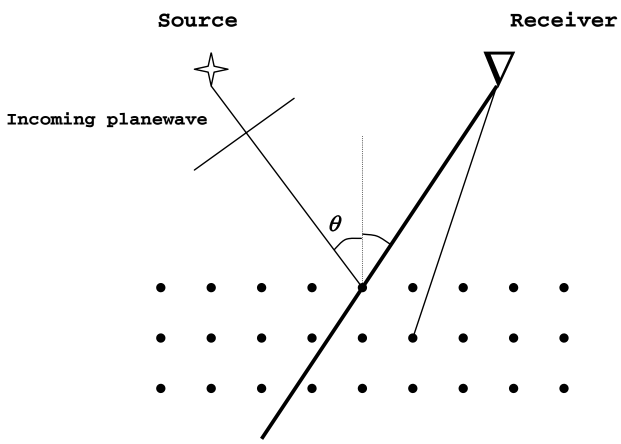

FIG. 5.2. *The physical interpretation of the approximation of the second integral in the calculation of P_1.*

As the figure shows, each scatterer behaves as a point source producing a wave propagating in all directions described by the Green's function given by (3.2). However, when the integral is approximated using saddle point techniques, only the direction of propagation bringing in the highest contribution is kept. The result given by (4.13),

$$G_0 \sim \frac{1}{4\pi i}\left(\frac{2\pi c_0}{i\omega d'}\right)^{1/2} e^{ik_0 d'}, \tag{5.1}$$

represents the part arriving from the scatterer to the receiver along the straight line connecting them, multiplied by a coefficient which accounts for the dismissal of all the other directions.

The approximation of the second integral in the expression of P_1 picks out the most important contribution arriving at the receiver from the totality of incoming rays. Here, the main contribution is found to be the one from the rays that make an angle equal to the incident's planewave angle with the vertical (see Figure 5.2); this can be seen from the expression of the saddle point for the x' integration:

$$x_g - x'_s = |z_g - z'| \frac{k'}{\nu_0}. \tag{5.2}$$

The last integral in the expression of P_1 is a 1D integral along the thick line shown in Figure 5.2. Even though the parameter of integration is z', there is a certain relation between z' and x', given by (5.2), such that the direction of integration is tilted at an angle equal to the incident angle rather than vertical. The lack of symmetry in this last integral is expected since the model is not symmetric: the discussion here is for a planewave component of a line source and a line receiver. It is anticipated that the symmetry would be recovered in the line source–line receiver case.

6. Convergence at the critical angle. The forward scattering series for the reflected wave-field for the model discussed in this paper is (see Matson [7])

$$P(x_g, z_g < z_1|k;\omega) = e^{ikx_g} e^{i\nu_0(2z_1 - z_g)} \left[\frac{1}{4}\frac{k_0^2\alpha}{\nu_0^2} + \frac{1}{8}\left(\frac{k_0^2\alpha}{\nu_0^2}\right)^2 + \frac{5}{64}\left(\frac{k_0^2\alpha}{\nu_0^2}\right)^3 + \cdots \right]. \tag{6.1}$$

The ratio test shows convergence for $|\frac{k_0^2\alpha}{\nu_0^2}| < 1$, divergence for $|\frac{k_0^2\alpha}{\nu_0^2}| > 1$, and is inconclusive for $|\frac{k_0^2\alpha}{\nu_0^2}| = 1$. When $c_0 < c_1$, this last condition is equivalent to $\frac{k_0^2\alpha}{\nu_0^2} = 1$, which in turn is equivalent to $\theta = \theta_c$; i.e., the incident angle is the critical angle. In other words, the forward series is convergent for precritical incidence and divergent for postcritical incidence; no information is found about the critical incidence. For a critical incident planewave, the series becomes

$$P(x_g, z_g < z_1|k;\omega) = e^{ikx_g} e^{i\nu_0(2z_1 - z_g)} \left[\frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + \cdots \right]. \tag{6.2}$$

Rewrite

$$R = \frac{1}{4} + \frac{1}{8} + \frac{5}{64} + \frac{7}{128} + \cdots = \sum_{n=2}^{\infty} \frac{1}{n!}\frac{1\cdot 3\cdot 5 \ldots (2n-3)}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{\Gamma(n+1/2)}{(n+1)!\Gamma(1/2)}. \tag{6.3}$$

Note that the series has the form $\sum_{n=2}^{\infty} a_n$ with $a_n = \frac{1}{n!}\frac{1\cdot3\cdot5\ldots(2n-3)}{2^{n-1}}$, and so

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n-1} - 1\right) = \frac{3}{2} > 1. \tag{6.4}$$

Hence Raabe's convergence test shows convergence. (For a full discussion of this convergence test, see [3].) The conclusion is that the forward scattering series for this model converges at the critical angle as well. Note that, in this case, the sum of the series, which corresponds to the reflection coefficient, is $R = 1$.

7. Postcritical divergence. For a $c_0 < c_1$ model, the forward series converges for precritical and critical incidence and diverges for postcritical incidence. From wave nonperturbative theory, the reflection coefficient R, which should be constructed by the forward scattering series, is

- $R = \frac{\nu_0 - \nu_1}{\nu_0 + \nu_1} < 1$ for precritical incidence. In this case both ν_0 and ν_1 are real.

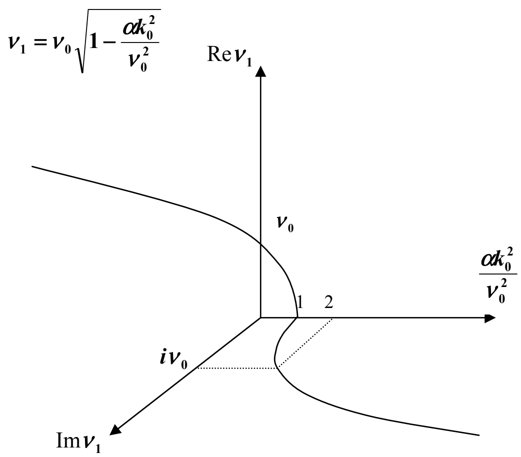

FIG. 7.1. *The graph of ν_1 as a function of $\frac{\alpha k_0^2}{\nu_0^2}$.*

- $R = 1$ for critical incidence. In this case $\nu_1 = 0$.
- $R = \frac{\nu_0-\nu_1}{\nu_0+\nu_1}$ for postcritical incidence. In this case ν_1 is purely imaginary, and hence R is complex. However, $|R| = 1$, and the complexity of R is attributed to a phase-shift of the emerging wave after hitting the interface due to the evanescent waves created in the second medium.

The term $\alpha k_0^2/\nu_0^2 = 1 - \nu_1^2/\nu_0^2$ is > 1 exactly when ν_1 becomes imaginary. In fact, if for this case one writes $R = e^{i\varepsilon}$, where ε is the phase-shift of the wave-field, then $\alpha k_0^2/\nu_0^2 = 1 + \tan^2 \varepsilon/2$, enforcing the earlier statement that the divergence is due to the phase-shift of the reflected wave. In other words, it is the impossibility of constructing a complex number ν_1 as a series of real numbers (powers of ν_0) which leads to the divergence of the series. The graph of ν_1 as a function of $\alpha k_0^2/\nu_0^2$ is shown in Figure 7.1.

For $c_0 < c_1$we have that $\alpha > 0$, so we are looking at the positive x-axis of the graph; if the velocity model is fixed, αk_0^2 is a constant. The vertical wavenumber of the propagating wave in the actual medium, ν_1, is equal to ν_0 when $\alpha k_0^2/\nu_0^2 = 0$, i.e., at normal incidence. When $\alpha k_0^2/\nu_0^2 = 1$ (at critical incidence), ν_1 is zero, showing that there is no propagation into the second medium. When $\alpha k_0^2/\nu_0^2 > 1$ (postcritical incidence), ν_1 is complex, and it becomes unrecoverable by a Taylor series written at $\alpha k_0^2/\nu_0^2 = 0$; the series is now divergent. For $c_0 > c_1$ it seems like this problem does not exist. In this case there is no critical angle, and so the vertical wavenumber ν_1 never becomes complex. However, the series inherits the divergent behavior for $\alpha k_0^2/\nu_0^2 < -1$ due to the singularity at $\alpha k_0^2/\nu_0^2 = 1$. For any value of $\alpha k_0^2/\nu_0^2$ outside the unit sphere centered at the origin the series will diverge due to that same singularity.

8. Conclusion. We have shown that the interchange of certain integrals in the calculation of terms in the forward scattering series yields the same result as the far field approximations of those integrals. The later approach allows the study of the restrictions imposed on the model by the former approach and provides new insights and physical interpretations for the terms in the forward scattering series. It is also anticipated that the new method would be more practical in the study of more complicated models (e.g., line source and receiver).

We have also proved the convergence of the forward scattering series at critical angle for the model of Matson [7] and provided an explanation for the divergence of the series for postcritical incident angles. The divergence is due to the inability of the forward scattering series to construct a complex vertical wavenumber from a series of real terms. Several possibilities for extending this result exist. First, one could introduce imaginary terms in the calculated series by using more than just the leading asymptotic behavior of the integral representation of the Hankel function, or of the dx' integral involved in the calculations. Second, one could try to make use of the evanescent part of the wave-field emanating from the scatterers to construct a complex vertical wavenumber. The evanescent part is always discarded when the asymptotic behavior of the integral representation of the Hankel function is considered; using it is attractive because it makes sense intuitively to construct an evanescent wave in the actual medium using evanescent waves in the reference medium. Third, an imaginary term in the reference velocity, and hence complex terms in the forward scattering series, could be brought in by the introduction of an absorptive reference medium. These ideas will be considered in future research.

Appendix. In section 3 we indicated how to calculate the first few terms in the forward scattering series for the reflected wave-field in a 2D vertically varying medium. We stated there that the calculated series for the scattered field for that specific model is (see (3.14))

$$P(x_g, z_g|\, k;\omega) = e^{ikx_g}e^{i\nu_0(2z_1-z_g)}\left[\frac{1}{4}\frac{k_0^2\alpha}{\nu_0^2}+\frac{1}{8}\left(\frac{k_0^2\alpha}{\nu_0^2}\right)^2+\frac{5}{64}\left(\frac{k_0^2\alpha}{\nu_0^2}\right)^3+\cdots\right], \tag{A.1}$$

which is recognized to be the Taylor series for $\sqrt{1-k_0^2\alpha/\nu_0^2}$ about $\frac{k_0^2\alpha}{\nu_0^2}=0$ after some algebraic operations are performed on it. In this section we provide a rigorous proof of this statement. The proof will proceed as follows: first we will write down the general term for the transmitted wave-field and show by induction that the expression is correct; then we will use it to calculate the general term for the reflected wave-field and show that it corresponds to the general term in the aforementioned Taylor series. The need for the general term for the transmitted field is obvious since the iteration step occurs in the transmitted wave rather than the reflected one. Once the general term for the transmitted wave-field, P_n^T, is obtained, the general term for the reflected wave-field, P_n^R, is obtained by calculating

$$\begin{aligned}&P_{n+1}^R\left(x_g, z_g < z_1|\, k;\omega\right)\\ &\quad=\int_{z_1}^{\infty}dz'\int_{-\infty}^{\infty}dx'\frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\frac{e^{ik_g(x_g-x')}e^{i\nu_{0g}|z_g-z'|}}{2i\nu_0}dk_g\right)k_0^2\alpha P_n^T\left(x', z'|\, k;\omega\right).\end{aligned} \tag{A.2}$$

To simplify the writing we introduce the notation

$$\frac{k_0^2\alpha}{\nu_0^2} = X \tag{A.3}$$

and

$$S_n = \frac{X^n}{2^n n!}\left(1+R\right)^{n+1}, \tag{A.4}$$

where

$$(1+R) = \frac{2}{X}\left[1 - Taylor\left(\sqrt{1-X}\right)\right] \tag{A.5}$$

and $Taylor\left(\sqrt{1-X}\right)$ stands for the Taylor series of $\sqrt{1-X}$ about $X = 0$. Notice that S_n is a series in X of lowest order n. Also denote by S_n^j the coefficient of the jth order in S_n, and notice that all these coefficients are zero for $j < n$.

We will prove by induction that the general term for the transmitted wave-field P_n^T for $n \geq 1$ is

$$P_n^T\left(x_g, z_g > z_1 \middle| k; \omega\right) = e^{ikx_g} e^{i\nu_0 z_g} X^n \sum_{l=0}^{n} \left[-i\nu_0\left(z_g - z_1\right)\right]^l S_l^n. \tag{A.6}$$

The first step of the induction is to verify this relation for $n = 1$, i.e., to check that

$$P_1^T = e^{ikx_g} e^{i\nu_0 z_g} X \left\{S_0^1 + \left[-i\nu_0\left(z_g - z_1\right)\right] S_1^1\right\}. \tag{A.7}$$

Note that $S_0^1 = 1/4$ and $S_1^1 = 1/2$, and hence this is the expression (2.25) found in Matson [7]. For the second step of the induction we assume that the relation (A.6) for P_n^T is true, and we calculate P_{n+1}^T and show that it has the same form; i.e., we want to prove that

$$P_{n+1}^T\left(x_g, z_g > z_1 \middle| k; \omega\right) = e^{ikx_g} e^{i\nu_0 z_g} X^{n+1} \sum_{l=0}^{n+1} \left[-i\nu_0\left(z_g - z_1\right)\right]^l S_l^{n+1}. \tag{A.8}$$

We have

$$\begin{aligned} P_{n+1}^T &= \int_{z_1}^{\infty} dz' \int_{-\infty}^{\infty} dx' \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}\left|z_g - z'\right|}}{2i\nu_0} dk_g\right) k_0^2 \alpha P_n^T\left(x', z' \middle| k; \omega\right) \\ &= \int_{z_1}^{\infty} dz' \int_{-\infty}^{\infty} dx' \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}\left|z_g - z'\right|}}{2i\nu_0} dk_g\right) \\ &\qquad \times k_0^2 \alpha e^{ikx'} e^{i\nu_0 z'} X^n \sum_{l=0}^{n} \left[-i\nu_0\left(z' - z_1\right)\right]^l S_l^n. \end{aligned} \tag{A.9}$$

We now solve the dk_g and the dx' by either one of the two methods described in the text and obtain

$$\begin{aligned} P_{n+1}^T &= \int_{z_1}^{\infty} dz' \frac{k_0^2 \alpha}{2i\nu_0} e^{ikx_g} e^{i\nu_0\left|z_g - z'\right|} e^{i\nu_0 z'} X^n \sum_{l=0}^{n} \left[-i\nu_0\left(z' - z_1\right)\right]^l S_l^n \\ &= e^{ikx_g} X^{n+1} \frac{\nu_0}{2i} \int_{z_1}^{\infty} dz' e^{i\nu_0\left|z_g - z'\right|} e^{i\nu_0 z'} \sum_{l=0}^{n} \left[-i\nu_0\left(z' - z_1\right)\right]^l S_l^n \\ &= e^{ikx_g} X^{n+1} \frac{(-i\nu_0)}{2} \sum\nolimits_{l=0}^{n} \int_{z_1}^{\infty} dz' e^{i\nu_0\left|z_g - z'\right|} e^{i\nu_0 z'} \left[-i\nu_0\left(z' - z_1\right)\right]^l S_l^n. \end{aligned} \tag{A.10}$$

We split the integral into two integrals in order to be able to evaluate the absolute value and get

$$P^T_{n+1} = e^{ikx_g} X^{n+1} \frac{(-i\nu_0)}{2} \sum_{l=0}^{n} \left\{ \int_{z_1}^{z_g} dz' e^{i\nu_0 z_g} \left[-i\nu_0 \left(z' - z_1\right)\right]^l S^n_l \right.$$
$$\left. + \int_{z_g}^{\infty} dz' e^{i\nu_0 (2z' - z_g)} \left[-i\nu_0 \left(z' - z_1\right)\right]^l S^n_l \right\}. \tag{A.11}$$

The first integral has an easy solution; the second is a bit more tedious since it involves integration by parts. After solving the two integrals, we find

(A.12)

$$\begin{aligned}
P^T_{n+1} = e^{ikx_g} e^{i\nu_0 z_g} X^{n+1} \Bigg\{ & \sum_{l=0}^{n} \frac{S^n_l}{2(l+1)} \left[-i\nu_0 \left(z_g - z_1\right)\right]^{l+1} \\
& + \frac{(-i\nu_0)}{2} \Bigg[S^n_0 (-i\nu_0)^0 \left(-\frac{1}{2i\nu_0} \right) \\
& + S^n_1 (-i\nu_0)^1 \left(-\frac{1}{2i\nu_0} (z_g - z_1) + \frac{1}{(2i\nu_0)^2} \right) \\
& + S^n_2 (-i\nu_0)^2 \left(-\frac{1}{2i\nu_0} (z_g - z_1)^2 + \frac{2}{(2i\nu_0)^2} (z_g - z_1) - \frac{2}{(2i\nu_0)^3} \right) \\
& \vdots \\
& + S^n_n (-i\nu_0)^n \left(\frac{-1}{2i\nu_0} (z_g - z_1)^n + \frac{n}{(2i\nu_0)^2} (z_g - z_1)^{n-1} + \cdots + \frac{(-1)^{n+1} n!}{(2i\nu_0)^{n+1}} \right) \Bigg] \Bigg\}.
\end{aligned}$$

Grouping together the terms with like powers of $[-i\nu_0 (z' - z_1)]$ in the expression above, we find

(A.13)

$$\begin{aligned}
P^T_{n+1} = e^{ikx_g} e^{i\nu_0 z_g} X^{n+1} \Bigg\{ & \left[-i\nu_0 \left(z_g - z_1\right)\right]^{n+1} \frac{S^n_n}{2(n+1)} \\
& + \left[-i\nu_0 \left(z_g - z_1\right)\right]^{n} \left(\frac{S^n_{n-1}}{2n} + \frac{S^n_n}{2} \frac{1}{2} \right) \\
& + \left[-i\nu_0 \left(z_g - z_1\right)\right]^{n-1} \left(\frac{S^n_{n-2}}{2(n-1)} + \frac{S^n_n}{2} \frac{n}{2^2} + \frac{S^n_{n-1}}{2} \frac{1}{2} \right) \\
& + \left[-i\nu_0 \left(z_g - z_1\right)\right]^{n-2} \left(\frac{S^n_{n-3}}{2(n-2)} + \frac{S^n_n}{2} \frac{n(n-1)}{2^3} + \frac{S^n_{n-1}}{2} \frac{n-1}{2^2} + \frac{S^n_{n-2}}{2} \frac{1}{2} \right) \\
& \vdots \\
& + \left[-i\nu_0 \left(z_g - z_1\right)\right]^{1} \left(\frac{S^n_0}{2} + \frac{S^n_n}{2} \frac{n(n-1)\dots 2}{2^n} + \frac{S^n_{n-1}}{2} \frac{(n-1)\dots 2}{2^{n-1}} + \cdots + \frac{S^n_1}{2} \frac{1}{2} \right) \\
& + \left[-i\nu_0 \left(z_g - z_1\right)\right]^{0} \left(0 + \frac{S^n_n}{2} \frac{n!}{2^{n+1}} + \frac{S^n_{n-1}}{2} \frac{(n-1)!}{2^n} + \cdots + \frac{S^n_1}{2} \frac{1!}{2^2} + \frac{S^n_0}{2} \frac{1}{2} \right) \Bigg\}.
\end{aligned}$$

We next show that the coefficients of $[-i\nu_0 (z_g - z_1)]^j$ in the above expression are exactly equal to S^{n+1}_j, and hence this last expression is the one required for the second step of the induction (see (A.8)).

For the first coefficient recall that, by definition, we have

$$S_n = \frac{X^n}{2^n n!} (1+R)^{n+1}, \tag{A.14}$$

and hence we can write

$$S_{n+1} = \frac{X^{n+1}}{2^{n+1}(n+1)!} (1+R)^{n+2} = \frac{X}{2(n+1)} (1+R) S_n. \tag{A.15}$$

This is an equality of two series, which implies that the coefficients of identical powers from both sides are equal. By equating the coefficients of the $n+1$ power, we obtain

$$S_{n+1}^{n+1} = \frac{1}{2(n+1)} S_n^n. \tag{A.16}$$

For the second coefficient we start with the identity

$$S_n = \frac{X^n}{2^n n!} (1+R)^{n+1} \tag{A.17}$$

and rewrite it as

$$S_n = \frac{X^n}{2^n n!} (1+R)^n (1+R) = \frac{X}{2n} S_{n-1} + \frac{X^n}{2^n n!} R (1+R)^n. \tag{A.18}$$

By equating the coefficients of the $n+1$ power from both sides, we find

$$S_n^{n+1} = \frac{1}{2n} S_{n-1}^n + \frac{1}{4} S_n^n, \tag{A.19}$$

where we have used that the coefficient of the first power of X in the expression for R is $1/4$.

For the third coefficient we start with the identity

$$S_{n-1} = \frac{X^{n-1}}{2^{n-1}(n-1)!} (1+R)^n \tag{A.20}$$

and rewrite it as

(A.21)
$$S_{n-1} = \frac{X}{2(n-1)} S_{n-2} + \frac{X^{n-1}}{2^{n-1}(n-1)!} R (1+R)^{n-2} + \frac{X^{n-1}}{2^{n-1}(n-1)!} R^2 (1+R)^{n-2}.$$

By equating the coefficients of the $n+1$ power from both sides, we find

$$S_{n-1}^{n+1} = \frac{1}{2(n-1)} S_{n-2}^n + \frac{1}{4} S_{n-1}^n + \frac{n}{8} S_n^n. \tag{A.22}$$

For this last expression we have used again the fact that the coefficient of the first power of X in the expression for R is $1/4$.

The procedure outlined for these first three coefficient can be continued without difficulty to show that all the coefficients in the expression (A.13) coincide with those in (A.8). This concludes the second step of the induction and hence the proof that the expression for the transmitted wave-field P_n^T is

$$P_n^T (x_g, z_g > z_1 | k; \omega) = e^{ikx_g} e^{i\nu_0 z_g} X^n \sum_{l=0}^{n} \left[-i\nu_0 (z_g - z_1)\right]^l S_l^n. \tag{A.23}$$

The general term in the forward scattering series representation (3.14) for the reflected wave-field can hence be calculated using the following formula:

(A.24)

$$P_{n+1}^{R}\left(x_g, z_g < z_1 \middle| k; \omega\right) = \int_{z_1}^{\infty} dz' \int_{-\infty}^{\infty} dx' \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}\left|z_g - z'\right|}}{2i\nu_0} dk_g \right) k_0^2 \alpha P_n^T\left(x', z' \middle| k; \omega\right).$$

Introducing the expression for P_n^T, we find

$$P_{n+1}^{R} = \int_{z_1}^{\infty} dz' \int_{-\infty}^{\infty} dx' \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \frac{e^{ik_g(x_g - x')} e^{i\nu_{0g}\left|z_g - z'\right|}}{2i\nu_0} dk_g \right) k_0^2 \alpha e^{ikx'} e^{i\nu_0 z'} X^n \times \sum_{l=0}^{n} \left[-i\nu_0\left(z' - z_1\right)\right]^l S_l^n. \tag{A.25}$$

Solving for the dx' and the dk_g integrals gives

$$P_{n+1}^{R} = e^{ikx_g} X^{n+1} \frac{(-i\nu_0)}{2} \int_{z_1}^{\infty} dz' \sum_{l=0}^{n} \left[-i\nu_0\left(z' - z_1\right)\right]^l S_l^n e^{i\nu_0(2z' - z_g)}. \tag{A.26}$$

Notice that this integral has been dealt with before: it is the integral appearing in the second part of (A.11), and its solution is given in the second part of (A.12). However, the limits of integration are different: the solution for our integral may be obtained from the second part of (A.12) by replacing z_g with z_1. This substitution cancels most of the terms, and the result is

$$P_{n+1}^{R} = e^{ikx_g} e^{i\nu_0 z_g} X^{n+1} \frac{(-i\nu_0)}{2} \left[-S_0^n \frac{1}{2i\nu_0} - S_1^n \frac{1}{2^2 i\nu_0} - \cdots - S_n^n \frac{n!}{2^{n+1} i\nu_0} \right] \tag{A.27}$$

or

$$P_{n+1}^{R} = e^{ikx_g} e^{i\nu_0 z_g} X^{n+1} \frac{1}{4} \left[S_0^n + \frac{S_1^n}{2^1} 1! + \frac{S_2^n}{2^2} 2! + \cdots + \frac{S_n^n}{2^n} n! \right]. \tag{A.28}$$

Again, the sum inside the square brackets is an expression that we have already analyzed before: it is the coefficient of $\left[-i\nu_0\left(z_g - z_1\right)\right]^0$ in (A.13). It was shown there that

$$\frac{1}{4} \left[S_0^n + \frac{S_1^n}{2^1} 1! + \frac{S_2^n}{2^2} 2! + \cdots + \frac{S_n^n}{2^n} n! \right] = S_0^{n+1}, \tag{A.29}$$

and hence the expression for P_{n+1}^R becomes

$$P_{n+1}^{R}\left(x_g, z_g < z_1 \middle| k; \omega\right) = e^{ikx_g} e^{i\nu_0 z_g} X^{n+1} S_0^{n+1}. \tag{A.30}$$

Recall from (A.4) and (A.5) that S_0^{n+1} represents the coefficient of the $n+1$ degree in the series for $1+R$, and hence it is the coefficient of the $n+1$ degree in the Taylor series for $\sqrt{1 - k_0^2\alpha/\nu_0^2}$ about $\frac{k_0^2\alpha}{\nu_0^2} = 0$ after some algebraic operations are performed on it. The total scattered field P is the summation of all P_n^R and hence it represents the full Taylor series for $\sqrt{1 - k_0^2\alpha/\nu_0^2}$ about $\frac{k_0^2\alpha}{\nu_0^2} = 0$ after some algebraic operations are performed on it.

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