Reverse time migration and Green's theorem: Part II-A new and consistent theory that progresses and corrects current RTM concepts and methods

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ABSTRACT

In this paper, part II of a two paper set, we place Green's theorem based reverse time migration (RTM), for the first time on a firm footing and technically consistent math-physics foundation. The required new Green's function for RTM application is developed and provided, and is neither causal, anticausal, nor a linear combination of these prototype Green's functions, nor these functions with imposed boundary conditions. We describe resulting fundamentally new RTM theory and algorithms, and provide a step-by-step prescription for application in 1D, 2D and 3D, the latter for an arbitrary laterally and vertically varying velocity field. The original RTM method of running the wave equation backwards with surface reflection data as a boundary condition is not a wave theory method for wave-field prediction, neither in depth nor in reversed time. In fact, the latter idea corresponds to Huygens Principle which evolved and was corrected and became a wave theory predictor by George Green in 1826. The original RTM method, where (1) 'running the wave equation backward in time', and then (2) employing a zero lag cross-correlation imaging condition, is in both of these ingredients less accurate and effective than the Green's theorem RTM method of this two paper set. Furthermore, *all* currently available Green's theorem methods for RTM make fundamental conceptual and algorithmic errors in their Green's theorem formulations. Consequently, even with an accurate velocity model, current Green's theorem RTM formulations can lead to image location errors and other reported artifacts. Addressing the latter problems is a principal goal of the new Green's theorem RTM method of this paper. Several simple analytic 1D examples illustrate the new RTM method. We also compare the general RTM methodology and philosophy, as the high water mark of current imaging concepts and application, with the next generation and emerging Inverse Scattering Series imaging concepts and methods.

INTRODUCTION

An important and central concept resides behind all current seismic processing imaging methods that seek to extract useful subsurface information from recorded seismic data. That concept has two ingredients: (1) from the actual recorded surface seismic experiment and data, to predict what an experiment with a source and receiver at depth would record, and (2) exploiting the fact that a coincident source receiver experiment at depth, would, for small recording times, be an indicator of only local earth mechanical property changes at the coincident source-receiver position. These two ingredients, a wave-field prediction, and an imaging condition, reside behind all current leading edge seismic migration algorithms. The purpose of this two paper set is to advance our understanding, and provide concepts and new algorithms for the first of these two ingredients: subsurface wave-field prediction from surface wave-field measurements, when the wave propagation between source and target and/or target and receiver is not a one-way propagating wave in terms of depth..

As with all current migration methods, an accurate velocity model is required for this procedure to deliver an accurate structure map, that is, the spatial configuration of boundaries in the subsurface that correspond to reflectors where rapid changes in physical properties occur.

In this paper, we for the first time place Reverse Time Migration (RTM) on a firm theoretical footing derived from Green's theorem. Green's theorem provides a useful framework for deriving algorithms to predict the wave-field at depth from surface measurements. There is much current interest and activity with RTM in exploration seismology.

The original RTM was pioneered, developed and applied by Dan Whitmore and his AMOCO colleagues in the 1980's (Whitmore (1983)), for exploration in the overthrust belt. The traditional seismic thinking that used a wave traveling from source down to the reflector and then up from the reflector to the receiver was extended to allow, e.g., waves to move down and up from source to a reflector and down and then up from reflector to the receiver. For one-way wave propagation, a single step in depth corresponds to one step in time, with a fixed sign in the relationship between change in depth and change in time. Hence, for one-way waves, we can equivalently go down the up wave in space or take a step backwards in time. For two-way wave propagation, reversing time or extrapolating down an upcoming wave are *not* equivalent. And to image a reflector that reflected a turning wave requires a non-one-way wave model that reversed time can satisfy. In wave theoretic downward continuation migration, the source wave-field and receiver wave-fields are each extrapolated to the subsurface using one-way wave equations to obtain an experiment with coincident sources and receivers at depth.

The idea behind the two-way wave extrapolators (Whitmore (1983), McMechan (1983), Baysal et al. (1983)) is to handle waves propagating in any direction, including overturning waves and prismatic waves. The most common implementation uses finite-difference techniques to solve the wave equation, which in the acoustic case is given by

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) P(\mathbf{r},t) = 0 \quad , \tag{1}$$

where P can be either the source or receiver wave-field. To calculate the source wave-field, standard forward modeling injecting a user defined source signature into the model at the actual source position is done. For the receiver wave-field, the wave equation is run backwards in time and the recorded wave-field is injected into the model at the receiver positions as a boundary condition. The injection of the recorded wave-field is done starting with later times and finishing with the early times. That idea of using the measured values of the wave-field as the boundary conditions for a wave equation run backwards in time corresponds to Huygens' principle (Huygens, 1690). The image, $I(\mathbf{x})$ is generated using a zero lag cross-correlation imaging condition,

$$I(\mathbf{x}) = \int_0^{t_{max}} dt\, S(\mathbf{x}, t) R(\mathbf{x}, t_{max} - t), \tag{2}$$

where the maximum recording time is t_{max}, $S(\mathbf{x}, t)$ is the modeled source wave-field and $R(\mathbf{x}, t_{max} - t)$ is the receiver wave-field (Fletcher et al., 2006). There are other imaging conditions cited in the literature, among them the deconvolution imaging condition (Zhang et al., 2007) but at this point in time it seems that the cross-correlation imaging condition is often employed. The latter imaging principle is not equivalent to the downward continuation of sources and receivers at depth and seeking a zero time result from a coincident source-receiver experiment.

One of the disadvantages of RTM is that it requires the availability of a large amount of memory which increases with respect to the frequencies we want to migrate (Liu et al., 2009). As a consequence, memory availability has been a limitation to the application of this technology, especially to high resolution data from large 3D acquisitions. Nevertheless, recent improvements in computer hardware have enabled different implementations of RTM throughout the energy industry and there is a renewed interest in this technology due to its ability to accommodate and image in media where waves turn, as e.g. can occur in subsalt plays. Several efforts have been aimed at improving the efficiency of the algorithm and dealing with the high storage cost for 3D implementation. For example, Toselli and Widlund (2000) used domain deconvolution which splits the computations across multiple nodes to improve the efficiency of the algorithm, and Symes (2007) introduced optimal checkpointing techniques to deal with the storage requirements, although, this type of technique can increase the computation cost. These are examples of improvements directly related to the numerical implementation of the RTM algorithm. Other efforts to deal with the practical requirements of RTM are based on changes in the theoretical approach to the problem. One example is the work of Luo and Schuster (2004) where a target oriented reverse time datuming (RTD) technique based on Green's theorem is proposed. RTD can also be seen as a bottom-up shooting approach for RTM. Using RTD's formulation, only the velocity model above the datum is used to calculate the Green's function. No velocity under the datum is required, making the modeling more efficient. This formulation also allows for target oriented RTM and/or inversion. In target oriented RTM, the idea is to redatum the data into a mathematical surface (referred to as the datum surface) within the earth's subsurface and use RTM below the datum surface to obtain a local RTM image of a given target area below the datum (Dong et al., 2009). In target oriented inversion, the inversion is carried out only for a target area below the datum. Target oriented inversion has also been proposed using the CFP domain.

The current formulation of RTD or bottom-up shooting for RTM, uses a high frequency approximation to Green's theorem (interferometry equation) and measurements at the measurement surface. This formulation presents several approximations which can impact the quality of the redatuming or the migration (if an imaging condition is applied after RTD).

1. The first approximation is related to the measurement surface. Green's theorem based al-

gorithms, in principle, require measurements over a closed surface. The fact that we only measure the wave-field in a limited surface has an effect on the quality of the redatuming and can create artifacts in both the redatumed data and the migration. Directly addressing that issue is one of the principle aims of this paper. These measurements can be interchanged for sources at the surface using reciprocity principles.

2. The second approximation is the high frequency, one-way wave approximation commonly used in interferometry. This approximation allows us to remove the need for the normal derivative of the pressure field at the measurement surface. The normal derivative is required by Green's theorem in its most common form, which is the one used by (Luo and Schuster, 2004) in their RTD formulation. As an analogy to interferometry, when used with two-way waves, this high frequency, one-way wave approximation will create spurious multiples in the redatumed wave-field within the earth's subsurface (see e.g. Ramírez and Weglein (2009)).

Dong et al. (2009) deal with the effect of these approximations by smoothing the model, and, hence, reducing the effect of the one-way wave approximation. However, smoothing the model does not solve completely the problems created by the use of approximations. The redatumed wave-field will contain artifacts. Some of these artifacts will be imaged and stacking will not remove these artifacts completely.

Some indication of the level of current interest in RTM can be gleaned by: (1) the number of papers devoted to that subject in recent SEG and EAGE meetings, and subsalt workshops and (2) the November 2010 Special Section of The Leading Edge on Reversed Time Migration with an Introduction by Etgen and Michelena (2010) and papers by Zhang et al. (2010), Jin and Xu (2010), Crawley et al. (2010), and Higginbotham et al. (2010).

PROPAGATION FOR RTM IN A ONE DIMENSIONAL EARTH: USING GREEN'S FUNCTIONS TO AVOID THE NEED FOR DATA AT DEPTH, NEW NONCAUSAL OR CAUSAL GREEN'S FUNCTIONS

Green's theorem in 3D in the $(\mathbf{r},\omega)$ domain to determine a wave-field, $P(\mathbf{r},\omega)$ for $\mathbf{r}$ in V is given by

$$\begin{aligned} P(\mathbf{r},\omega) &= \int_V d\mathbf{r}'\,\rho(\mathbf{r}',\omega)G_0(\mathbf{r},\mathbf{r}',\omega) \\ &+ \oint_S dS'\,\mathbf{n}\cdot\left(P(\mathbf{r}',\omega)\nabla' G_0(\mathbf{r},\mathbf{r}',\omega) - G_0(\mathbf{r},\mathbf{r}',\omega)\nabla' P(\mathbf{r}',\omega)\right) \quad . \end{aligned} \tag{3}$$

In 1D in the slab $a \le z \le b$, (3) becomes

$$\begin{aligned} P(z,\omega) &= \int_a^b dz'\,\rho(z',\omega)G_0(z,z',\omega) \\ &+ \Big|_a^b \left(P(z',\omega)\frac{dG_0}{dz'}(z,z',\omega) - G_0(z,z',\omega)\frac{dP}{dz'}(z',\omega)\right) \quad . \end{aligned} \tag{4}$$

Assuming no sources in the slab, the 1D homogeneous wave equation is

$$\left(\frac{d^2}{dz'^2} + k^2\right)P(z',\omega) = 0 \quad , \text{ for } a < z' < b \tag{5}$$

with general solution

$$P(z',\omega) = Ae^{ikz'} + Be^{-ikz'} \text{ for } a < z' < b \tag{6}$$

where $k = \omega/c$. Given the conventions positive z' increasing downward and time dependence $e^{-i\omega t}$ in Fourier transforming from ω to t, the first term in (6) is a downgoing wave and the second term is an upgoing wave.

The equation for the corresponding Green's function is

$$\left(\frac{d^2}{dz'^2} + k^2\right) G_0(z,z',\omega) = \delta(z-z') \quad , \tag{7}$$

with causal and anticausal solutions

$$G_0^+(z,z',\omega) = \frac{1}{2ik} e^{ik|z-z'|} \quad , \tag{8}$$

$$G_0^-(z,z',\omega) = -\frac{1}{2ik} e^{-ik|z-z'|} \quad . \tag{9}$$

Eq. (4) suggests that the Green's function we need is such that it and its derivative vanish at $z' = b$. Such a Green's function removes the need for measurements at $z' = b$. Eq. (7) is an inhomogeneous differential equation with general solution $A_1 e^{ikz'} + B_1 e^{-ikz'} + G_0(z,z',\omega)$ where the first two terms are the general solution to the homogeneous differential equation and the third term is any particular solution to the inhomogeneous differential equation. The choice $G_0(z,z',\omega) = G_0^+(z,z',\omega)$ gives the following general solution of (7):

$$G_0(z,z',\omega) = A_1 e^{ikz'} + B_1 e^{-ikz'} + \frac{1}{2ik} e^{ik|z-z'|} \quad . \tag{10}$$

Its derivative is

$$\begin{aligned} \frac{dG_0}{dz'}(z,z',\omega) &= A_1 e^{ikz'} ik + B_1 e^{-ikz'}(-ik) \\ &+ \frac{1}{2ik} e^{ik|z-z'|} ik \ \operatorname{sgn}(z-z')(-1) \quad . \end{aligned} \tag{11}$$

Now we impose boundary conditions in order to find A_1 and B_1. The requirement that (10) and (11) vanish at $z' = b$ gives

$$\begin{aligned} 0 &= A_1 e^{ikb} + B_1 e^{-ikb} + \frac{1}{2ik} e^{ik \overbrace{|z-b|}^{b-z}} \\ 0 &= A_1 e^{ikb} ik + B_1 e^{-ikb}(-ik) + \frac{1}{2ik} e^{ik \overbrace{|z-b|}^{b-z}} ik \ \underbrace{\operatorname{sgn}(z-b)}_{-1}(-1) \\ A_1 e^{ikb} + B_1 e^{-ikb} &= -\frac{1}{2ik} e^{ik(b-z)} \\ A_1 e^{ikb} - B_1 e^{-ikb} &= -\frac{1}{2ik} e^{ik(b-z)} \\ 2A_1 e^{ikb} &= -2\frac{1}{2ik} e^{ik(b-z)} \\ A_1 &= -\frac{1}{2ik} e^{-ikz} \quad , \qquad (12) \\ 2B_1 e^{-ikb} &= 0 \\ B_1 &= 0 \quad . \qquad (13) \end{aligned}$$

Substituting (12) and (13) into (10) gives

$$\begin{aligned} G_0(z,z',\omega) &= -\frac{1}{2ik}e^{-ikz}e^{ikz'} + \frac{1}{2ik}e^{ik|z-z'|} \\ &= -\frac{1}{2ik}(e^{-ik(z-z')} - e^{ik|z-z'|}) \quad . \end{aligned} \tag{14}$$

Note the following about (14):

1. When $z' = b$, $G_0(z,b,\omega)$ vanishes:

$$G_0(z,b,\omega) = -\frac{1}{2ik}(e^{-ik(z-b)} - e^{ik\overbrace{|z-b|}^{b-z}}) = -\frac{1}{2ik}(\underbrace{e^{-ik(z-b)} - e^{-ik(z-b)}}_{0}) \quad . \tag{15}$$

2. When $a < z' < b$, $G_0(z,z',\omega)$ is neither causal nor anticausal due to the presence of the term $-1/(2ik)\, e^{-ik(z-z')}$.

3. When $z' = a$, $G_0(z,a,\omega)$ is the sum of anticausal and causal terms, but not in general or at any other depth.

$$G_0(z,a,\omega) = -\frac{1}{2ik}(e^{-ik\underbrace{(z-a)}_{|z-a|}} - e^{ik|z-a|}) = \underbrace{-\frac{1}{2ik}e^{-ik|z-a|}}_{anticausal} + \underbrace{\frac{1}{2ik}e^{ik(z-a)}}_{causal} \quad . \tag{16}$$

4. Normally one uses Dirichlet or Neumann or Robin boundary conditions on the surface S (in our 1D case at both a and b). Constructing the Green's function (14) has enabled us to use both Dirichlet and Neumann boundary conditions on part of the surface S (in our 1D case only at b).

The Green's function for two-way propagation that will eliminate the need for data at the lower surface of the closed Green's theorem surface is found by finding a general solution to the Green's function for the medium in the finite volume model and imposing both Dirichlet and Neumann boundary conditions at the lower surface. We confirm that the Green's function (14), when used in Green's theorem, will produce a two-way wave for $a < z < b$ with only measurements on the upper surface. Substituting (6), (14), and their derivatives into (4) gives $P(z,\omega) = Ae^{ikz} + Be^{-ikz}$, i.e., we recover the original two-way wave-field. The details are in Appendix A.

A and B can be derived from the measured data $P(a)$ and $P'(a)$:

$$\begin{aligned} P(a) &= Ae^{ika} + Be^{-ika} \\ P'(a) &= Ae^{ika}ik + Be^{-ika}(-ik) \\ \frac{P'(a)}{ik} &= Ae^{ika} - Be^{-ika} \\ 2Ae^{ika} &= P(a) + \frac{P'(a)}{ik} \\ A &= e^{-ika}\frac{ikP(a) + P'(a)}{2ik} \quad , \qquad (17) \\ 2Be^{-ika} &= P(a) - \frac{P'(a)}{ik} \\ B &= e^{ika}\frac{ikP(a) - P'(a)}{2ik} \quad . \qquad (18) \end{aligned}$$

In a homogeneous medium the 3D equivalent of (5) is

$$(\nabla'^{\,2} + k^2)P(x', y', z', \omega) \quad = \quad 0 \quad , \tag{19}$$

where $k = \omega/c$. Fourier transforming over x' and y' gives

$$\left(\frac{d^2}{dz'^{\,2}} \underbrace{-k_{x'}^2 - k_{y'}^2 + \frac{\omega^2}{c^2}}_{\equiv k_{z'}^2} \right) P(k_{x'}, k_{y'}, z', \omega) \quad = \quad 0 \quad , \tag{20}$$

which looks like the 1D problem

$$\left(\frac{d^2}{dz'^{\,2}} + k_{z'}^2 \right) P(k_{x'}, k_{y'}, z', \omega) \quad = \quad 0 \quad , \tag{21}$$

with general solution

$$P(k_{x'}, k_{y'}, z', \omega) \quad = \quad A e^{ik_{z'}z'} + B e^{-ik_{z'}z'} \quad . \tag{22}$$

We illustrate, in the next section, a more complicated 1D example, where the finite volume contains a reflector.

RTM AND GREEN'S THEOREM: TWO-WAY WAVE PROPAGATION IN A 1D FINITE VOLUME THAT CONTAINS A REFLECTOR

Consider a single reflector example with the following properties: z increases downward, the source is located at depth z_s (where $0 < z_s < a$), the receiver is located at depth z_g (where $z_s < z_g < a$), for $0 \leq z \leq a$ the medium is characterized by c_0, for $z > a$ the medium is characterized by c_1, and the reflection coefficient R and transmission coefficient T at the interface ($z = a$) are given by

$$R \quad = \quad \frac{c_1 - c_0}{c_0 + c_1} \quad , \tag{23}$$

$$T \quad = \quad \frac{2c_0}{c_0 + c_1} \quad . \tag{24}$$

Assume the source goes off at $t = 0$. Then the wave-field P for $z < a$, i.e., above a, is given by

$$P \quad = \quad \frac{e^{ik|z-z_s|}}{2ik} + R\,\frac{e^{-ik(z-a)}}{2ik} e^{ik(a-z_s)} \quad . \tag{25}$$

In the time domain, the front of the plane wave travels with $\delta(t - |z - z_s|/c_0)$ out from the source. Hence, the first term in (25) for P is the incident wave-field (an impulse) and for the second term in P

$$\delta\left(t - \underbrace{\frac{|a - z_s|}{c_0}}_{\text{from source to reflector}} - \underbrace{\frac{|z - a|}{c_0}}_{\text{from reflector to field point } z} \right) . \tag{26}$$

Fourier transforming gives:

$$\begin{aligned} \int_{-\infty}^{\infty} e^{i\omega t} \delta\left(t - \frac{|a - z_s|}{c_0} - \frac{|z - a|}{c_0} \right) dt \quad &= \quad e^{\left(i\omega\left(\frac{|a-z_s|}{c_0} + \frac{|z-a|}{c_0} \right) \right)} \\ \text{for } z < a \quad &= \quad e^{ik(a-z_s) - ik(z-a)} = e^{-ik(z-(2a-z_s))} \quad . \end{aligned} \tag{27}$$

Therefore $2a - z_s - z$ is the travel path from the source to the reflector and up to the field point z. If instead of an incident Green's function we choose a plane wave, we drop the $1/(2ik)$ and set $z_s = 0$, and then the incident plane wave passes the origin $z = 0$ at $t = 0$.

The transmitted wave field is for $z > a$

$$P = \frac{1}{2ik} T e^{ik|a-z_s|} e^{ik_1(z-a)} \quad , \tag{28}$$

with R and T given by (23); then (25) and (28) provide the solution for the total wave field everywhere.

Now we introduce Green's theorem. The total wave-field P satisfies

$$\left\{ \frac{d^2}{dz'^2} + \frac{\omega^2}{c^2(z')} \right\} P = \delta(z' - z_s) \quad , \tag{29}$$

and the Green's function G will satisfy

$$\left\{ \frac{d^2}{dz'^2} + \frac{\omega^2}{c^2(z')} \right\} G = \delta(z - z') \quad , \tag{30}$$

where

$$c(z') = \begin{cases} c_0 & z' < a \\ c_1 & z' > a \end{cases} \quad . \tag{31}$$

The solution for P is given in (25) and (28). The solution for G will be determined below. G_H and G_P are a homogeneous solution and particular solution, respectively, of the following differential equations:

$$\left\{ \frac{d^2}{dz^2} + \frac{\omega^2}{c^2(z)} \right\} G_H = 0 \quad , \tag{32}$$

$$\left\{ \frac{d^2}{dz^2} + \frac{\omega^2}{c^2(z)} \right\} G_P = \delta \quad . \tag{33}$$

A particular solution, G_P, can be given by (25) and (28) and with z' replacing z_s, we find

$$P(z, z_s, \omega) = \Big|_A^B \{P(z', z_s, \omega) \frac{dG}{dz'}(z, z', \omega) - G(z, z', \omega) \frac{dP}{dz'}(z', z_s, \omega)\} \quad , \tag{34}$$

where $A = z_g$, the depth of the MS, and $B > a$ is the lower surface of Green's theorem.

The 'source' at depth z is within $[A, B]$ and either above or below the reflector at $z = a$; conditions will be placed on the solution, G, (for a source within the volume) for the field point of G at depth z' to satisfy at B. That is

$$(G(z, z', \omega))_{z'=B} = 0 \quad , \tag{35}$$

$$\text{and} \quad \left(\frac{dG}{dz'}(z, z', \omega) \right)_{z'=B} = 0 \quad . \tag{36}$$

First pick the 'source' in the Green's function to be above the reflector, then (25) and (28) provide a specific solution when we substitute for z_s in P, the parameter, z, and for z in P the parameter, z'. The latter allows a particular solution for G for the case that z in G is within $[A, B]$ but above $z = a$. Please note: The physical source is *outside* the volume, but the 'source' in the Green's function is inside the volume. Also, note that for the case of the (output point, z) 'source' in the Green's function to be below the reflector and within $[A, B]$ that a different solution for P other than what is given in (25) and (28), would be needed for a particular solution of G. The latter would require a solution for P where the source is in the lower half space.

What about the general solution for G_H?

$$\left\{ \frac{d^2}{dz'^2} + \frac{\omega^2}{c^2(z')} \right\} G_H = 0 \quad . \tag{37}$$

The general solution to (37) is, for *any* incident plane wave, $A(k)$ for $[A, B]$

$$G_H = \begin{cases} A_1 e^{ikz'} + B_1 e^{-ikz'} & z' < a \\ C_1 e^{ik_1 z'} + D_1 e^{-ik_1 z'} & z' > a \end{cases} . \tag{38}$$

The general solution to (37) has to allow the possibility of an incident wave from either direction, *that's what general solution means!* The general solution for G for the single reflector problem and the source, z, above the reflector is given by

$$G(z, z', \omega) = \begin{cases} \frac{e^{ik|z'-z|}}{2ik} + R\frac{e^{-ik(z'-a)}}{2ik} e^{ik(a-z)} + A_1 e^{ikz'} + B_1 e^{-ikz'} & z' < a \\ \frac{T}{2ik} e^{ik|a-z|} e^{ik_1(z'-a)} + C_1 e^{ik_1 z'} + D_1 e^{-ik_1 z'} & z' > a \end{cases} \tag{39}$$

for the source z being above the reflector and we choose C_1 and D_1 such that

$$G(z, B, \omega) = 0 \quad , \text{and} \tag{40}$$

$$\left[\frac{dG}{dz'}(z, z', \omega) \right]_{z'=B} = 0 \quad , \tag{41}$$

where, for $z' > a$, G_P is $T/(2ik)e^{ik|a-z|}e^{ik_1(z'-a)}$. Eq. (39) will be the Green's function needed in Green's theorem to propagate/predict above the reflector at $z' = a$ where P is given by (25) and (28). To determine A_1 and B_1, make the solutions for $z' < a$ and $z' > a$ and their derivatives match at $z' = a$ (conditions of continuity across the reflector). The details are in Appendix B. In practice, the deghosted scattered wave is *upgoing* (one-way) and finding its vertical derivative is simply $ik_z \times P$. Deghosting precedes migration.

For downward continuing past the reflector, as previously stated, the P solution needed for the particular solution of the Green's function starts with a source in the lower half space where $k_1 = \omega/c_1$. That's how it works. In practice for a $v(x, y, z)$ medium a modeling will be required that *imposes* a double vanishing boundary condition at depth to produce the Green's function for RTM.

MULTIDIMENSIONAL RTM

Consider a volume V inside a homogeneous medium; V is bounded on the left by $x' = A$, on the right by $x' = L_1$, on the top by $z' = B$, and on the bottom by $z' = L_2$ (Fig. 1). We want to use Green's theorem to estimate the wave-field P in V which requires we measure P and $\partial P/\partial n$ on the boundary S of V. However, we can place receivers only at $z' = B$. Can we construct a Green's function G such that it and its normal derivative $\partial G/\partial n$ vanish on three sides of V so that P can be estimated in V using only the measurements on $z' = B$?

G can be written as the sum of a homogeneous solution G_H and a particular solution G_P where G satisfies the partial differential equation $(\nabla^{'\,2} + k^2)G = \delta(\mathbf{r} - \mathbf{r}^{'})$ and G_H satisfies the partial differential equation $(\nabla^{'\,2} + k^2)G_H = 0$. We try solutions of the form:

$$G(\mathbf{r}^{'}, \mathbf{r}, \omega) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(x') Z_n(z') + G_P(\mathbf{r}^{'}, \mathbf{r}, \omega) \quad , \tag{42}$$

$$G_H(\mathbf{r}^{'}, \mathbf{r}, \omega) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(x') Z_n(z') \quad , \tag{43}$$

with the boundary conditions that G and $\partial G/\partial n$ vanish at $x' = A$, $z' = L_2$, and $x' = L_1$, i.e.,

$$\text{at } x' = A \;\; G = 0 \text{ and } -\frac{\partial G}{\partial x'} = 0, \tag{44}$$

$$\text{at } z' = L_2 \;\; G = 0 \text{ and } \frac{\partial G}{\partial z'} = 0, \text{ and} \tag{45}$$

$$\text{at } x' = L_1 \;\; G = 0 \text{ and } \frac{\partial G}{\partial x'} = 0. \tag{46}$$

Substituting (43) into $(\nabla^{'\,2} + k^2)G_H = 0$ gives:

$$\begin{aligned}
0 &= \left(\frac{\partial^2}{\partial x'^{\,2}} + \frac{\partial^2}{\partial z'^{\,2}} + k^2\right) X_m(x') Z_n(z') \\
&= X_m''(x') Z_n(z') + X_m(x') Z_n''(z') + k^2 X_m(x') Z_n(z') \\
&= \frac{X_m''(x')}{X_m(x')} + \frac{Z_n''(z')}{Z_n(z')} + k^2 \\
&\Longrightarrow \frac{Z_n''(z')}{Z_n(z')} = -\lambda^2 \\
0 &= Z_n''(z') + \lambda^2 Z_n(z') \\
Z_n(z') &= C_1 e^{i\lambda_n z'} + C_2 e^{-i\lambda_n z'} && (47) \\
0 &= X_m''(x') + (\underbrace{k^2 - \lambda^2}_{\equiv \mu^2}) X_m(x') \\
X_m(x') &= C_3 e^{i\mu_m x'} + C_4 e^{-i\mu_m x'} \quad , && (48)
\end{aligned}$$

where $\mu_m^2 \longrightarrow X_m(x')$ and $\lambda_n^2 \longrightarrow Z_n(z')$. We assume $X_m(x')$ and $Z_n(z')$ are orthonormal and complete and $\mu^2 \equiv k^2 - \lambda^2 \geq 0$.

The boundary conditions on the left are $G(A, z') = 0$ and $G_{x'}(A, z') = 0$, on the right $G(L_1, z') = 0$ and $G_{x'}(L_1, z') = 0$, and on the bottom $G(x', L_2) = 0$ and $G_{z'}(x', L_2) = 0$. Substituting these boundary conditions into (42) gives:

$$
\begin{aligned}
0 &= G(A,z',x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(A) Z_n(z') + G_P(A,z',x,z) \quad , && (49)\\
0 &= G_{x'}(A,z',x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X'_m(A) Z_n(z') + \frac{d}{dx'} G_P(A,z',x,z) \quad , && (50)\\
0 &= G(L_1,z',x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(L_1) Z_n(z') + G_P(L_1,z',x,z) \quad , && (51)\\
0 &= G_{x'}(L_1,z',x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X'_m(L_1) Z_n(z') + \frac{d}{dx'} G_P(L_1,z',x,z) \quad , && (52)\\
0 &= G(x',L_2,x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(x') Z_n(L_2) + G_P(x',L_2,x,z) \quad , && (53)\\
0 &= G_{z'}(x',L_2,x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(x') Z'_n(L_2) + \frac{d}{dz'} G_P(x',L_2,x,z) \quad . && (54)
\end{aligned}
$$

Eq. (49) is:

$$
-G_P(A,z',x,z) = \sum_{m,n} A_{m,n}(\mathbf{r}) X_m(A) Z_n(z') \quad . \tag{55}
$$

Multiplying by $Z_s(z')$, integrating, and substituting (47) and (48) give:

$$
\begin{aligned}
&- \int_{L_2}^{B} G_P(A,z',x,z) Z_s(z')\, dz' = \sum_{m} A_{m,s}(\mathbf{r}) X_m(A)\\
&- \int_{L_2}^{B} G_P(A,z',x,z)(C_1 e^{i\lambda_s z'} + C_2 e^{-i\lambda_s z'})\, dz'\\
&= \sum_{m} A_{m,s}(\mathbf{r})(C_3 e^{i\mu_m A} + C_4 e^{-i\mu_m A}) \quad . && (56)
\end{aligned}
$$

In similar fashion we get:

$$
\begin{aligned}
- & \int_{L_2}^{B} \frac{d}{dx'} G_P(A, z', x, z)(C_1 e^{i\lambda_s z'} + C_2 e^{-i\lambda_s z'})\, dz' \\
= & \sum_m A_{m,s}(\mathbf{r}) i\mu_m (C_3 e^{i\mu_m A} - C_4 e^{-i\mu_m A}) \quad , \qquad (57) \\
- & \int_{L_2}^{B} G_P(L_1, z', x, z)(C_1 e^{i\lambda_s z'} + C_2 e^{-i\lambda_s z'})\, dz' \\
= & \sum_m A_{m,s}(\mathbf{r})(C_3 e^{i\mu_m L_1} + C_4 e^{-i\mu_m L_1}) \quad , \qquad (58) \\
- & \int_{L_2}^{B} \frac{d}{dx'} G_P(L_1, z', x, z)(C_1 e^{i\lambda_s z'} + C_2 e^{-i\lambda_s z'})\, dz' \\
= & \sum_m A_{m,s}(\mathbf{r}) i\mu_m (C_3 e^{i\mu_m L_1} - C_4 e^{-i\mu_m L_1}) \quad , \qquad (59) \\
- & \int_{A}^{L_1} G_P(x', L_2, x, z)(C_3 e^{i\mu_s x'} + C_4 e^{-i\mu_s x'})\, dz' \\
= & \sum_m A_{m,s}(\mathbf{r})(C_1 e^{i\lambda_n L_2} + C_2 e^{-i\lambda_n L_2}) \quad , \qquad (60) \\
- & \int_{A}^{L_1} \frac{d}{dz'} G_P(x, L_2, x, z)(C_3 e^{i\mu_s x'} + C_4 e^{-i\mu_s x'})\, dz' \\
= & \sum_m A_{m,s}(\mathbf{r}) i\lambda_n (C_1 e^{i\lambda_n L_2} - C_2 e^{-i\lambda_n L_2}) \quad . \qquad (61)
\end{aligned}
$$

The $A_{m,s}$ coefficients are determined by the imposed Dirichlet and Neumann boundary conditions on the base and walls of the finite volume.

GENERAL STEP-BY-STEP PRESCRIPTION FOR RTM IN A FINITE VOLUME WHERE THE VELOCITY CONFIGURATION IS $C(X,Y,Z)$

Step (1) For a desired downward continued/migration output point (x,y,z) for determining $P(x,y,z,\omega)$

$$\left\{\nabla'^2 + \frac{\omega^2}{c^2(x',y',z')}\right\} G_0(x',y',z',x,y,z,\omega) = \delta(x-x')\delta(y-y')\delta(z-z') \quad , \qquad (62)$$

for a source at (x,y,z) and P is the physical/causal solution satisfying

$$\left\{\nabla'^2 + \frac{\omega^2}{c^2(x',y',z')}\right\} P(x',y',z',x_s,y_s,z_s,\omega) = A(\omega)\delta(x'-x_s)\delta(y'-y_s)\delta(z'-z_s). \qquad (63)$$

G_0 is the auxiliary or Green's function satisfying

$$\left\{\nabla'^2 + \frac{\omega^2}{c^2(x',y',z')}\right\} G_0(x,y,z,x',y',z',\omega) = \delta(x-x')\delta(y-y')\delta(z-z') \quad , \qquad (64)$$

for (x,y,z) in V *and* G_0 and $\nabla' G_0 \cdot \hat{n}'$ are both zero for (x',y',z') on the lower surface S_L and the walls S_W of the finite volume. The solution for G_0 in V and on S can be found by a numerical modeling algorithm where the 'source' is at (x,y,z) and the field, G_0, at (x',y',z') and $\nabla G_0 \cdot \hat{n}$ are both imposed to be zero on S_L and S_W. Once that model is run for a source at (x,y,z) for $G_0(x',y',z',x,y,z,\omega)$ [for every eventual wave prediction point, (x,y,z), for P] where G_0 satisfies

Dirichlet and Neumann conditions for (x', y', z') on S_L and S_W we output $G_0(x', y', z', x, y, z, \omega)$ for (x', y', z') on S_U (the measurement surface).

Step (2) Downward continue the receiver

$$\begin{aligned} P(x, y, z, x_s, y_s, z_s, \omega) &= \int \left\{ \frac{\partial G_0^{DN}}{\partial z'}(x, y, z, x', y', z', \omega) P(x', y', z', x_s, y_s, z_s, \omega) \right. \\ &- \left. \frac{\partial P}{\partial z'}(x', y', z', x_s, y_s, z_s, \omega) G_0^{DN}(x, y, z, x', y', z', \omega) \right\} dx'dy' \quad , \end{aligned} \tag{65}$$

where z' = fixed depth of the cable and (x_s, y_s, z_s) = fixed location of the source. This brings the receiver down to (x, y, z), a point below the measurement surface in the volume V.

Step (3) Now downward continue the source

$$\begin{aligned} P(x_g, y_g, z, x, y, z, \omega) &= \int \left\{ \frac{\partial G_0^{DN}}{\partial z_s}(x, y, z, x_s, y_s, z_s, \omega) P(x_g, y_g, z, x_s, y_s, z_s, \omega) \right. \\ &- \left. \frac{\partial P}{\partial z_s}(x_g, y_g, z, x_s, y_s, z_s, \omega) G_0^{DN}(x, y, z, x_s, y_s, z_s, \omega) \right\} dx_s dy_s. \end{aligned} \tag{66}$$

$P(x_g, y_g, z, x, y, z, \omega)$ is a downward continued receiver to (x_g, y_g, z) and the source to (x, y, z) and change to midpoint offset $P(x_m, x_h, y_m, y_h, z_m, z_h = 0, \omega)$ and

$$\begin{aligned} &\int d\omega \left\{ \frac{\partial G_0^{DN}}{\partial z_s}(x, y, z, x_s, y_s, z_s, \omega) P(x_g, y_g, z, x_s, y_s, z_s, \omega) \right. \\ &- \left. \frac{\partial P}{\partial z_s}(x_g, y_g, z, x_s, y_s, z_s, \omega) G_0^{DN}(x, y, z, x_s, y_s, z_s, \omega) \right\} \quad , \end{aligned} \tag{67}$$

and Fourier transform over x_m, x_h, y_m, y_h to find $\widetilde{P}(k_{x_m}, k_{x_h}, k_{y_m}, k_{y_h}, k_{z_m}, z_h = 0, t = 0)$ the RTM uncollapsed migration for a general $v(x, y, z)$ velocity configuration.

RTM AND INVERSE SCATTERING SERIES (ISS) IMAGING: NOW AND THE FUTURE

In practice, RTM is often applied using a wave equation that avoids reflections at reflectors above the target. Impedance matching at boundaries in the modeling, allows density and velocity to both have rapid variation at a reflector, but are arranged so that the normal incidence reflection coefficient will be zero. The result ia a smooth 'apparent velocity' that can support diving waves, but seeks to avoid the discontinuous velocity model commitment that including reflections would require. In RTM, including those reflections above the reflector to be imaged, drives a need for an accurate and discontinuous velocity model. The ISS imaging methods welcome (and require) all the reflectors above the one reflector being imaged, *without* implying a concomitant need for an accurate discontinuous velocity model.

One way to view the RTM to Inverse Scattering Series (ISS) (Weglein et al., 2003) imaging step is as removing reflectionless reflectors by an 'impedance matching' differential equation in RTM to avoid the need for a commitment to an accurate and discontinuous velocity. With ISS imaging we have the opposite situation: the welcome of *all* reflections to the imaging of any reflector, and without the need to know or determine the discontinuous velocity model. That is the next step, and our first field data tests with ISS imaging are underway. In the interim, we thought it useful to provide an assist to current best imaging RTM practice. We will be returning reflections to reflectors thereby turning the problem, observation, and obstacle in current RTM into the instrument of significant imaging progress, and without the need for a velocity model, discontinuous or otherwise.

SUMMARY

Migration and migration-inversion require velocity information for location and beyond velocity only for amplitude analyses at depth. So when we say the medium is 'known,' the meaning of known depends on the goal: migration or migration-inversion. Backpropagation and imaging each evolved and then extended/generalized and merged into migration-inversion (Fig. 2).

For one-way wave propagation the double downward continued data, D is

$$D(\text{at depth}) = \int_{S_s} \frac{\partial G_0^{-D}}{\partial z_s} \int_{S_g} \frac{\partial G_0^{-D}}{\partial z_g} \, D \, dS_g \, dS_s \quad , \tag{68}$$

where D in the integrand $= D$(on measurement surface), $\partial G_0^{-D}/\partial z_s =$ anticausal Green's function with Dirichlet boundary condition on the measurement surface, $s =$ shot, and $g =$ receiver. For two-way wave double downward continuation:

$$\begin{aligned} D(\text{at depth}) &= \int_{S_s} \left[\frac{\partial G_0^{DN}}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_0^{DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_0^{DN} \right\} dS_g \right. \\ &+ \left. G_0^{DN} \frac{\partial}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_0^{DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_0^{DN} \right\} dS_g \right] dS_s \quad , \end{aligned} \tag{69}$$

where D in the integrands $= D$(on measurement surface). G_0^{DN} is *neither* causal nor anticausal. G_0^{DN} is not an *anti*causal Green's function; it is not the inverse or adjoint of any physical propagating Green's function. It is the Green's function needed for RTM. G_0^{DN} is the Green's function for the model of the finite volume that vanishes along with its normal derivative on the lower surface and the walls. If we want to use the anticausal Green's function of the two-way propagation with Dirichlet boundary conditions at the measurement surface then we can do that, but we will need measurements at depth and on the vertical walls. To have the Green's function for two-way propagation that doesn't need data at depth and on the vertical sides/walls, that requires a non-physical Green's function that vanishes along with its derivative on the lower surface and walls.

In the Inverse Scattering Series (ISS) model (sketch 4 in Fig. 2) the Lippmann-Schwinger (LS) equation over all space, rather than Green's theorem, is called upon and the Lippmann-Schwinger equation requires no imposed boundary conditions on S since all boundary conditions are already incorporated in LS from linearity/superposition and causality. See, e.g., Weglein et al. (2003), Stolt and Jacobs (1980), Weglein et al. (2009), and Weglein et al. (2006).

The appropriate Green's function, for a closed surface integral in Green's theorem, with an arbitrary and known medium within the volume can be satisfied with *any* Green's function satisfying the propagation properties within the volume and with Dirichlet, Neumann, or Robin boundary conditions on the closed surface. The issue and/or problem in exploration reflection seismology is the measurements are only on the upper surface.

Why Green's theorem for migration algorithms?

1. Allows a wave theoretical platform/framework for wave-field prediction from surface measurements that builds on quantitative and potential field theory history and evolution.

2. Allows $(\mathbf{x}, \omega)$ processing without transform artifacts and yet is wave theoretic in a $(\mathbf{x}, \omega)$ world where up-down is not so simple to define as in $(\mathbf{k}, \omega)$. Deghosting (Zhang (2007)) and wavelet estimation (Weglein and Secrest (1990)) are other examples where Green's theorem provides $(\mathbf{x}, \omega)$ advantage.

3. Allows avoidance of very common pitfalls and erroneous algorithm derivations based on qualitative (at best) methods launched from Huygens' principle or discrete matrix inverses and it allows the wave theoretic imaging conditions introduced by Clayton and Stolt (1981) and Stolt and Weglein (1985) to be used rather than the lesser cross-correlation of wave-field imaging concepts.

Backpropagation is quantitative from Green's theorem rather than these G_0^{-1}, G_0^*, less wave theoretic more generalized inverse, discrete matrix thinking approaches for backpropagation. For RTM and Green's theorem the data, D, at depth is definitely *not*

$$D \neq \int G_{0S}^{-1} \int G_{0R}^{-1} D \qquad \text{(Huygens)} \quad , \tag{70}$$

where G_0^- indicates an anticausal Green's function. This is OK with Huygens but *violates Green's theorem* and the equation is not dimensionally consistent with the right hand side not having the dimension of data, D. The data, D, at depth for *one-way* waves is

$$D = \int_{S_s} \frac{\partial G_0^{-D}}{\partial z_s} \int_{S_g} \frac{\partial G_0^{-D}}{\partial z_g} D \, dS_g \, dS_s \qquad \text{(Green)} \quad , \tag{71}$$

where $D =$ Dirichlet boundary condition on top and G_0^- anticausal. This is OK with Green but not for two-way RTM propagation. The data, D, at depth for *two-way* waves is

$$\begin{aligned} D \quad &= \quad \int_{S_s} \left[\frac{\partial G_0^{DN}}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_0^{DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_0^{DN} \right\} dS_g \right. \\ &+ \quad \left. G_0^{DN} \frac{\partial}{\partial z_s} \int_{S_g} \left\{ \frac{\partial G_0^{DN}}{\partial z_g} D + \frac{\partial D}{\partial z_g} G_0^{DN} \right\} dS_g \right] dS_s \qquad \text{(Green)} \quad , \end{aligned} \tag{72}$$

where $DN =$ Dirichlet and Neumann boundary conditions to be imposed on bottom and walls and G_0^{DN} is neither causal nor anticausal nor a combination. Please see Fig. 3.

COMMENTS AND FUTURE DEVELOPMENTS

In this manuscript, we provide a firm foundation for RTM based on Green's theorem. As in the case of interferometry (Ramírez and Weglein (2009)) misuse, abuse and/or misunderstanding of Green's theorem in RTM has also led to strange and curious interpretations, and to opinions being offered about the cause of artifacts and observed problems and communicating 'deep new insights' that are neither new nor accurate. We communicate here to simply understand and stick with

Green's theorem as the guide and solution in both cases, interferometry and RTM. The original RTM methods of running the wave equation backwards with surface reflection data as a boundary condition is not a wave theory method for wave-field prediction, neither in depth nor in reversed time. In Huygens' principle the wave-field prediction doesn't have the dimension of a wave-field. In fact that idea corresponds to the Huygens' principle idea (Huygens (1690)) which was made into a wave theory predictor by George Green in 1826.

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APPENDIX A: CONFIRMATION THAT THE GREEN'S FUNCTION EQ. (14), WHEN USED IN GREEN'S THEOREM, WILL PRODUCE A TWO-WAY WAVE FOR $A < Z < B$ WITH ONLY MEASUREMENTS ON THE UPPER SURFACE.

$$
\begin{aligned}
& P(z,\omega) \\
= \;& \int_a^b dz' \left(\underbrace{\frac{-1}{2ik} e^{-ikz} e^{ikz'} + \frac{1}{2ik} e^{ik|z-z'|}}_{G_0(z,z',\omega)} \right) \underbrace{\rho(z',\omega)}_{0} \\
& + |_a^b \left(\underbrace{(Ae^{ikz'} + Be^{-ikz'})}_{P(z',\omega)} \left(\underbrace{\frac{-1}{2ik} e^{-ikz} e^{ikz'} ik + \frac{1}{2ik} e^{ik|z-z'|} ik \;\; \mathrm{sgn}(z-z')(-1)}_{\frac{dG_0(z,z',\omega)}{dz'}} \right) \right. \\
& \left. - \left(\underbrace{\frac{-1}{2ik} e^{-ikz} e^{ikz'} + \frac{1}{2ik} e^{ik|z-z'|}}_{G_0(z,z',\omega)} \right) \underbrace{(Ae^{ikz'} ik + Be^{-ikz'}(-ik))}_{\frac{dP(z',\omega)}{dz'}} \right) \\
= \;& \frac{-1}{2} |_a^b (\cancel{Ae^{ik(2z'-z)}} + A \;\; \mathrm{sgn}(z-z') e^{ikz'} e^{ik|z-z'|} \\
& + Be^{-ikz} + B \;\; \mathrm{sgn}(z-z') e^{-ikz'} e^{ik|z-z'|} \\
& - \cancel{Ae^{ik(2z'-z)}} + Ae^{ikz'} e^{ik|z-z'|} + Be^{-ikz} - Be^{-ikz'} e^{ik|z-z'|}) \\
= \;& \frac{-1}{2} (A \underbrace{\mathrm{sgn}(z-b)}_{-1} e^{ikb} e^{ik \overbrace{|z-b|}^{b-z}} \\
& + \cancel{Be^{-ikz}} + B \underbrace{\mathrm{sgn}(z-b)}_{-1} e^{-ikb} e^{ik \overbrace{|z-b|}^{b-z}} \\
& + Ae^{ikb} e^{ik \overbrace{|z-b|}^{b-z}} + \cancel{Be^{-ikz}} - Be^{-ikb} e^{ik \overbrace{|z-b|}^{b-z}} \\
& - A \underbrace{\mathrm{sgn}(z-a)}_{1} e^{ika} e^{ik \overbrace{|z-a|}^{z-a}} \\
& - \cancel{Be^{-ikz}} - B \underbrace{\mathrm{sgn}(z-a)}_{1} e^{-ika} e^{ik \overbrace{|z-a|}^{z-a}} \\
& - Ae^{ika} e^{ik \overbrace{|z-a|}^{z-a}} - \cancel{Be^{-ikz}} + Be^{-ika} e^{ik \overbrace{|z-a|}^{z-a}}) \\
= \;& \frac{-1}{2} (- \cancel{Ae^{ik(2b-z)}} - Be^{-ikz} + \cancel{Ae^{ik(2b-z)}} - Be^{-ikz} \\
& - Ae^{ikz} - \cancel{Be^{-ik(2a-z)}} - Ae^{ikz} + \cancel{Be^{-ik(2a-z)}}) \\
= \;& \frac{-1}{2} (-2Ae^{ikz} - 2Be^{-ikz}) \\
= \;& Ae^{ikz} + Be^{-ikz}
\end{aligned}
$$

APPENDIX B: EVALUATING A_1, B_1, C_1, AND D_1 IN EQ. (39).

For $z' > a$ choose C_1 and D_1 such that

$$
\begin{aligned}
G(z, B, \omega) &= 0 \quad \text{and} \\
\left[\frac{dG}{dz'}(z, z', \omega)\right]_{z'=B} &= 0 \\
\implies \frac{T}{2ik} e^{ik|a-z|} e^{ik_1(B-a)} + C_1 e^{ik_1 B} + D_1 e^{-ik_1 B} &= 0 \\
\frac{T}{2ik} e^{ik|a-z|} e^{ik_1(B-a)} ik_1 + C_1 e^{ik_1 B} ik_1 + D_1 e^{-ik_1 B}(-ik_1) &= 0 \\
\implies C_1 e^{ik_1 B} + D_1 e^{-ik_1 B} &= -\frac{T}{2ik} e^{ik|a-z|} e^{ik_1(B-a)} \\
C_1 e^{ik_1 B} - D_1 e^{-ik_1 B} &= -\frac{T}{2ik} e^{ik|a-z|} e^{ik_1(B-a)}
\end{aligned}
$$

Adding and subtracting give

$$
\begin{aligned}
2C_1 e^{ik_1 B} &= -\frac{2T}{2ik} e^{ik|a-z|} e^{ik_1(B-a)} \\
C_1 &= -\frac{T}{2ik} e^{ik|a-z|} e^{-ik_1 a} \\
2D_1 e^{-ik_1 B} &= 0 \\
D_1 &= 0
\end{aligned}
$$

For $z' = a$ choose A_1 and B_1 such that

$$
\begin{aligned}
G(z, a, \omega)|_{z'<a} &= G(z, a, \omega)|_{z'>a} \quad \text{and} \\
\left[\frac{dG}{dz'}(z, z', \omega)\right]_{z'<a \text{ at } z'=a} &= \left[\frac{dG}{dz'}(z, z', \omega)\right]_{z'>a \text{ at } z'=a}
\end{aligned}
$$

$$
\begin{aligned}
\implies & \frac{e^{ik|a-z|}}{2ik} + R\frac{\overbrace{e^{-ik(a-a)}}^{1}}{2ik} e^{ik(a-z)} + A_1 e^{ika} + B_1 e^{-ika} \\
&= \frac{T}{2ik} e^{ik|a-z|} \underbrace{e^{ik_1(a-a)}}_{1} + \underbrace{C_1}_{-(T/2ik)e^{ik|a-z|}e^{-ik_1 a}} e^{ik_1 a} + \underbrace{D_1}_{0} e^{-ik_1 a}
\end{aligned}
$$

$$
\begin{aligned}
& \frac{e^{ik|a-z|}}{2ik} ik\, sgn(a-z) + R\frac{\overbrace{e^{-ik(a-a)}}^{1}}{2ik}(-ik) e^{ik(a-z)} + A_1 e^{ika} ik + B_1 e^{-ika}(-ik) \\
&= \frac{T}{2ik} e^{ik|a-z|} \underbrace{e^{ik_1(a-a)}}_{1} ik_1 \\
&+ \underbrace{C_1}_{-(T/2ik)e^{ik|a-z|}e^{-ik_1 a}} e^{ik_1 a} ik_1 + \underbrace{D_1}_{0} e^{-ik_1 a}(-ik_1)
\end{aligned}
$$

$$\implies \frac{e^{ik|a-z|}}{2ik} + \frac{R}{2ik}e^{ik(a-z)} + A_1 e^{ika} + B_1 e^{-ika} = \frac{T}{2ik}e^{ik|a-z|} - \frac{T}{2ik}e^{ik|a-z|} = 0$$

$$\frac{1}{2}\, sgn(a-z)e^{ik|a-z|} - \frac{R}{2}e^{ik(a-z)} + A_1 ike^{ika} - B_1 ike^{-ika} = \frac{Tik_1}{2ik}e^{ik|a-z|} - \frac{Tik_1}{2ik}e^{ik|a-z|} = 0$$

$$\implies A_1 e^{ika} + B_1 e^{-ika} = -\frac{e^{ik|a-z|}}{2ik} - \frac{R}{2ik}e^{ik(a-z)}$$

$$A_1 e^{ika} - B_1 e^{-ika} = -\frac{1}{2ik}\, sgn(a-z)e^{ik|a-z|} + \frac{R}{2ik}e^{ik(a-z)}$$

Adding and subtracting give

$$\begin{aligned}
2A_1 e^{ika} &= -\frac{1}{2ik}e^{ik|a-z|}(1 + sgn(a-z)) \\
A_1 &= -\frac{1}{4ik}e^{-ika}e^{ik|a-z|}(1 + sgn(a-z)) \\
2B_1 e^{-ika} &= -\frac{1}{2ik}e^{ik|a-z|}(1 - sgn(a-z)) - \frac{2R}{2ik}e^{ik(a-z)} \\
B_1 &= -\frac{1}{4ik}e^{ika}e^{ik|a-z|}(1 - sgn(a-z)) - \frac{2R}{4ik}e^{ik(2a-z)} = -\frac{1}{4ik}(e^{ika}e^{ik|a-z|}(1 - sgn(a-z)) + 2Re^{ik(2a-z)})
\end{aligned}$$

Check:

$$\begin{aligned}
&G(z,a,\omega)|_{z'<a} - G(z,a,\omega)|_{z'>a} \\
=&\frac{e^{ik|a-z|}}{2ik} + \frac{R}{2ik}e^{ik(a-z)} + \left(-\frac{1}{4ik}e^{-ika}e^{ik|a-z|}(1 + sgn(a-z))\right)e^{ika} \\
&+ \left(-\frac{1}{4ik}(e^{ika}e^{ik|a-z|}(1 - sgn(a-z)) + 2Re^{ik(2a-z)})\right)e^{-ika} \\
&- \frac{T}{2ik}e^{ik|a-z|} - \left(-\frac{T}{2ik}e^{ik|a-z|}e^{-ik_1 a}\right)e^{ik_1 a} - (0)e^{-ik_1 a} \\
=&\underbrace{\frac{1}{2ik}e^{ik|a-z|}}_{\text{cancels 1}} \underbrace{+\frac{R}{2ik}e^{ik(a-z)}}_{\text{cancels 2}} \underbrace{-\frac{1}{4ik}e^{ik|a-z|}}_{\text{cancels 1}} \underbrace{-\frac{1}{4ik}e^{ik|a-z|}sgn(a-z)}_{\text{cancels 3}} \\
&\underbrace{-\frac{1}{4ik}e^{ik|a-z|}}_{\text{cancels 1}} \underbrace{+\frac{1}{4ik}e^{ik|a-z|}sgn(a-z)}_{\text{cancels 3}} \underbrace{-\frac{2R}{4ik}e^{ik(a-z)}}_{\text{cancels 2}} \\
&\underbrace{-\frac{T}{2ik}e^{ik|a-z|}}_{\text{cancels 4}} \underbrace{+\frac{T}{2ik}e^{ik|a-z|}}_{\text{cancels 4}} \\
=&0 \quad \text{as desired}
\end{aligned}$$

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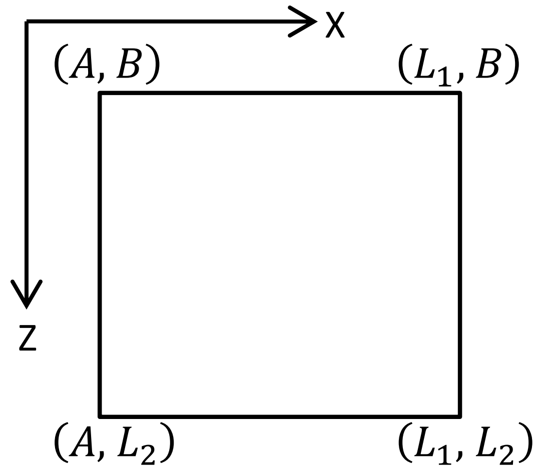

Figure 1: *Two dimensional finite volume model*

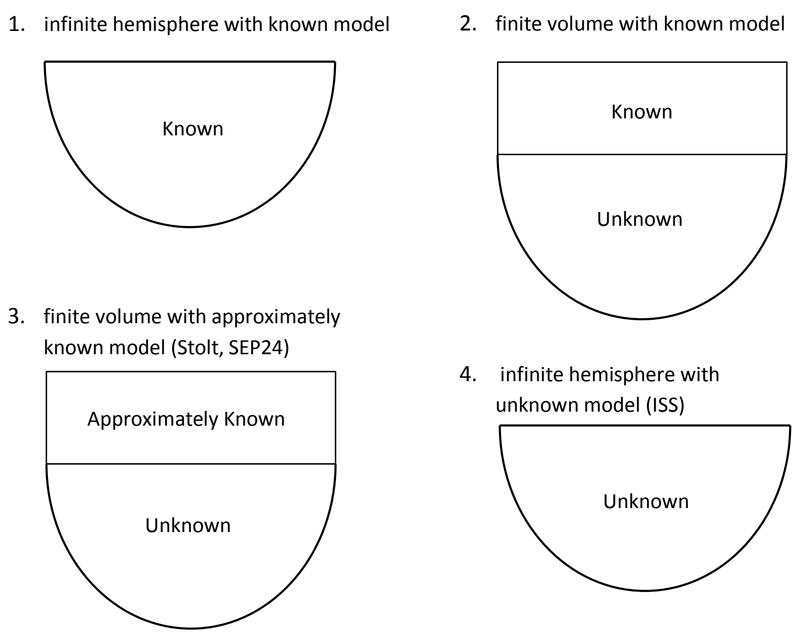

Figure 2: *Backpropagation model evolution*

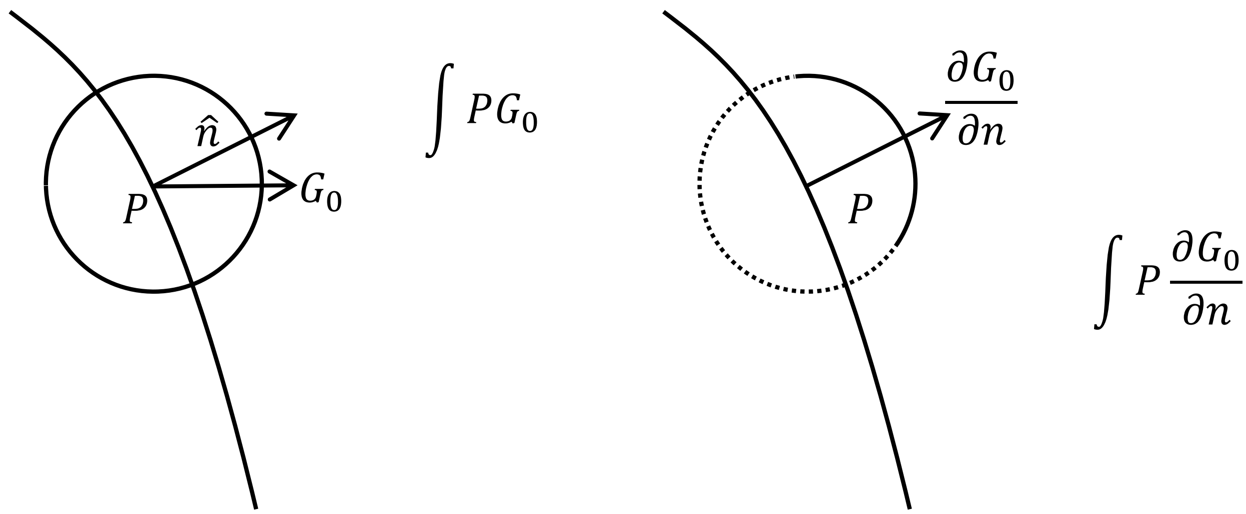

Figure 3: *Qualitative vs. quantitative wave propagation:*
(Left) Huygens (1690), and e.g., Whitmore (1983), McMechan (1983), Fletcher et al. (2006), Berkhout (1997), Claerbout (1992), Dong et al. (2009), Luo and Schuster (2004)
(Right) Green (1828), and e.g., Morse and Feshbach (1953), Born and Wolf (1999), Stolt (1978), Schneider (1978), Esmersoy and Oristaglio (1988), Weglein and Secrest (1990), Weglein et al. (1997), Liu et al. (2006), Ramírez and Weglein (2009), and Weglein et al. (2011)